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APPREHENDING MATHEMATICAL STRUCTURE: A CASE STUDY OF COMING TO UNDERSTAND A COMMUTATIVE RING

ABSTRACT. Abstract algebra courses tend to take one of two pedagogical routes: from examples of mathematics structures through definitions to general theorems, or directly from definitions to general theorems. The former route seems to be based on the implicit pedagogical intention that students will use their understanding of particular examples of an algebraic structure to get a sense of those properties which form the basis of the fundamental definitions. We will explain the transition from examples to abstract algebra as a series of shifts of attention and in this paper we will use a case study to examine the initial shift, which we will call *apprehending a structure*, and examine how one student came to apprehend the structure of the commutative ring $\mathbf{Z}_{99}$.

KEY WORDS: abstract algebra, advanced mathematical thinking, representational re-description, structural thinking, apprehending structure

The teaching of abstract algebra is a disaster, and this remains true almost independently of the quality of the lectures.

Leron and Dubinsky (1995)

1. INTRODUCTION

To someone with a strong background in abstract algebra, just encountering the phrase 'the ring $\mathbf{Z}_{99}$' may well bring with it a flood of images and ideas about zero elements, unit elements, zero-divisors and so on. For them, $\mathbf{Z}_{99}$ is an *instance* of a commutative ring and as such it *inherits* all the properties of commutative rings they have learned (and perhaps proved) in abstract situations, along with ones which may be peculiar to this instance (such as, because 99 is composite, having zero-divisors). To teachers, however, encouraging students to engage in tasks such as the investigation of the properties of $\mathbf{Z}_{99}$ can be intended to serve a complementary purpose: to enable the students to attend to the interrelationships between the elements of $\mathbf{Z}_{99}$ which appear as consequences of the modular operations, as a preparation for the later development of formal definitions related to commutative rings.

This paper will explore the extent to which this complementary purpose is fulfilled for one student and how she begins to make sense of *structure*. In that sense, it complements papers which focus on how undergraduate students make sense of the *processes* and *objects* of their mathematical

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studies (Asiala et al., 1996; Gray and Tall, 1994; Sfard, 1991) and how they come to understand the fundamental notions of formal proof (Alibert and Thomas, 1991; Harel and Sowder, 1998; Moore, 1994; Selden and Selden, 2002). First the paper will explore the concept of the acquisition of structural sense, particularly in the context of abstract algebra, and define what we mean by 'apprehending structure' in terms of relationships between objects. We will then explore the pedagogical routes to abstract algebra and the teaching intentions that implicitly inform them. In analyzing the data from our case study, we will suggest that Karmiloff-Smith's (1992) notion of representational redescription provides a suitable way of accounting for the development of structural apprehension.

The main part of the paper will contextualize the idea of apprehending structure through a detailed examination of one student's development as she tries to make sense of a structure which was intended to be a conceptual basis of a commutative ring. Over a period of three years, from her first encounter with the structure, to her communicating her discoveries and understandings in a diploma thesis, data was collected in a variety of ways. Looking at both small scale conceptions of objects and particular operations within the structure and the conception of the structure as a whole, we will show that there is a consistent pattern in the shift of her attention from particular objects and operations to the interrelationships of objects caused by the operations. This is accompanied by an increasing consciousness and ability to communicate that shifted attention.

Finally, we will consider the implications of our perspective for differing pedagogical strategies for teaching abstract algebra.

2. Abstract algebra

The existing research literature concerning how students learn abstract algebra splits into concerns about common student misconceptions, and explorations of innovative teaching experiments based on the genetic decomposition of fundamental algebraic concepts. Many of the fundamental misconceptions, such as confusing a theorem with its converse (Hazzan and Leron, 1996) or difficulties distinguishing between a set and an element of the set (Selden and Selden, 1978), may be of more general concern than the cognitive development of abstract algebra. Others, such as a lack of focus on the structure's given operations (Hart, 1994) and using techniques and ideas from systems other than the one under consideration (Hazzan, 1994), seem more fundamentally related to the nature of learning abstract algebra and it is on these ideas that our theoretical account and case study analysis will focus.

A second stream of literature depends upon the development of innovative courses in abstract algebra, founded upon the idea of a genetic decomposition of concepts using the notion that the transition from action through process to object conceptions are often fundamental in many areas of mathematics (Asiala et al., 1996). In early work on developing a curriculum based on programming operations (such as coset multiplication), and tests for properties (such as associativity), Dubinsky et al. (1994) hoped to help students move through these various conceptions. The research focused on the students' conceptions in this type of course and demonstrated significant difficulties with making process-object transitions and with understanding structural concepts which involve working with substructures, such as cosets, quotient groups and normality (Asiala et al., 1997). Other investigations in the same theoretical tradition have focused on student difficulties with particular algebraic structures or examples, such as permutation and symmetry (Asiala et al., 1998) or the dihedral group D_4 (Zazkis et al., 1996).

Hazzan (1999), in a more general exploration of abstract algebra, introduces the notion of *reducing abstraction* as a 'plausible theoretical framework to explain students' ways of thinking in abstract algebra situations'. Reducing abstraction may involve basing arguments on familiar objects and operations (rather than the more abstract or unusual ones intended by the teacher), interpreting definitional properties (such as commutativity) as procedural instructions and working with an element from a set rather than the set as a whole or working with an instance rather than the generality. The opportunity to reduce abstraction may depend to a large extent on the way the teaching is organized.

3. Attentional shifts in learning abstract algebra

We contend that there are two basic teaching strategies observable in most textbooks and descriptions of courses, and that these can be related to implicit intentions concerning the problem of reducing abstraction. Teaching the definitions of algebraic structures before examining examples might encourage students to work at high levels of abstraction from the beginning. Teaching from examples of structures to definitions of those structures, however, carries the implicit pedagogical intention of encouraging students to attend to aspects of the particular which will appear as important facets of the general (Mason and Pimm, 1984). Indeed, Skemp argues that this examples-to-definitions route is cognitively necessary:

> Concepts of a higher order than those which people already have cannot be communicated to them by a definition, but only by arranging for them to encounter a suitable collection of examples.

> Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner.
>
> (Skemp, 1971)

For example, in working with sets of permutations and the operation of composition, students are encouraged to attend to permutations which reverse the action of others, to a permutation which appears to have no effect, to the importance of the order in which permutations are composed, etc. This, it is supposed, will enable them to make sense of the properties which define 'group' or are important in the general analysis of groups.

