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DIDACTIC RESTRICTIONS ON THE TEACHER'S PRACTICE: THE CASE OF LIMITS OF FUNCTIONS IN SPANISH HIGH SCHOOLS

ABSTRACT. The Anthropological Theory of Didactics describes mathematical activity in terms of *mathematical organisations* or *praxeologies* and considers the teacher as the *director of the didactic process* the students carry out, a process that is structured along six dimensions or *didactic moments.* This paper begins with an outline of this epistemological and didactic model, which appears as a useful tool for the analysis of mathematical and teaching practices. It is used to identify the main characteristics of the mathematical organisation around the *limits of functions* as it is *proposed to be taught* at high school level. The observation of an *empirical didactic process* will finally show how the internal dynamics of the didactic process is affected by certain mathematical and didactic constraints that significantly determine the teacher's practice and ultimately the mathematical organisation *actually taught.*

KEY WORDS: Anthropological Theory of Didactics, Epistemological Approachs, mathematical organisation, praxeology, didactic moments, didactic transposition, limit of functions

1. INTRODUCTION

The main purpose of this paper is to show how teachers' practices are strongly conditioned by different restrictions, of mathematical origin, related to the particularities of the considered content, and of didactic origin, implied by the organisation of mathematics teaching. The case of the teaching of limits of functions in Spanish high schools will highlight these restrictions. Some of them – maybe the most well known (see for instance Artigue, 1998; Ferrini-Mundi and Graham, 1994; Williams, 1991) – refer to the particularities of the notion of limit and to the difficulties of its introduction as a functional tool to enhance students' mathematical problem solving ability. Other restrictions come from the mathematical knowledge as it is proposed to be taught in official syllabi and textbooks, related to, for instance, the difficulty of giving sense to the teaching of limits of functions when these are presented as a tool to study the continuity of functions. There are, moreover, didactic restrictions, which affect the teacher's practice at a more general level and can be linked to the atomisation of mathematical curriculum and to the limited scope for action traditionally assigned to the teacher.

Educational Studies in Mathematics (2005) 59: 235–268
DOI: 10.1007/s10649-005-5889-z

In the first section we present the main elements of the Anthropological Theory of Didactics in accordance with the recent works of Yves Chevallard (1997, 1999, 2002a and 2002b), which constitutes the theoretical basis of our research. The problem of teaching 'limits of functions' is then presented, in the second section, in terms of the three steps of the process of didactic transposition: the 'scholarly' mathematical knowledge, the mathematical knowledge as it is designed to be taught and the way it is actually taught by a concrete teacher in a concrete classroom. The third section presents this last component from the observation of an empirical didactic process that took place during 14 sessions in a Spanish high school class (15 to 16-year-old students). The particular way the observed teacher directs his students' practice is described in Section 4 referring to the dynamics of the *didactic moments* as proposed by Chevallard (1999). This brings us finally, in Section 5, to a paradigmatic example of some visible didactic restrictions, which affect the teacher's practice at the different levels of generalisation.

2. Fundamental elements of the anthropological theory of didactics

2.1. *Mathematical organisations*

What we call the *Epistemological Program* in didactics of mathematics – to be distinguished from the *Cognitive Program* (Gascón, 1998 and 2003b) – is the program of research which stems from the work of Guy Brousseau[1], and is prompted by the conviction that the construction of models of mathematical activity to study phenomena related to the diffusion of mathematics in social institutions constitutes the first step in mathematics education research. Within the Epistemological Program, the Anthropological Theory of Didactics proposed by Chevallard (1997, 1999, 2002a and 2002b) offers a general epistemological model of mathematical knowledge where mathematics is seen as a human activity of *study of types of problems.* Two inseparable aspects of mathematical activity are identified. On the one hand, there is the *practical block* (or know-how) formed by *types of problems* or *problematic tasks* and by the *techniques* used to solve them. Doing mathematics consists in studying (in order to solve) some problems of a given type. For instance, in upper secondary school, possible types of problems related to limits of functions are: to calculate the limit of a function, to demonstrate the existence of a limit, to define the notion of limit of a function, to check the validity of a proof, etc. The term 'technique' is used here in a very broad sense to refer to what is done to deal with a problematic task. There are different techniques to calculate the limit of a function (depending on the kind of function and on the way it is given), to do a proof, to propose a definition, etc. Some techniques are of algorithmical nature, but most are not; some are well known and easy to characterise, while others are not. The anthropological approach assumes that any 'way of working', the accomplishment of any task or the resolution of any problem requires the existence of a technique, even if this technique can be difficult to describe or show to others (even to ourselves).

A second anthropological assumption is that human practices rarely exist without a *discursive environment,* the aim of which is to describe, explain and justify what is done. Consequently, on the other hand, there is the *knowledge block* of mathematical activity that provides the mathematical discourse necessary to justify and interpret the practical block. This discourse is structured in two levels: the *technology* ('logos' – discourse – about the 'techne'), which refers directly to the technique used, and the *theory* that constitutes a deeper level of justification of practice. Thus, for instance, we can explain the calculation of the limit of a function referring to different technological ingredients, such as 'infinitesimals of equivalent order' or the '$\varepsilon - \delta$ definition' or 'elimination of indeterminations'. These different technological ingredients can make sense and be justified in turn by a discourse of a second level whose aim is to provide a framework of notions, properties and relations to locate, establish and generate technologies, techniques and problems.

Types of problems, techniques, technologies and theories are the basic elements of the anthropological model of mathematical activity. They are also used to describe the mathematical knowledge that is at the same time a means and a product of this activity. Types of problems, techniques, technologies and theories form what is called mathematical *praxeological organisations* or, in short, *mathematical organisations* or *mathematical praxeologies.* The word 'praxeology' indicates that practice (*praxis*) and the discourse about practice (*logos*) always go together, even if it is sometimes possible to find local know-how which is (still) not described and systematised, or knowledge 'in a vacuum' because one does not know (or one has forgotten) what kinds of problems it can help to solve.