Dreyfus, however, argues that, for some students, the route through examples may be superfluous or even a distraction:

> [One student] was able to learn directly about abstract algebraic structures, and concrete representations tended to disturb him, if anything. When asked if a given structure was a ring, he immediately realized he needed to prove three things and proceeded to prove these with maximal reliance on earlier proved structural results. A similar situation may pertain to advanced mathematics students who have had the opportunity to acquire considerable experience with abstraction; this experience is likely to make some of the above stages superfluous and, for complex mathematical structures, even a hindrance to abstraction ... abstraction from one, or even from zero cases, may be seen to be easier for such students.
>
> (Dreyfus, 1991)

We suggest that moving from examples to formality involves multiple shifts of attention (in the sense of Mason, 1989). In coming to understand a particular set and operation as an *example* of a structure, the student has to make sense of it in two stages. Clearly they must first understand how each operation works and what the objects in the set are. This is not necessarily straightforward: coming to understand, say, permutations as objects which the operation of composition can act upon is far from automatic (Asiala et al., 1998) and certainly encapsulating permutations as objects seems necessary for the second stage of understanding the structure of a given group of permutations.

This second stage involves attending to interrelationships between objects which are the consequences of the operations.[1] In particular, the students should come to attend to interrelationships amongst elements within the particular structure which are important in the general theory. In the case of using particular permutation groups to motivate the definition and subsequent analysis of group theory, we, as teachers, would like our students to attend not to the particular objects and operation, but to the fact that imposing the operation on the set of objects creates interrelationships which are important, such as associativity, inverses, etc. Thus, to understand initial examples in preparation for working with the general and formal, we want students to shift their attention from the objects and operations to their relationships.

Subsequent to this, as they are introduced to the formal definitions of, say, a group, we need them to shift their attention again from the particular relationships, which are observed in the example, to general statements of relationships. These general statements need to be seen as definitions which can be used as the basis for deductions that reveal other general relationships amongst elements and properties of the structure (theorems). The power of much abstract algebra comes from this shift – working from the generality of the definitions to discover further relationships which tell us about properties which may have been hidden in examples of the structures (for example that certain sets of permutations may be collected together and form structures of their own as, say, quotient groups). The shift of attention from the inter-relationships in the example to those inter-relationships as examples of the defining properties of the abstract structure involves a focus on what Dorier (1995) calls the 'unifying and generalizing concept'. Sierpinska sees this shift as both necessary and problematic:

> Examples are, in understanding abstract concepts, the indispensable prop and the necessary obstacle. It is on the basis of examples that we make our first guesses. When we start to probe our guesses, the fundamental role is slowly taken over by the definitions.
>
> (Sierpinska, 1994)

This shift of attention acts rather like Deacon's shift from indexical to symbolic cognition (Deacon, 1997). Deacon contends that symbolic thinking catalysed the co-evolution of language and the brain and that systems of symbols connected, through indexical links, to systems of referents provide us with ways of working at ever more abstract levels. Importantly, the generality of the more abstract thinking applies to *all* applicable examples.

In Deacon's terms, then, the unstated (and unexplored) pedagogical intention behind an examples-to-definitions teaching strategy is to encourage students to focus attention on the interrelationships between objects in preparation for providing indexical links between signs like 'commutative', 'identity', 'inverses', etc. which will later form the basis of the formal symbol system of, say, group theory. Leron and Dubinsky (1995) also note the importance of the development of indexical links in the learning of abstract algebra through an innovative course: "If the students are asked to construct the group concept on the computer (by programming it), there is a good chance that a parallel construction will occur in their mind".

Thus, the transition from working with an example structure to working abstractly involves an intricate sequence of shifts of attention. These are as follows:

1. Seeing the elements in the set as objects upon which the operations act (which may involve a process-object shift).
2. Attending to the interrelationships between elements in the set which are consequences of the operations.
3. Seeing the signs used by the teacher in defining the abstract structure as abstractions of the objects and operations, and seeing the names of the relationships amongst signs as the names for the relationships amongst the objects and operations.
4. Seeing other sets and operations as *examples* of the general structure and as *prototypical* of the general structure.
5. Using the formal system of symbols and definitional properties to derive consequences and seeing that the properties inherent in the theorems are properties of all examples.

4. Apprehending structure and representational redescription

To some extent this repetitive shifting of attention follows roughly the same pattern as Piaget and Garcia's (1989) movement between stages of operational thinking from intra-, to inter- and then trans-operational thinking.

> Oversimplifing drastically, one might say that in the first stage one performs actions within the objects, with attention to the properties of the objects themselves. Next, one shifts attention to relationships and transformations between objects and invariances across objects. Lastly, one builds a higher level structure that embodies these relationships as its elements and one attends to the properties of this structure. (Kaput, 1994)

It is important to note that, unlike process-object theories, we do not see all of the shifts as involving the reification of new *objects* that have their genesis in processes on previous objects. The first shift described above (seeing the elements in the set as objects upon which the operations act) may indeed involve such a reification. The later stages, however, need not. The final three stages involve relating the definitional properties of the (teacher-defined) abstract algebraic structure with the interrelationships noted in the initial example (and in subsequent instances of the structure).

However, our focal point in this paper is the second shift of attention. We will define the phrase 'apprehending structure'[2] to encapsulate this shift: by it, we will mean *the shift of attention from the familiarity and specificity of objects and operations to the sense of interrelationships between the objects caused by the operations*. For example, this may involve moving from a focus on permutations and composing, to relationships between certain pairs of permutations (which we will eventually call inverse pairs), between one particular permutation (which we will come to call the identity) and all others, etc. We argue that this shift of attention is an act of what Karmiloff-Smith calls 'representational redescription' (Karmiloff-Smith, 1992).

In an attempt to reconcile nativism and constructivism, Karmiloff-Smith argues that learners pass through multiple cycles of re-representing, in different formats, that which they have already internally represented: in Karmiloff-Smith's terms, the representation has come to be redescribed. She does not argue that this is the only source of new material for the mind, but it is an important one for us in this context, as we believe that we can account for the move from objects and operations to relationships and, ultimately, to expressions of generality and deduction as cycles of representational redescription. Arguing against theorists who posit a 'massively modular' structure to the brain (such as Cosmides and Tooby, 1994), Karmiloff-Smith suggests "the brain is not prestructured with ready-made representations; it is channeled to progressively *develop* representations via interaction with both the external environment and its own internal environment."