The more elementary praxeologies or mathematical organisations are said to be *punctual* if they are based around what is considered a unique type of problems in a given institution. Thus, at high school level, 'to calculate the limit of rational functions at infinity' or 'to demonstrate the existence of the limit of a function using a numerical sequence' can be at the origin of punctual mathematical organisations. When a mathematical organisation (henceforth abbreviated as MO) is obtained by the integration of a certain set of *punctual* MOs in such a way that all of them may be explained using the same technological discourse, it can be said that one has a *local* MO characterised by its technology. For instance, the above mentioned punctual MO can be integrated into a local MO around the calculation of limits of functions, under the technology of the 'algebra of limits', but it can also be integrated into a different local MO depending on the technological discourse used to describe and justify the techniques and also on the different punctual mathematical organisations that are linked together. Going one step ahead, the integration of a number of local MOs accepting the same theoretical discourse gives rise to a *regional* MO. In the same way that a punctual MO can be integrated into different local MOs, a local MO can also be integrated into different regional MOs.

Given this (short) presentation of the general anthropological model of mathematical activity, we can now ask what is needed to create or re-create mathematical organisations? How can one pass from an initial problematic question to the practical and theoretical knowledge structured in a MO? What conditions allow the development of institutionalised mathematical activities? In other words, what are the means available to the mathematician or the mathematics student to carry out a mathematical activity giving an answer to certain problematic questions and crystallising in a MO?

2.2. *Didactic organisations and the moments of the didactic process*

In the Anthropological Theory of Didactics, the process of creation or re-creation of a mathematical organisation is modelled by the notion of *process of study* or *didactic process*. This process presents a non-homogeneous structure and is organised into six distinct *moments*, each of which is characterised depending on the studied mathematical organisation. Each moment has a specific function to fulfill which is essential for a successful completion of the didactic process. These six moments are: the moment of the *first encounter*, the *exploratory* moment, the *technical* moment, the *technological–theoretical* moment, the *institutionalisation* moment, and the *evaluation* moment. According to Chevallard (1999, pp. 250–255, our translation):

> The *first moment* of study is that of the *first encounter* with the organisation O at stake. Such an encounter can take place in several ways, although one kind of encounter or 're-encounter', that is inevitable unless one remains on the surface of O, consists of meeting O through at least one of the types of tasks T_i that constitutes it.[...] The *second moment* concerns the *exploration* of the type of tasks T_i and *elaboration of a technique* τ_i relative to this type of tasks.[...] The *third moment* of the study consists of the *constitution of the technological–theoretical environment* [...] relative to τ_i. In a general way, this moment is closely interrelated to *each* of the other moments. [...] The *fourth* moment concerns the *technical work*, which has at the same time to improve the technique making it more powerful and reliable (a process which generally involves a refinement of the previously elaborated

technique), and develop the mastery of its use. [. . .] The *fifth moment* involves the *institutionalisation*, the aim of which is to identify what the elaborate mathematical organisation 'exactly' is. [. . .] The *sixth moment* entails the *evaluation*, which is linked to the institutionalisation moment [. . .]. In practice, there is always a moment when a balance has to be struck, since this moment of reflection when one examines the *value* of what is done, is by no means an invention of the school, but is in fact on a par with the 'breathing space' intrinsic to every human activity.

It is clear that a 'complete' realisation of the six moments of the didactic process must give rise to the creation of a MO that goes beyond the simple resolution of a single mathematical task. It leads to the creation (or re-creation) of at least the first main elements of a *local* MO, structured around a technological discourse.

The Anthropological Theory of Didactics considers that the notion of praxeological organisation can be applied to any form of human activity, and not only to mathematics. In particular, it can be used to describe the teacher's and the student's practice in terms of *didactic praxeologies* or *didactic organisations*. A didactic praxeology is used when a person or group of persons want to have an appropriate MO available (the *mathematician*'s or *student*'s didactic praxeology) or to help others to do it (the *teacher*'s didactic praxeology). As any praxeology, it has a *practical block* composed of types of *didactic problematic tasks* and *didactic techniques*, and a *knowledge block* formed by a *didactic technological–theoretical* environment. Given the growing interest and necessity to conduct research on teachers and their role in the didactic relationship, the analysis of teachers' didactic praxeologies appears to be a relevant and productive field of investigation for today's didactics of mathematics.

The work presented here began with an observation of two teaching processes about limits of functions.[2] Its main goal was to study how *institutional restrictions* could affect the spontaneous practice of the observed teachers. We are presenting here only one of the observed didactic processes, which will show, not only the kind of analysis we can provide using the Anthropological Theory of Didactics, but also how this analysis allows us to highlight the *didactic restrictions* that affect teachers' practices. Two kinds of didactic restrictions are identified here:

(1) *Specific didactic restrictions* arising from the precise nature of the knowledge to be *taught*. In this study – those related to the content of the *limits of functions* as proposed by official syllabi and textbooks in Spanish secondary schools.
(2) *Generic didactic restrictions* the mathematics teacher encounters when facing the problem of how to teach any proposed mathematical topic in a school institution.

We will show that the conjunction of the two kinds of restrictions determine to a large extent the knowledge related to limits of functions that can be *actually taught* in the classroom. This will provide a first delimitation of the field of possible didactic organisations that can be set up in the considered school institution.

3. The problem of teaching 'limits of functions'

The problem of teaching 'limits of functions' in secondary schools constitutes a particular case of the *teacher's praxeological problem*. According to the Anthropological Theory of Didactics, this problem consists, essentially, in creating, through a *didactic process*, a specific mathematical organisation in a particular educational institution (Chevallard, 2002a and 2002b; Bosch and Gascón, 2002). To solve this problem, the teacher has some 'given data', such as curricular documentation, textbooks, assessment tasks, national tests, etc., where some components of a mathematical organisation, as well as some pedagogical elements and indications on how to conduct the study can be found. This is how the educational institution 'informs' the teacher about what mathematics to teach and how to do so. Nonetheless, it is clear that an important part of the teacher's problem lies in decoding the information provided by curricular documentation in order to elaborate, in collaboration with the students, a mathematical organisation complete enough to allow the development of a quite coherent mathematical study process.

When considering the 'teaching of limits of functions' as a *research* problem in mathematics education, we need to understand also the choices made by teachers and the institutional restrictions acting upon them. Given that teaching and learning are not isolated but take place in a complex process of didactic transposition (Chevallard, 1985), we need to adopt a broader point of view to make a distinction between: (1) the 'scholarly' mathematical knowledge; (2) the mathematical knowledge 'to be taught' and (3) the mathematical knowledge as it is actually taught by teachers in their classrooms. Figure 1 illustrates these three steps of the didactic transposition process and it includes the 'reference' mathematical knowledge (Bosch and Gascón, 2004) that constitutes the basic theoretical model for the researcher and is elaborated from the empirical data of the three corresponding institutions: the mathematical community, the educational system and the classroom.

3.1. *The reference mathematical organisation*

Spanish official programs and textbooks propose a set of mathematical elements (types of problems, techniques, notions, properties, results, etc.) that

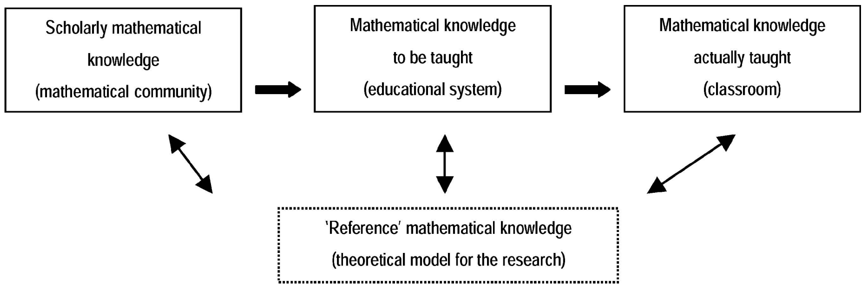


Figure 1. The process of didactic transposition

consititutes the knowledge *to be taught* about the limits of functions. As researchers, we need to interpret these as components of a MO which we will call the *reference mathematical organisation*. This MO constitutes our epistemological model of the 'scholarly knowledge' that legitimates the knowledge to be taught. It is the broader map with reference to which we can interpret the mathematical contents that are proposed to be studied at school. The reference mathematical organisation we are considering here about limits of functions includes and integrates in a regional organisation *two different local* mathematical organisations MO_1 and MO_2 that will assume different roles.

The first mathematical organisation, MO_1, can be named the *algebra of limits*. It starts from the supposition of the existence of the limit of a function and poses the problem of how to determine its value – how to calculate it – for a given family of functions. The two main types of problems or problematic tasks T_i of MO_1 are as follows:

T1: Calculate the limit of a function $f(x)$ as $x \to a$, where a is a real number.
T2: Calculate the limit of a function $f(x)$ as $x \to \pm\infty$.

In both cases the function $f(x)$ is supposed to be given by its algebraic expression and the techniques used to calculate the limits are based on certain algebraic manipulations of this expression (factoring, simplifying, substituting x by a, etc.). For instance:

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

$$\lim_{x\to +\infty} \frac{x^2 + 3x + 2}{x^2 + 1} = \lim_{x\to +\infty} \frac{(x^2 + 3x + 2)/x^2}{(x^2+1)/x^2} = \lim_{x\to +\infty} \frac{1 + \frac{3}{x} + \frac{2}{x^2}}{1 + \frac{1}{x^2}} = 1$$

There is also a third type of tasks, much less important, that links the calculation of the limit to the graph of the function:

T3: Determine the limit of a function given its Cartesian graph $y = f(x)$.

Although neither this problem nor the corresponding techniques (based on the reading and interpretation of the graph) are of a proper algebraic nature, in practice this third type of tasks always appears closely related – and even subordinated – to the first ones.

What is, in MO_1, the *minimum technological discourse* needed to generate, explain and justify the properties of limits of functions that are used to calculate them[3], and what is the theoretical foundation of this discourse? A good illustration of the knowledge block of this 'algebra of limits' can be found in the work of Serge Lang (1986) who, for instance, proposes a small axiomatic system to introduce the properties of the notion of limit that will constitute the 'primary resource' of the techniques used to calculate them. This technological ingredient can be informally stated using the following terms:

(1) The limit of the sum of two functions equals the sum of their limits.
(2) The limit of the product of two functions equals the product of their limits.
(3) The limit of the quotient of two functions equals the quotient of their limits.
(4) Inequalities between functions are preserved in the 'passage to limits'.
(5) The limit of a function comprised between two other functions with the same limit equals the value of this limit.

The knowledge (technology and theory) and the know-how (problems and techniques) of MO_1 do not exhaust the mathematical contents that are supposed to be taught in Spanish high schools. Therefore, we need to consider a second component of the reference model, MO_2, which can be designated as the *topology of limits*. This mathematical organisation emerges from the question of the nature of the mathematical object 'limit of a function' and aims to address the problem of the *existence of limit* with respect to different kinds of functions. Some types of problematic tasks T_i that constitute MO_2 are as follows:

T_1: Show the existence (or non-existence) of the limit of a function $f(x)$ as $x \to a$, where a can be a real number, or $x \to +\infty$.
T_2: Show the existence (or non-existence) or one-sided limits for certain kinds of functions (such as monotonic functions).
T_3: Show the properties (1)–(5) used above to justify the way certain limits of functions are calculated.

The *mathematical techniques* usually brought into play in proving these rules are based on the '$\varepsilon - \delta$ inequalities' or on the consideration of special kinds of convergent sequences. A single example can indicate the 'nature' of this work and its difference from the one done in MO_1:

Show that the function $f(x) = \sin(x)$

does not have a limit when $x \to +\infty$.

Let us consider the sequence $x_n = (\pi/2 + 2\pi n)$.

We know that $\lim_{n\to+\infty} x_n = +\infty$

We have: $f(x_n) = \sin(\pi/2 + 2\pi n) = 1$

for all n, which implies $\lim_{n\to+\infty} f(x_n) = 1$

Let us consider the sequence $x'_n = (-\pi/2 + 2\pi n)$.

We know that $\lim_{n\to+\infty} x'_n = +\infty$.