It is not clear how this process of channeling takes place. Tomasello (1999) argues that it is channeled by a *single* innate structure which encourages us to see others as intentional beings and to create, with them, joint attentional scenes. Informally, we ask ourselves, unconsciously, 'what would I have intended me to attend to if I had made that utterance' every time we hear someone speak to us. In formal learning contexts (such as undergraduate abstract algebra classes, where the teacher starts by asking us to examine particular sets of objects and operations) asking that question of ourselves makes us consider what aspects of the sets and objects the teacher wishes us to attend to and thus, through the process of representational redescription, we may come to attend to their interrelationships – that is we begin to apprehend the structure.

The process by which knowledge is redescribed is broken down by Karmiloff-Smith into four stages, though we will argue that only two (the first and last) can be seen clearly in students' behaviour and, while intermediate aspects can be seen in our data, the distinction between the middle two stages is perhaps too subtle for naturalistic studies such as ours to uncover clearly.

1. Initially information about the structure is encoded as separately stored procedures, with no intra- or inter-domain connections. At this phase, the learner might appear to have a set of actions on pre-existing objects (in the sense of Dubinsky, 1991).
2. These actions become internally redescribed. This redescription is an act of abstraction which retains only some of the aspects of the full

procedures (it is an internal *description* of the procedure, no longer the procedure itself). At this phase, such a description is unconscious, but may manifest itself in the wise choice of objects in the structure with which to work.

3. Learners are able to consciously access the redescription of the procedures, so that they have an appreciation of the relationships within the structure sufficient to guide them in solving structural problems. However, they may not be able to verbalise or symbolically express these relationships. Learners may, for example, note similarities or relationships between objects in the structure, but might not be able to articulate the nature of the similarity or relationship.
4. The final phase is the ability to communicate directly about the relationships and properties of the structure.

Despite the difficulty in discerning unconscious redescriptions from conscious but unsharable ones, the general flow from focusing on existing objects upon which one acts, through an uncommunicable appreciation of the relationships inherent in the structure which develops from the actions on the objects, to a naming and communication about the relationships seems to be a reasonable account for the transition which is the focus of this paper: the apprehension of mathematical structure.

5. THE STUDY

The data for this paper was taken from one student who took a particularly significant role in a larger study about the learning of abstract algebra. All the students were training to become secondary mathematics teachers in the Czech Republic. Traditionally these students encounter concepts of abstract algebra structures quite formally. However, they appear often to acquire them by memorisation and learning the rules of manipulating symbols. Not least because of their intended future career, their emphasis on learning without an apparent appreciation of meaning was of concern. Therefore, it was decided to carry out a research project on the ways in which they link or do not link their formal knowledge of algebra acquired in their university lectures to a particular example of an abstract structure acquired informally by their own investigations. A research tool was thus developed which involved an instance of what mathematicians would see rapidly as an important algebraic structure: $\mathbf{Z}_{99}$.

The students were introduced to a structure consisting of a set, $\underline{\text{A}}_2 = \{1, 2, \ldots, 99\}$, and two operations $\oplus$, $\otimes$ (called z-addition and z-multiplication respectively). These operations were defined in terms of what was called a reduction mapping on the naturals: $r\colon \mathbf{N} \to \mathbf{N}$,

$r(n) = n - 99 \times [n/99]$, (where $[y]$ is the integer part of y). Initially the reduction mapping was introduced as a verbal instruction rather than in mathematical symbols, and was accompanied by examples:

For a natural number $n < 100$, $r(n) = n$. If $n \geq 100$, we split n into pairs of digits starting from the units digit and add the pairs together. We repeat the procedure until we get an element of $\underline{A}_2$. For example,

$$r(682) = r(6 + 82) = 88,$$
$$r(7\,945) = r(79 + 45) = r(124) = r(1 + 24) = 25.$$

The two operations on $\underline{A}_2$ were then defined and illustrated as:

$$\forall x, y \in \underline{A}_2, x \oplus y = r(x + y) \quad \text{and} \quad x \otimes y = r(x \times y).$$

For instance, $72 \oplus 95 = r(167) = 68$, $72 \otimes 95 = r(6\,840) = 9$.

It is worth noting that, while the structure is disguised so as not to cue students instantly into seeing the structure as a supposedly familiar one, it was expected that the students would come to see it as a representation of $\mathbf{Z}_{99}$. They did not. Certainly all the students would have met some aspects of modular arithmetic prior to encountering this structure and it was felt that they might make some connections between their formally taught abstract algebra and this situation. Again, they did not.

There were originally 12 participants all of whom took part in the first stage of the research reported here. However, one of them, whom we will call 'Molly', became extremely interested in the work and took up an offer to continue to work on the topic with a view, eventually, to writing her diploma thesis about it. In all, from her first encounter to the submission of her diploma thesis, she worked on and off with the structure for three and a half years. As a result, her case study database is extremely rich and diverse. She was communicative, conscientious, had a good rapport with the interviewer and was willing to speak freely about her thinking. She claimed to like mathematics and particularly enjoyed solving mathematical problems.

In terms of her knowledge of abstract algebra, when she started working on the problem, she had briefly met the concept of group, ring and congruences modulo n (with examples of small n). In her third year (which corresponds to weeks 78–110 of the study, see Figure 1), she took a course in which structures with one and two operations were investigated in more detail.

The data collection consists of three main forms: Interviews, Writing and Investigations.