We have: $f(x'_n) = \sin(-\pi/2 + 2\pi n) = -1 \quad$ for all n,

which implies $\lim_{n\to+\infty} f(x'_n) = -1$.

We have two sequences that tend to infinity and whose images through f converge to different points. Thus the limit of $f(x) = \sin(x)$ for $x \to +\infty$ does not exist.

The technological discourse of MO_2 is centred on the properties of limits of sequences and the classic $\varepsilon - \delta$ definition of limit. It provides the technical resources needed to solve the problems of the existence of limits. This technology is based on a theory of real numbers structured as a metric space with its different properties: density, completeness, existence of the supremum of every bounded non-empty subset of **R**, Cauchy sequences, etc.

We have, in short, a *reference* MO that integrates, at least, two *local* mathematical organisations, MO_1 and MO_2, which have the following relationships:

(a) Far from being distinct, MO_1 and MO_2 appear to be closely related. As shown, the proof of the rules that support the calculation techniques of MO_1 (that is the *technology* of MO_1) can be considered as a mathematical technique in MO_2 (that is, a part of the *practical block* of MO_2).[4] In fact, it can be stated that MO_1 is partly contained in MO_2.
(b) MO_1 and MO_2 share the same theory of real numbers. Thus, it is possible to state that they can be integrated into the same *reference regional* MO that includes both MO_1 and MO_2 and other MOs. This *regional* MO can, for instance, be the organisation that deals with the question of *differentiability* of certain kinds of functions.

3.2. *The mathematical knowledge to be taught*

The above description of the *reference* MO is now used to describe the mathematical knowledge to be taught about the limits of functions as it appears in the official curriculum of Spanish high schools. The types of problematic tasks most frequently presented in curricular materials and textbooks are as follows:

T_1: Determine the limit of a function $f(x)$ as $x \to a$, with a real and $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials or simple irrational functions.

T_2: Determine the limit of a function $f(x)$ as $x \to \pm\infty$, and $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials or simple irrational functions.

T_3: Determine the limit of a function at a point from its graph $y = f(x)$.

T_4: Study the continuity of $f(x)$.

The first three types of tasks are particular cases of the constitutive tasks of MO_1. T_4 tasks do not correspond directly to the determination of a limit but are totally subordinate to it. The common techniques introduced to calculate these kinds of limits, for the most part, are based on some algebraic manipulations of the expression of $f(x)$ or on a direct reading of its graph $y = f(x)$. These 'curricular' tasks and techniques make up the practical block of the knowledge *to be taught* and correspond to the practical block of MO_1, mainly. In the following map (Figure 2) $MO'_1 = [T/\tau//]$ is used to indicate the trace left by MO_1 in the textbooks. The letters 'T' and 'τ' indicate the types of problems and techniques of MO_1, while the blanks indicate that the technology and the theory corresponding to this practice are practically *absent* from the curriculum, in the sense that, if they appear, they are not supposed to be used by the students but only presented by the teacher. As such, the reconstruction of MO_1 can be accomplished in the curriculum only in part.

The technological-theoretical discourse proposed by syllabi and textbooks to present, explain and justify this practice clearly comes from MO_2 and, as previously indicated, focuses on the problem of the *existence of the limit* of a function. It uses the standard mathematical discourse but is not accompanied by any mathematical practice within the students' reach. Following the notation proposed by Chevallard (1999), we will use the letters 'θ' and 'Θ' to indicate, respectively, the technology and the theory of a given MO. In our case, the trace left by MO_2 in the textbooks is indicated by $MO'_2 = [//\theta/\Theta]$ in the map and is weaker than the one left by MO_1. It contains only a few technological elements (some definitions and supposedly meaningful comments) whose function is mainly ornamental.

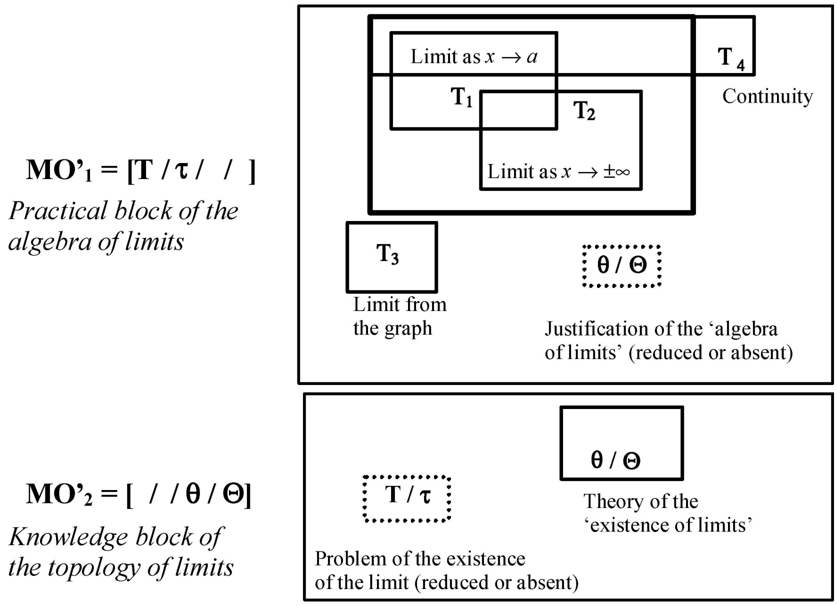


Figure 2. Map of the knowledge to be taught

The blanks, again, indicate an absence. In this case, what is lacking is the practical block of MO_2.

In summary, the considered mathematical knowledge to be taught is composed of the *disjoint union* of the traces left by MO_1 and MO_2 (see Figure 2).

The fact that MO_1 and MO_2 appear *completely disconnected* in the curriculum is mainly due to the absence of both the technological-theoretical block of MO_1 and the practical block of MO_2. The curriculum does not propose the creation of a technological discourse appropriate for the practical block of MO_1, the computation of limits effectively developed by students. Neither does it allow a practice that could be related to the standard mathematical theory about limits of functions (the 'scholarly knowledge') that is proposed instead and which is the technological–theoretical block of *another* mathematical organisation, MO_2. We are not considering here the origin of this *phenomenon of curricular 'two-sidedness' about the limits of functions*, that has to be found in a complex historical process that constitutes the first step of the didactic transposition (see Figure 1).[5] But we want to mention two of its didactic consequences.

The first has already been identified and concerns the major difficulties a teacher will certainly have to face when choosing the concrete mathematical components to teach. In other words, what types of problems must be proposed, what types of techniques can be used to solve them and what kind of explanations and justifications are necessary. The most likely scenario is taking MO_1 (where the existence of limits is not a problem in itself) as the knowledge *to be taught,* for it is the local MO closer (in terms of mathematical components) to the set of tasks, techniques, technologies and theories proposed by the curriculum. But this choice will not remove the difficulties and even contradictions due to the absence of the proper technology of MO_1 and to the presence of technological elements 'external' to MO_1.

There is, however, another phenomenon related to the above-mentioned 'two-sidedness' of the mathematical knowledge to be taught. When looking at the problem of the 'meaning' of limits of functions at secondary school, one can notice that it is precisely the missing technological–theoretical block of MO_1 – how to explain and justify the existence of the limit of a function and the algebraic properties used to determine it – that constitutes the *raison d'être* or the *rationale* of MO_2. In these circumstances, the teacher will encounter the difficulty of motivating the definition of the limits of functions as they are proposed by the curriculum, since this motivation has to be found in a broader MO that includes MO_1 and MO_2 as closely linked components. The same kind of difficulty will appear later at the university level: usually, the knowledge to be taught is mainly based on MO_2 but the practice that motivate this knowledge – the technology of MO_1 – has not been sufficiently developed before.

4. THE MATHEMATICAL KNOWLEDGE ACTUALLY TAUGHT

4.1. *Description of the didactic process*

As indicated before, our research included the observation of a class of Spanish secondary school students (15 to 16-year-olds) during the study of the topic 'limits of functions'. The observation started with the teacher's preparation of the subject and finished with the last session of revising and preparing for the final exam. The main steps of the experimental process are summarised below:

(a) *Data collected about the teacher's performance*

- Videorecording of all the sessions.
- Notes collected during classroom observation.
- Transcript of an interview with the teacher at the end of the process.
- Teachers' didactic materials (books, textbooks, personal notes).

(b) *Data collected about the students' performance*

- Students' notes related to the study process (6 students randomly chosen out of 34).
- Students' solutions to the initial and final exams on the studied topic.
- Students' answers to a questionnaire proposed by the researchers at the end of the teaching process.

(c) *First instrument for the analysis of the global didactic process*
We elaborated two different tables to organise and analyse the collected information of the observed didactic process. Table I included the full transcript of the teaching process according to the following general headings:

TABLE I
Transcript of the teaching process

Episode	Didactic moment	Main player	Mathematical objects	Observed didactic activities

The first column, '*episode*', contained a first intuitive breakdown of the teaching process. The second column, '*didactic moment*', shows the dominant category of moment of study as a summary information, which can help the observer to understand the development of the teaching process. The '*main player*' is the person (teacher or student) who has the responsibility of the specific mathematical task developed (even if it is through interaction with others). '*Mathematical objects*' are those that explicitly appear in the teacher's or students' public discourse (oral or written) in the considered episode. The '*observed didactic activities*' column contains the details of the transcribed and observed public activities in the classroom. This table offers details of the sequence of lessons and a first sequence of the didactic process organised into episodes and *moments*, including the essential components of the *created* MO. As an illustration, we present a small part of the table in the Appendix (it is based on our analysis of the first class).

In fact, this first table only presents 'raw material' which can appear in a non-structured and non-analysed form at the beginning. It does, however, provide the empirical foundation for the second step of the analysis (Table II), the aim of which is to specify the framework of the didactic process in terms of moments.

(d) *Second instrument for the analysis of the global didactic process*
Table II presented a more detailed analysis of the didactic process, and contained the following six columns:

TABLE II
Analysis of the teaching process

Session	Type of mathematic problem	Mathematical techniques	Technological theoretical elements	Dominant moment and sub-moments	Elements of the local didactic techniques

The purpose of this table was to describe the didactic process in terms of the *reference* MO (that is, of the components of MO1 and MO2), taking into account the mathematical elements that have been more or less explicitly present in the class. The table indicates how the didactic process developed, how the different moments were linked, which mathematical objects (types of problems, techniques, technological ingredients, theoretical principles, etc.) appeared, what was their function at every didactic moment, etc. This gives a description of the *practical block of the didactic praxeology*. In order to approach the way the teacher describes and justifies the observed practices – the *knowledge block of the didactic praxeology* – a type of *reference didactic organisation* is also required (Bosch and Gascón, 2002). Given that there are no didactic theoretical models, which are sufficiently well developed to describe the didactic technologies, our study of the teacher's spontaneous didactic technology is only exploratory and preliminary in nature. Empirical data used for the interpretation of the knowledge block of the didactic organisation come from a semi-structured interview with the observed teacher at the end of the didactic process. Themes addressed during this interview were the preparation, planning and management of the didactic process, general matters about the taught MO (components and criteria for their selection), students' difficulties and links with other topics of the syllabus. The interviews had a common structure and specific script and the dialogue with each teacher was freely conducted.

4.2. *The mathematical practice developed in the classroom*

While the knowledge *to be taught* can be reproduced from textbook elements and curriculum documents, the MO *actually taught* appears in students' notes and in the specific teaching practices carried out by the teacher in the classroom. It is clear that the latter heavily depends on the former. They do not however necessarily coincide because the knowledge *to be taught* is not always clearly fixed in curriculum documents and also because of the strong restrictions on the day-to-day teaching praxeologies.