Nine one-to-one semi-structured, clinical interviews (in the sense of Ginsburg, 1981) each lasting between 15 and 60 minutes were carried out

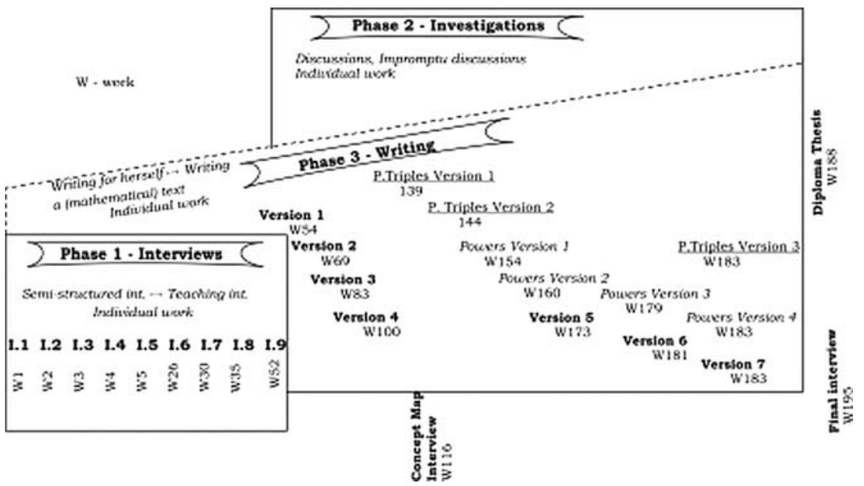

Figure 1. Overview of data collection.

during which Molly was introduced to the structure by defining $\underline{A}_2$, the reduction mapping (as an instruction) and z-addition and z-multiplication. Then she was given some problems to solve, starting with additive and multiplicative equations. Later, she posed her own tasks related to the structure. The interviews were recorded and later transcribed.

Under the heading of 'writing', the case study database includes Molly's writing for herself and her mathematical description of results she found about the structure which culminated in her writing a diploma thesis. There were 7 substantial pieces of writing, many being drafts of the final thesis. The writing comprises her descriptions of basic properties of the structure, of her investigation of general powers and of Pythagorean triples in the structure. As Morgan (1998) suggests, as well as being a good source of data for the study, "writing can actually help students in their learning of mathematics, in particular in supporting moves towards the symbolization of generalizations, but also in supporting reflection and the development of problem solving processes."

Alongside much of the writing, Molly was involved in a number of self-directed investigations into the structure which led to impromptu discussions with the interviewer, and short pieces of writing. Detailed field notes were taken of all of these encounters.

In addition, Molly attended a 'concept map interview', in which she was asked to report on a concept map (in the sense of Novak, 1990) which she had previously completed relating her current ideas about the structure, and a 'final interview' where she answered questions about her views of mathematics and mathematics teaching, her experiences as a learner across both school and university and about her experience with this research study.

Figure 1 provides an overview of Molly's encounters with the structure, the data collected and the times at which the encounters and data collection took place. The data were analysed using methods adapted from grounded theory (Glaser and Strauss, 1967) together with atomic analysis, used particularly for the analysis of students' written solutions (Hejný, 1992). The analysis began by exploring the interviews, coding initially on mathematical aspects in the transcripts. The transcripts were then summarised and emergent phenomena noted. These emerging phenomena led to discerning clear pathways of the development of particular aspects of a mathematical structure. This style of analysis and the codes and the categories were then applied to the remaining data using the constant comparative methods discussed by Glaser and Strauss (1967), leading to a substantive theory.

The analysis of Molly's mathematical writing was done from the viewpoint of the following five categories whose choice was inspired by van Dormolen (1986) and Morgan (1998): structure of writing, style of writing, presentation of new concepts, vocabulary and symbols, mathematical validity. In this paper we will explore Molly's development from the point of view of how she comes to apprehend the structure of $\underline{A}_2$.

6. Apprehending structure

The notion of apprehending structure as the shift of attention from the familiarity and specificity of objects and operations to the sense of interrelationships between the objects caused by the operations can be applied at many scales. One may attend to particular interrelationships caused when one looks at operations in particular ways (which we will call 'small scale' apprehensions of structure), or one may attend to the way these particular interrelationships are linked within the structure as a whole ('large scale').

It should be noted that the separation of small scale and large scale conceptions is necessarily artificial: it is clearly not the case that Molly first gained a 'full' understanding of one small scale conception, then a 'full' understanding of another, and so on until she attained a full apprehension (or 'comprehension') of the whole structure. The development of her conceptions overlap and have been disentangled here only so as to make the developments more accessible.

We will examine two 'small scale' examples of Molly's apprehension of some of the structure of $\underline{A}_2$ under the defined operations – developing inverse operations (called z-subtraction and z-division) and apprehending the structure of zero-divisors in $\underline{A}_2$. We will also give an account of Molly's developing understanding of the structure as a whole. In both of these we will see the shift of attention from objects and operations to interrelationships, as a passage through stages of representational redescription.

6.1. *Inverse operations*

Having worked with the teacher-given definitions of $\underline{A}_2$, $\oplus$, $\otimes$ and r, Molly had already begun to attend to structural aspects of $\underline{A}_2$ which were distinct from the structural aspects of ordinary arithmetic in **N** and **R**, such as the importance of the object 99. The concept of z-subtraction, however, did not occur spontaneously for her in these initial explorations. It had been expected that this might occur as she attempted to solve some additive equations, but Molly instead developed what we called the 'strategy of inverse reduction' (SIR) in which, on trying to solve an equation like $x \oplus 25 = 6$, she considered an 'inverse reduction' of 6 (e.g. $r(105) = 6$) and implicitly solved $x + 25 = 105$ with her knowledge of ordinary arithmetic.

Having avoided the expected spontaneous development, she was prompted (at the end of her second interview session) to 'look at z-subtraction' in preparation for interview 3. She was quickly able to categorise three different situations shown in Figure 2.[3]

During the same interview, as she discussed the last part of Figure 2, she said: "If a is smaller than b, (pause) I will get a negative number (pause)