In the observed didactic process, nine types of problems appeared, with different subtypes of problems that are indicated with a quotation mark and the corresponding subindex:[6]

Π_0: Calculate the slope of the straight line $y = ax + b$.
Π'_0: Calculate the slope of straight lines that intersect a curve $y = f(x)$ at a given fixed point x_0 and at a point near this one.
Π_1: Calculate the slope of the straight line tangent to the curve $y = f(x)$ at a given point a.
*Π_2:** Calculate the limit of a function $f(x)$ when $x \to a$ (a real).
*Π'_2:** Determine the points where a function is not defined and calculate the limit of the function at these points.
*Π_3:** Study functions defined piecewise with rational and irrational 'pieces'.
*Π'_3:** Study the limits of functions with rational and irrational 'pieces'.
*Π_4:** Given the graph of a function, determine the limits of the function at certain determined points.
*Π'_4:** Given the graph of a function, determine the points where the function does not have a limit.
*Π_5:** Study the continuity of a function at a point.
*Π_6:** Study the type of discontinuity of a function at a point.
*Π'_6:** Given the graph of a function, identify the points where it is not continuous and determine the type of discontinuity.
*Π''_6:** Given a function, find its points of discontinuity.
*Π_7:** Study of the conditions under which a function has a limit at a given point.
*Π'_7:** Study the conditions for $f(x)$ to be continuous at a given point.
*Π_8:** Calculate the limit of an irrational function $f(x)$ when $x \to \infty$ with an indetermination $\infty - \infty$.
*Π_9:** Calculate the limit of a sum, difference, product, division or composition of elementary functions.

Most of the activity carried out during the didactic process is aimed at preparing for the emergence of what has to be the *constitutive* tasks of the *actually taught* MO. Thus, most of the types of problems and techniques appear for technological reasons only, for instance, to explain the functioning of a particular technique or to give meaning to a theoretical question. As such, these elements are intended to disappear from the didactic process. Due to the particular circumstances of the specific 'history' of the didactic process, there are also some elements that will be impossible to introduce and which will end up being simple added elements to the created MO.

The types of problems that finally make up the actually created MO in the first observed class are marked by an asterisk. The correspondence between the types of problems Π_k that appeared in the class and the curricular tasks

T_i analysed in Section 3.1 is as follows:

– $\Pi_0$and $\Pi_1$do not correspond to any of the types of tasks T_i
– $^*\Pi_2$, Π_3 and $^*\Pi_9$ correspond to T_1
– $^*\Pi_4$ corresponds to T_3
– $^*\Pi_5$, $^*\Pi_6$ and Π_7 correspond to T_4
– $^*\Pi_8$ corresponds to T_2

The mathematical techniques used in the class to solve these types of problems are not detailed here, because we consider that they can be easily inferred from the delimitation of the types of problems presented above. Consequently, only an example of one of these with its specifications is presented:

τ_2: Replace x by a in the expression of $f(x)$ and manipulate it arithmetically to obtain the final numerical result.
τ'_2: Calculate the table of values of $f(x)$ taking values of x close to a $(x > a)$ and deduce the value of the right-hand limit.
τ''_2: Calculate the table of values of $f(x)$ taking values of x close to a $(x < a)$ and deduce the value of the left-hand limit.
τ'''_2: Factor the expression of $f(x)$, simplify it and write it distinguishing two cases: $f(x)$ equals the simplified function when $x \neq a$ and $f(x)$ is not defined when $x = a$. Graph the simplified function and determine its limit using τ_2.

In short, one can say that the mathematical organisation developed in the observed class corresponds to the *practical block* of MO_1. However, two types of problems that do not strictly belong to MO_1 – Π_5 and Π_6 that correspond to T_4 – can be identified. Nevertheless, the associated techniques are 'low level' variations of mathematical techniques pertaining to MO_1. Thus, it can be said that the mathematical organisation actually developed does not go much beyond the practical block of MO_1.

4.3. *'Raison d'être' and technology of the studied mathematical organisation*

In the observed didactic process, the *raison d'être* of the mathematical knowledge actually taught responds to the single question of the *calculation of the limit of a function at a point or at infinity*, under the assumption that these limits (at least the one-sided ones) exist or are infinite. Because mathematical technology corresponding to this practice is absent from the curricular documents and textbooks, the observed teacher chooses to *describe and institutionalise* the used techniques as *rather transparent rules*, as if they did not need any justification. This approach of denying the problem is a common strategy in institutionalised human activities when the means of solving it – or approaching it successfully – are not available. Thus, the limit of the quotient of two functions, for instance, is never used to compare their asymptotical behaviour. On the other hand, the technological elements introduced by the teacher are not integrated in the students' practice. They belong to MO_2, the mathematical answer to the question of the existence of the limit, and this question cannot be really raised in the classroom.

5. Dynamics of the didactic process

The didactic process followed in the observed class to recreate the knowledge *to be taught* will be examined now. Our aim is to compare the relations that must be established between the different moments of the theoretical process with the empirical didactic process as it was *actually lived* in the classroom. After using the reference MO to describe both the knowledge *to be taught* (from curricular documents) and the MO *actually taught* (from the students' notes and the teaching practices observed in the classroom), we intend to show how the aforementioned restrictions can affect the possible ways of organising the study of the limits of functions. Our analysis will suggest that any intent to recreate the knowledge *to be taught* in the Spanish high schools will result in a MO that is very close to the MO *actually taught* in the observed class.

5.1. *The moment of the first encounter and the confinement at the thematic level*

As mentioned in section 1.2, following the anthropological model, any didactic process requires a *first encounter* with the MO in question. In the case of the teaching of limits of functions, the ambiguity begins here because neither curriculum documents nor textbooks are explicit enough about this MO and do not answer questions such as: What mathematical knowledge should I teach? Which are its main components? Why is it important? Why is it useful? In this situation, the observed teacher initially proposes a number of type Π_0 problems ('find the slope of a straight line') in preparation for type Π_1 problems ('find the slope of the tangent to a curve in a given point', considering that Π_0 and Π_1 can provide a good first encounter with the MO (see Appendix). This assumptions seems *possible* because all the mathematical objects involved in Π_0 and Π_1 are expected to be unproblematic for the students. Nonetheless, as stated above, neither Π_0 nor Π_1 forms part of the curricular mathematical tasks defining the knowledge *to be taught* around the 'algebra of limits'. What role do those problems play in this case in the *taught* MO? We consider that, for the teacher, Π_0 and Π_1 are an attempt to 'give sense' to the study of the main types of problems that will constitute the *taught* MO. This can explain why he does not propose a first encounter through the 'algebraic' task of calculating the value of a rational function when the value of its denominator is 'very close' to zero (without being equal to zero), which constitutes the 'main mathematical task' of the teaching process (what students should learn to do). He chooses instead a 'geometrical' task which consists in finding the 'slope of a curve at a given point' through calculating the successive slopes of the secants to the curve that tend to the tangent line at this point. And, after this first encounter, the students will not meet this kind of geometrical task again.

5.2. *The exploratory moment and the elaboration of a technique*

Even if the teacher intends to manage the *exploration* of the type of problems Π_1 through the *elaboration and functioning of a technique* τ_1, his attempts rapidly fail almost certainly due to the following factors:

(a) The complexity of Π_1, which seems to be more appropriate for a first encounter with other MOs, for instance involving the 'derivative of a function at a point'.
(b) The lack of the necessary technological elements (related to curves in the plane and the foundations of real numbers) to give more stability and robustness to this mathematical activity.

From the beginning of the third session, the teacher proposes the *exploration* of a new type of problems Π_2 related to the calculation of the limit of a function as $x \to a$ (a real). Thus, the problem of the 'slope of a curve' at a given point completely disappears and the question 'what is the meaning of the limit of a function at a point' becomes secondary. What remains is the *problem of the calculation of the limit* considered as the value that a function 'approaches' when its argument 'approaches closer and closer to' a given real number a. It is this type of problems that will finally form the 'core' of the *actually taught* MO.

The *mathematical technique* τ_2 that is initially elaborated to begin the *exploration* of the first specimen of Π_2 is, in fact, a sub-technique of Π_1. It consists in carrying out algebraic manipulations with the expression of $f(x)$ (factor and simplify the common factors) and in substituting a for x in the simplified expression of $f(x)$. The use of τ_2 increases the need for constituting a technological–theoretical environment to provide a description, interpretation and justification of τ_2 and its numerous variations. In particular, the dynamics of the didactic process requires providing the answer to the following questions (among many others):

1. Why can we replace x by a in the simplified expression of $f(x)$ if we could not do it before?
2. Why, and in which cases, do we need the table of values of $f(x)$ to calculate the limit?
3. Why is it necessary to make a distinction between 'right-hand limit' and 'left-hand limit'?
4. When do we say that $f(x) \to \pm\infty$ as $x \to a$: only if $f(x)$ has a numerator that tends to a real number different from zero and a denominator that tends to zero?

5.3. *Trying to constitute a theoretical–technological environment*

In view of these technological necessities, the restrictions that strongly delimit the teacher's praxeology are clear. Given the structure of the mathematical objects proposed by curricular documents and textbooks around the limits of functions – what we refer to as the 'double-headed' MO $=$ $[\mathrm{T}/\tau//] \cup [//\theta/\Theta]$ – the teacher has no technological elements at his disposal that are 'coherent' and 'reliable' enough to be integrated into the mathematical milieu of the student and provide some answers to the questions generated by the activity. This is why the teacher has to find answers to such questions among the previously studied mathematical objects that are, presumably, unproblematic for the students.

The teacher, therefore, decides to consider functions of a particular form:

$$f(x) = (x-a)g(x)/(x-a),$$

a special subtype of the study of functions defined piecewise (Π_3). This option facilitates the justification of the technique τ_2''' used to compute the limit of $f(x)$ at $x = a$ based on the assumed obvious relationship between the graph of functions defined piecewise and their algebraic expressions. Nonetheless, this technique fails because functions defined piecewise have not been mastered by the students. As such, the mathematical activity around the block $[\Pi_3/\tau_3]$, that the teacher tried to situate at the technological level (towards τ_2''') has to be started over again, with a first encounter, an exploration of Π_3 and the carrying out of the technique τ_3 by the students.

There is a deeper reason, however, not related to students' knowledge, for the failure of the teacher's strategy. It is clear that, in the last analysis, reading a graph cannot justify the fact that x can be replaced by a to compute the limit of $f(x)$ as $x \to a$ (central point of τ_2'''). Thus, the teacher is forced to explicitly propose 'reading off the graph' as a justification (and, implicitly, as a new technique τ_4) when computing the limit of any kind of function given by its graph, and not only those defined piecewise.

It is, again, the internal dynamics of the didactic process that determines the presence of a new practical activity around the block $[\Pi_4/\tau_4]$, which is now extended to include a new punctual organisation $[\Pi'_4/\tau'_4]$. It consists in computing limits from the visual interpretation of the graph of a function. Thus, τ_4 and τ'_4 are completely *naturalised*, in the sense that they are invisible and always taken for granted. Because they cannot be really considered as techniques, it becomes impossible to carry out technical work to improve them and, consequently, it is also very difficult to *institutionalise* them.

It follows that, in short, the mathematical activity around $[\Pi_4/\tau_4]$ and $[\Pi'_4/\tau'_4]$ cannot be developed into a relatively 'complete' didactic process, in other words, a process that can integrate all the different didactic moments. What is more, this activity cannot be considered as a justification of the calculation of the limit of a function at a given point because it also gives rise to new technological needs.

A third reason also exists, equally related to the internal dynamics of the didactic process, that can explain why this activity is abandoned and there is a return to type Π_2 problems. It concerns the important technical and technological difficulties that would appear if the problems of Π_4 (where functions are given by their graphs) were connected to those of Π_2 (where functions are given by their analytical expression).

These restrictions may explain why, at the end of the fourth session, the teacher decided to return to the exploration of Π_2 problems with the intention of developing τ_2 (which generates several variations of this technique). This work had been interrupted in view of the need for a technological–theoretical environment. Nonetheless, even if this goal is not attained, the work with τ_2 could not be delayed any further. During the next three sessions, the teacher managed the technical work trying to present successive variations of τ_2 as if they were completely 'natural' and did not require any justification. Thus, for instance, faced with the disorientation of the students, the teacher concluded by saying that, to compute the limit of $f(x)$ as $x \to a$: 'Sometimes we can replace [x by a] and sometimes we can not.'