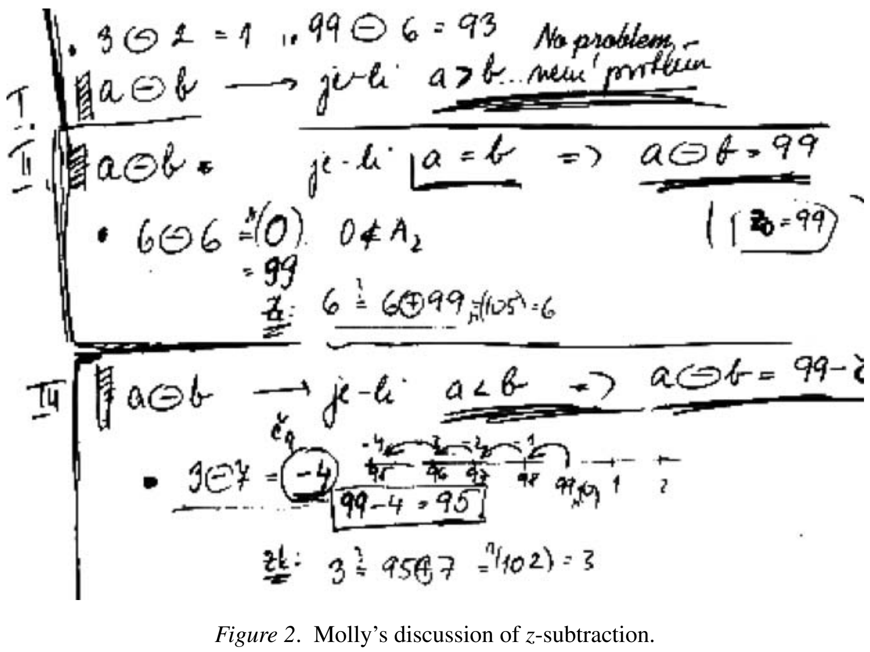

Figure 2. Molly's discussion of z-subtraction.

and I drew it on a number line because I know that the two will repeat itself all the time . . . I calculated an example 3 minus . . . 7 equals minus 4. . . . I simply went the four steps to the left from the zero and came to the result that $3 - 7$ is 95." Her representation of *z*-numbers on a number line suggests that while she retains some imagery for natural numbers she has a partial awareness of the cyclic nature of the structure she is now working with.

At this stage we contend that the fact that she saw three separate situations and used an adaptation of a common representation from familiar integer arithmetic meant that she was still focused on the objects and operations and, indeed, on the objects and operations as adaptations of integers and common arithmetic operations.

Separately from her work on *z*-subtraction, the notion of inverses appeared in a different context when Molly said that "opposite numbers"[4] in **Z** were, for instance, 3 and −3. When asked to consider the situation in $\underline{A}_2$, she gave an example in Figure 3 and stated that the opposite number exists to any *z*-number except 99 because $99 - 99 = 0$ which is not a *z*-number. Later she corrected herself and said that 99 is opposite to 99.

She then used this implicit additive-inverse notion in her subsequent use of a procedure for subtracting a bigger number: $4 \ominus 12 = 4 \oplus (99 - 12)$. At this stage, she did not draw attention to her new procedure for *z*-subtraction or make the link between *z*-additive-inverse and *z*-subtraction explicit. It can also be seen at the stage she drew her concept map where *z*-subtraction is only directly connected to what Molly called "negative numbers" (which she put in inverted commas suggesting she sees the relationship as *like* negative numbers in ordinary arithmetic), the number 99 and the property of being closed on a set.

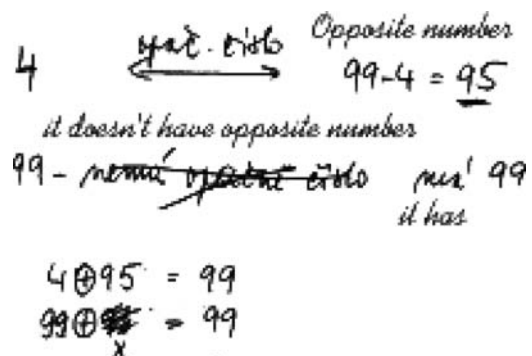

Figure 3. Molly's discussion of opposite numbers.

In her mathematical writing (version 1) as she tried to make explicit her understanding of z-subtraction for her perceived reader, she suggested a sequence of concepts as follows:

A. Negative number: "…Consider the property $x \oplus 99 = x$. If we add number 99 to a negative number, we will get a positive number (this positive number corresponds to that negative number). E.g. $r(-4) = \ominus 4 \oplus 99 = 99 \ominus 4 = 95$, so $r(-4) = 95$."[5]
B. Z-subtraction: "We already know how to change a negative number into a positive number in $\underline{A}_2$, so there is no problem. For example, $6 \ominus 24 = r(-18) \oplus 99 = 99 \ominus 18 = 81$."
C. Z-subtraction on a number line: "The operation of subtraction can be best imagined on a number line. There are only numbers from $\underline{A}_2$ on this line in such a way that they repeat in a cyclic way. So … 52, 53,…, 98, 99, 1, 2,…, 33, 34, 35,…, 14, 15,…, 99, 1, … .. On our example $6 \ominus 24$ we can show how z-subtraction functions on a number line. We will find number 6 (in any cycle) and move from this number 6 by 24 places to the left. We will get number 81."

In version 3 of her writing she tried to reconcile her first introduction of z-subtraction and the structural introduction as the addition of inverse.

"In the set of $\underline{A}_2$, classic negative numbers do not exist, that is why we will introduce *the opposite number* x' which will play a function of a negative number. … It holds $x \oplus x' = 99$, where $x' = 99 - x$, or $r(-x) = 99 - x$, where $x \in \underline{A}_2$."

In the subsequent writing, she kept the sections A, B, C and included another definition of the operation: "z-subtraction of two numbers can be introduced as addition of the opposite number: $\forall x, y \in \underline{A}_2, x \ominus y = x \oplus x'$."

Thus we have seen a shift of attention from the objects and operations (treated often as objects and operations from ordinary integer arithmetic), through increasingly explicit focuses on the relationships between objects (which are part of a new structure), finally to a clear and explicitly written attention to part of the structure of $\underline{A}_2$, namely to z-subtraction and its relationship with z-additive inverse (the 'opposite number').

This same sequence of representational redescription occurs as Molly shifts attention from objects and operations to relationships in the case of z-division:

Phase 1: Initially in working on multiplicative equations like $x \otimes 6 = 9$, she did not consider the possibility of z-division and worked with the SIR procedure. Her focus was on objects and operations (and often as

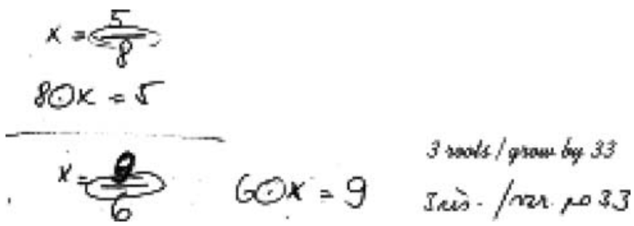

Figure 4. Z-division as solving equations.

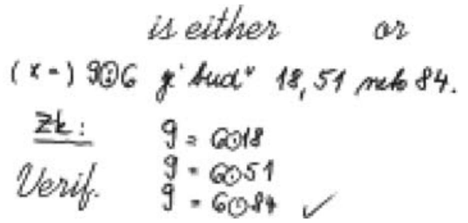

Figure 5. Noting different categories in *z*-division.

objects and operations familiar from ordinary arithmetic) (in interviews 2 and 3).

Phases 2–3: When she was prompted to consider *z*-division, she changed the problem into solving equations (Figure 4, interview 6) and noted that certain situations produce multiple answers (Figure 5, interviews 6 and 7).

She also noted the importance of the zero-element in $\underline{A}_2$ by stating that in these types of equations "*x* must not equal 99". It was not until version 3 of her writing, that she also *explicitly* included divisors of 99. Molly's understanding of *z*-division at this stage is confirmed by her concept map where this operation is directly connected to the closure property and equations only.

Phase 4: While it took her longer to get to the final stage than with *z*-subtraction, by version 6 of her mathematical writing she had both a procedural approach to *z*-division (converting into a multiplicative equation, with provisos about zero-divisors) and a clear focus on interrelationships within $\underline{A}_2$, by attending to *z*-division as multiplication by the *z*-multiplicative inverse.

There is a gap in time between her working with *z*-division and being able to explicitly discuss the link between *z*-division and multiplicative inverses. This might well be because she did not encounter any need to go beyond this procedural approach which worked for her. We suggest that getting to the final stage of her apprehension of z-division was also facilitated by the need to structure her mathematical knowledge about $\underline{A}_2$ for her diploma thesis.

6.2. *Zero-divisors*

The idea of inverse operations is one which Molly would have met in familiar algebraic structures such as integer addition and rational multiplication. In order to apprehend $\underline{A}_2$ as an algebraic structure, however, she has to shift her attention from the familiarity and specificity of the objects and operations to the interrelationships *between* the objects *caused* by the operations – in the case of $\underline{A}_2$ many new interrelationships are formed between the elements of 1, . . . ,99. One such relationship which will be of importance in later consideration of abstract ring theory is that of zero-divisor.

Molly was not asked any specific questions about zero-divisors: the concept appeared slowly and naturally from her work with the initial objects and operations. In the first few interviews she began to notice that certain of the objects behaved differently from others, particularly when working with specific multiplicative equations. She noticed that 3, 6, 9, 12 and 15 "cause the equation to have several solutions" (later adding 11 to this list).

In preparation for interview 4, she produced a classification of numbers according to the number of roots to a multiplicative equation involving them (Figure 6) which she later made more precise.

A set of numbers identified as the multiples of 3 appeared in interview 6 as numbers which present "problems when cancelling [multiplicative equations] by them". By interview 8, she extended this problem set to include multiples of 11 and at this stage she began to explicitly refer to them as "special numbers" or "basic numbers" because of their status in multiplicative equations.

In interview 8, when solving a quadratic equation of the form $a \otimes x \otimes x \oplus b \otimes x = 99$, she changed it to $x \otimes (a \otimes x \oplus b) = 99$

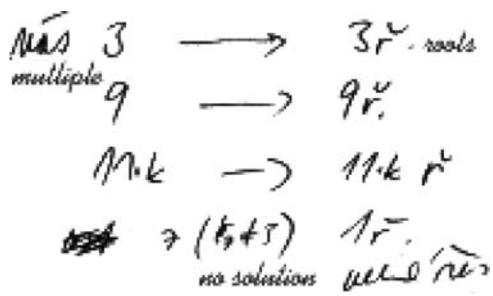

Figure 6. Classification according to numbers of roots of a multiplicative equation.

and adapted her knowledge from ordinary arithmetic that a product is zero if and only if one of the factors is zero (with 99 as the zero element in this structure) with no obvious sense of the possibility of zero-divisors at this stage (Figure 7).

In version 2 of her writing, however, there is a stress on multiples of 3 and 11 in the section of multiplicative equations, and in the section on quadratic equations we see a table which addresses zero-divisors (Figure 8 – note that in her typed written work, she uses '*' for z-multiplication and '\+' for z-addition).

Here, there seems to be some implicit attention to the divisors of 99 (the zero of $\underline{A}_2$ under these operations) though it may not be clear whether the link between 99 being simultaneously the zero and having divisors is clear to her explicitly at this point. In subsequent versions of her writing, however, she talked explicitly about 'divisors of 99' (Figure 9, writing version 3).

Thus again we see, for a structural conception, the shift from a focus on objects and operations (perhaps initially seen as objects and operations

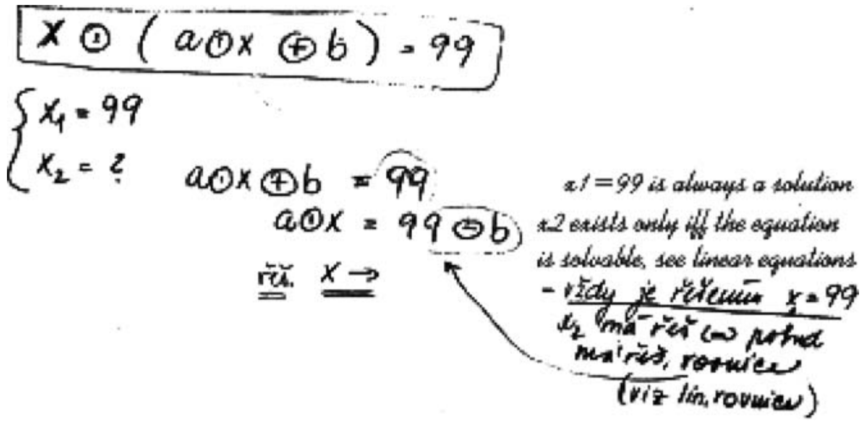

Figure 7. Lacking reference to zero-divisors.

x = 99	∨	a * x \+ b = 99
x = 3	∧	a * x \+ b = 33
x = 33	∧	a * x \+ b = 3
x = 9	∧	a * x \+ b = 11
x = 11	∧	a * x \+ b = 9

Figure 8. Zero-divisors and quadratic equations.

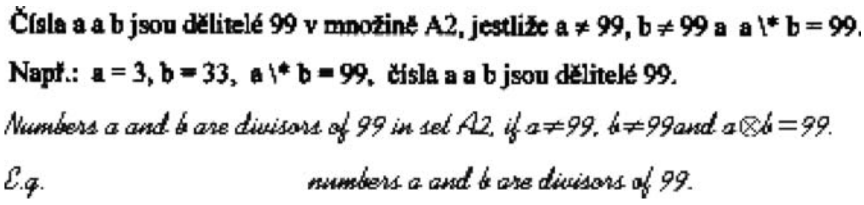
Čísla a a b jsou dělitelé 99 v množině A2, jestliže a ≠ 99, b ≠ 99 a a * b = 99.
Např.: a = 3, b = 33, a * b = 99, čísla a a b jsou dělitelé 99.
Numbers a and b are divisors of 99 in set A2, if a≠99, b≠99 and a⊗b = 99.
E.g. numbers a and b are divisors of 99.

Figure 9. Explicit reference to zero-divisors.

from ordinary arithmetic) to a focus on an important inter-relationship (zero-divisors) caused by the operations and that shift appears to have followed the general pattern of representational redescription: from the implicit to the explicit.

6.3. *Apprehending* $\underline{A}_2$

We can consider the apprehension of structure on a number of levels. In the previous two sections, we saw Molly shift her focus from numbers and the defined operations to seeing particular interrelationships in $\underline{A}_2$: inverse operations and zero-divisors. However, the implicit pedagogical intention of the examples-to-definitions approach is that a fully worked out example is meant to prepare the student for subsequent shifts of attention, first to the definitional properties of the abstract structure and then to proofs and theorems as simultaneously general and applicable to all examples of the structure. The student needs to see a wide network of interrelationships which form the structure as a whole.

Thus we are interested not just in Molly's shift of attention from objects and operations to inverse operations and zero-divisors, but in how she comes to apprehend $\underline{A}_2$ as a preparatory example of a commutative ring. The long period over which Molly works on the structure and the intensity with which she engages on the tasks are unusual, but they allow us to map her changing view of what she thinks she is working with. The vast amount of data gives considerable detail of her development (Stehlíková, 2004) and, as we mentioned earlier, the act of disentangling specific pieces of her development is somewhat artificial. However, we can give a sense of the general direction of her apprehension of the structure by looking, with a wider focus, on the ways in which she engaged with tasks across the whole period of the study. We suggest that this, too, shows a form of larger scale representational redescription in which she moves from working with the specific (and, in this case, familiar) objects and teacher-given operations, through an increasing, but unarticulated sense of the interrelationships between the objects which are consequences of the operations, to a sense of $\underline{A}_2$ as a new mathematical structure.

There are a number of stages which we highlight here in the process of Molly's apprehension of the structure of $\underline{\text{A}}_2$. They are roughly chronological, but overlapping, showing the longer term and slow development she went through.

- Initially her attention was on concrete mathematical tasks given by the interviewer. This involved solving equations, identifying z-divisibility tests, much of which was discussed above. (Interviews 1 to 7.)
- She moved on to working more generally: finding general solutions (or general solving strategies) by solving multiple concrete tasks which she posed herself (Interviews 3 to 9).

  For example, when given the task "classify multiplicative equations according to the number of their roots" she had to pose a number of particular multiplicative equation problems for herself to be able to discern the numbers which are important to the classification. After solving some equations which she got from the interviewer, she solved the equation $c \otimes x = d$ for $c = 1, 2 \ldots 13$ each time noting for which d the equation had a solution and how many solutions it produced. While doing it, she noticed that multiples of 3, 11 and 9 played the key role. She then explored some other equations for $d = 18, 27, 30, 33, 66$, etc. and gradually came to a complete classification.
- As time went on, she began exploring the problem situations given by the interviewer by solving specific tasks where it appears her burgeoning sense of the structure of $\underline{\text{A}}_2$ enabled her to choose tasks wisely. (Interview 8 and investigations.)

  For example, when given the quite open task to "consider quadratic equations" she had the opportunity to choose from a wide variety of initial strategies.

  She chose to consider a general equation $a \otimes x^2 \oplus b \otimes x \oplus c = 99$ (where x^2 means $x \otimes x$) and distinguished subtypes $a \otimes x^2 \oplus c = 99$ (and further into $a = 1$ and $a \neq 1$) and $a \otimes x^2 \oplus b \otimes x = 99$. For each subtype, she suggested a solving strategy in general and illustrated it on examples of equations. The strategy tended to involve changing the problem into solving linear equations. Then she solved some particular examples of general quadratic equations using the quadratic formula. Finally, she distinguished equations where a was a zero-divisor and showed how to change the formula so that we did not divide by zero-divisors (again by changing it into a linear equation). She looked into the problem of the number of roots and their mutual relationships and got some partial results.
- As she became further involved in the study, she began suggesting and elaborating her own problem situation. (Investigations.)

a		b		c
{1, 10, 89, 98}	•	{33, 66, 99}	•	{1, 10, 89, 98}
{2, 20, 79, 97}	-	{33, 66, 99}	-	{2, 20, 79, 97}
{3, 30, 36, 63, 69, 96}	-	{33, 66, 99}	-	{3, 30, 36, 63, 69, 96}
{4, 40, 59, 95}	-	{33, 66, 99}	-	{4, 40, 59, 95}
{5, 50, 49, 94}	-	{33, 66, 99}	-	{5, 50, 49, 94}

Figure 10. Pythagorean Triples in $\underline{A}_2$.

In week 135, she suggested that she would like to investigate Pythagorean triples (PT) and their behavior in $\underline{A}_2$. She first transferred the definition of PT into $\underline{A}_2$ and using her table of z-squares listed all possible triples for which $m \oplus n = z$, where m, n and z were z-squares, in a long and unorganized table. In the second table, she showed how these equalities can be used for generating all Pythagorean triples in $\underline{A}_2$. For example, from the equality $1 \oplus 99 = 1$, we get 48 triples (e.g. $1^2 \oplus 33^2 = 1^2$, $10^2 \oplus 33^2 = 1^2$).

She went on to make the presentation of them more concise by a table (shown in Figure 10) in which she worked with sets (z-square roots) rather than individual numbers.

The last table enabled her to see whether properties of PT that hold in **N** also hold in $\underline{A}_2$ and to suggest new properties of PT which only hold in $\underline{A}_2$. Finally, she formulated them.

- When she began to write drafts of her diploma thesis, she tended to order the mathematical concepts in what we could call a chronological and psychological way: that is, she tended to write the ideas in the order she had developed them or in the order of her own conceptual difficulty. (Writing versions 1–4.)

  In these early drafts, for example, she had initial sections titled ‘addition of numbers’ and ‘multiplication of numbers’, followed by sections on ‘z-zero’, ‘subtraction’ and ‘division’. These reflected the order in which she had encountered these ideas.
- As she moved to the end of the study, with her later drafts of her thesis and with advice from others, she reorganized the structure to order the mathematical concepts in what we could call a mathematical or logical way: considering the logical dependence of concepts on each other and ensuring that the foundational issues appear prior to the derived ones. (Writing versions 5–8.)

  In the later drafts, she began with a section on z-addition, but included in it subsections on z-zero and z-subtraction. Similarly, a section on

z-multiplication had subsections dealing with z-division. These appear to reflect a much greater sense of mathematical structure.

- One final aspect which we suggest highlights the larger scale apprehension of structure in Molly's work is the movement from procedure, to partial justification to near-formal proofs. (Writings 1–8.)

  In the earliest work and in the first version of her writing, she stated results and provided only some accompanying examples (albeit increasingly generic ones). For example, she stated that $x \oplus 99 = x$ with only one accompanying example.

  In version 2 of her writing, there was an attempt at the proof of some divisibility tests.

  By writing version 6, she had a number of more formal proofs (including proving that $x \oplus 99 = x$ using inverse reduction).

7. Discussion and conclusions

Across the data collected over the three and a half years of Molly's work, we can see the gradual development of a structural sense of $\underline{A}_2$. She has moved from adding numbers in $\underline{A}_2$ through using (without articulating) some relationships, finally to explicating and partially proving these relationships. Her attention has shifted from the (teacher-given) objects and operations to the general interrelationships between those objects which are consequences of the operations. She has begun to apprehend the structure of $\underline{A}_2$.

Recall, however, that this study was focused on just one shift of attention from a number of shifts needed to move along the route intended by examples-to-definitions pedagogies towards abstract algebra.

To get beyond apprehending the structure of an example to working abstractly, Molly still (at the time of her diploma submission) needed to:

- see the signs used by the teacher in defining the abstract structure as abstractions of the objects and operations, and see the names of the relationships amongst signs as the names for the relationships amongst the objects and operations.
- see other sets and operations as examples of the general structure and as prototypical of the general structure.
- use the formal system of symbols and definitional properties to derive consequences and seeing that the properties inherent in the theorems are properties of all examples.

However, one of the aspects of Molly's case which is interesting is that, over the course of the 188 weeks of work on $\underline{A}_2$, she obviously did a considerable amount of other mathematical activity as part of her degree. Much of that activity was very formal and some of it *was abstract algebra*!

At the same time as her detailed exploration of $\underline{A}_2$, Molly took formal mathematical courses which covered groups, rings and fields. Thus, she had access to the teacher-given signs used to define a commutative ring with identity and had at least seen the formal derivations of the central theorems and had important ideas highlighted. So Molly would appear to have access to the stimulus to make at least one further shift of attention: linking together her example structure and the formal, general mathematics. What surprised us most is that – despite Molly's enthusiasm, hard work and apparent level of intelligence – there is no evidence that she did so.

The way that Molly was encouraged to work with the structure was deliberately designed to downplay the role of the teacher as authority. While the structure itself, and the initial tasks, were set by the teacher, the direction her exploration took, the later choice of tasks and, importantly, what she *chose to attend to* were her own concern. The 'teacher' (the interviewer) tried, as much as possible, not to influence her.

We suggest that it is this which may be at the heart of both the time it took for Molly to attain a suitable apprehension of the structure of $\underline{A}_2$ and the lack of what were, for us, obvious links between the $\underline{A}_2$ structure and the formal mathematics she was learning simultaneously. One might argue that there was something deliberate in Molly's inability to link her detailed work with the $\underline{A}_2$ structure to her knowledge of formal abstract algebra, but we argue that without guidance about what to attend to, Molly (like many other students) was unable to spontaneously shift her attention in an appropriate direction.

Attention can easily shift in many different directions. Particularly after the initial explorations, Molly could choose to attend to any aspect of $\underline{A}_2$ that she wished. The choice to investigate Pythagorean triples and '*z*-pyramids' (neither of which would appear to be an obvious fruitful line of enquiry to an algebraist) indicate that her attention may still be guided by her thinking about the arithmetic of natural numbers. What seems important in the development of an examples-to-generality pedagogy is not the free-for-all of unguided discovery, but an emphasis on the guidance of *joint attention*: on teacher and learner making sense of structures together, with the teacher able to explicitly guide attention to, first, those aspects of the structure which will be the basis of later abstraction and, then, to the links between the formal and general with the specific example.

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We are grateful to Milan Hejný for his original development of the $\underline{A}_2$ structure. The paper was partially supported by a Czech grant agency GACR No. 406/02/0829.

NOTES

1. We make the distinction between inter-relationships between objects and properties of objects. A property of an object is something which pertains to it, in and of itself or its constituent parts, and not in its relationship to other objects. Thus being a complete linearly ordered field is a property of the reals, since it is a consequence of the structure of **R** and its elements. The statement '17 is prime', however much it might look like a statement about a property of 17, is either properly thought of as a property of **N** or as a relationship between 17 and other naturals and which is a consequence of the operation of multiplication.
2. We use the word '*ap*prehend' advisedly and in its original meaning of 'to lay hold of in the mind'. Sierpinska (1994) links Dewey's use of apprehension to Piaget's assimilation and *com*prehension to accommodation. In contrast, as part of the new term we are defining here ('apprehending structure' as a particular shift of attention) we associate 'apprehending' with laying hold mentally in part against 'comprehending' as 'embracing or understanding in all its compass and extent'.
3. Where appropriate, Molly's work is either translated into English, or (in scanned examples of her work) translations of important words are written in cursive script beside her Czech language originals. Also note that the Czech letter 'č' appears regularly. This is the first letter of the Czech word for 'number' – číslo – and would be used as English speakers would use '*n*'.
4. In the Czech language, the additive inverse is also often called "opačné číslo" (opposite number) in mathematics textbooks.
5. She only appeared to mean negative numbers which can result from subtracting two *z*-numbers.

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