5.4. *Concentrating the didactic process on the exploration and 'routinisation' of the techniques*

The last sessions were dedicated to what the teacher presented as a new and important type of problems: the study of the continuity of a function, Π_5. The first encounter is conducted through an oral discourse supported by graphs, in a very fast and natural manner. The continuity of a function f at a point a is not presented as a problematic question, but as an application of the computation of the limit of $f(x)$ as $x \to a$ and its comparison with $f(a)$. Thus, τ_5 (calculate the limits of a function at the boundary of its definition domain) and its variations τ_5^i therefore arise as simple 'applications' of previously explored techniques, and the *institutionalisation* of the produced work need only affect the new technological elements that are reduced to the identification of 'continuity' with 'regular behaviour of a function graph'. For instance:

θ_1 ='A function f is continuous at $x = a$ if the two lateral limits as $x \to a$ exists and are equal to $f(a)$'.
θ_2 = 'If a function is not defined at a point, it cannot be continuous at this point'.

Continuity problems are studied in a rather hasty way, as if they only consisted of a change of language in relation to the previously studied ones. In turn, the notion of 'elementary function' suggested by θ_2 assumes an unexpected technological role at the end of the process. Indeed, the teacher improperly extends θ_2 using an implicit definition of 'elementary function': if a function f does not exist at a point a, it cannot be continuous at a; but if f exists at a and is an elementary function, then the limits can be computed by replacing x by a. Subsequently f is continuous at a. As such, it is assumed that the students are already aware which functions are 'elementary' (for instance rational functions) and which are not (for instance defined piecewise). During the whole didactic process, the teacher attempts to justify, unsuccessfully, the mathematical work carried out around $[\Pi_2/\tau_2]$. But, this effort is useless without deeply modifying the knowledge to be taught.

In conclusion, in spite of the major effort made by the teacher to carry out a relatively 'complete' didactic process, the final result is disappointing. Whenever the teacher attempts to go beyond the *exploratory* moment and extend the merely routine use of a technique, he is led to building up the technological discourse using materials extracted from mathematical praxeologies that, in spite of the students' and the teacher's efforts, need to be further explored and 'routinised' by the students. In this way, it is not possible to reach a real technological questioning about the techniques proposed by the teacher. These are only superficially explored and weakly elaborated, thus becoming more rigid in the students' hands. This restricted dynamic of the didactic process involves the construction of a very incomplete and unstructured MO which, moreover, lacks a clear raison d'être, impeding the possibility of becoming globally institutionalised.

6. Didactic restrictions on the possible ways of teaching limits of functions

This last section considers different levels of didactic restrictions, from the most specific ones, arising at the thematic level of the limits of functions, to the most generic ones that arise at the level of school mathematics taken as a whole and even beyond. At every level, these restrictions affect the characteristics of the possible 'ways of teaching' limits in secondary schools, that is, the *possible didactic organisations* that a secondary school teacher can use. Consequently, these restrictions also affect the specific MOs that can actually be taught.

6.1. *Effect of 'thematic confinement' on the spontaneous didactic organisation*

The Anthropological Theory of Didactics suggests a *hierarchy of levels of co-determination* (Chevallard, 2002b) between the different possible levels of MOs that can be considered (punctual, local, regional, etc.) and the way its study is organised at school. This hierarchy is structured in a sequence of levels of MOs and didactic organisations which runs from the most generic level, the society, to the most specific one, a simple mathematical question to be studied, as shown below:

Society → School → Pedagogy → Discipline → Area
→ Sector → Theme → Question

Each level of codetermination introduces particular restrictions showing the mutual determination between mathematical organisations and didactic organisations. The structure of a MO at each level of the hierarchy determines the possible ways of organising its study and, reciprocally, the nature and the functions of a didactic organisation at each level determine, to a large extent, the kind of MOs that can be created (studied) in the considered institution.

This sequence of levels is relative not only to the considered question or group of questions, but also to the corresponding historical period and teaching institution. Let us consider the following mathematical question, for example:

How to compute the limit of a concrete function $f(x)$ as $x \to a$ or as $x \to \pm\infty$?

This question must satisfy certain conditions for it to be studied in a specific teaching institution. One condition is that it arises from a primary question in one of the higher levels of the hierarchy, which is higher than the discipline level, and that its study 'leads somewhere'. In other words, it must not be an 'isolated' question and therefore a 'dead-ended' one (Chevallard et al., 1997, p. 118).

Which institution can assume the responsibility for the questions proposed to be studied in schools (mathematical ones, for example) to satisfy these conditions? Can the teacher be responsible for making such a decision? In general terms, it can be noted that *this problem is out of the teacher's control*. According to Chevallard (2002b), the common situation is that the teacher 'neglects' the higher levels, both those concerning the society and the school, and even the level of the *sector* (in the case of the limits of functions, the sector may be the study of differentiability, for instance). When teachers prepare to teach the limits of functions, they do not decide to which sector the theme belongs. This decision is already taken. They only have to decide how to organise the study of the strict MO around the limits of functions. This 'confinement into themes' constitutes a didactic phenomenon labelled by Chevallard as the '*thematic autism*' of the teacher. It is related to the 'low level' status of the teaching profession.[7] The neglect of the higher levels is not absolute: sometimes mathematics teachers can pay some attention to the *discipline* level and even the school and social ones. However, they rarely express their concerns and opinions as teachers but simply as individuals or, at most, members of political groups or trade unions.

In short, the teacher is destined to go no further than the thematic level, and this situation has important consequences. In particular, the most significant outcome is the disappearance of the reasons of being of the studied MO (Chevallard, 2002b). Indeed, the majority of the mathematical questions that are proposed for study at schools are formed and disappear at the thematic level. Consequently, these questions are only vaguely connected to the higher levels of the organisation (sectors, areas and discipline), which are usually considered as transparent and unquestionable. Furthermore, school mathematical themes are not flexible enough to form local MOs and, therefore, cannot even be united in a functional way in either regional or global MOs. As a result, school mathematical questions are not only very weakly related to the higher levels of determination, but also appear in isolation from each other.

The teacher's responsibility is thus confined to the selection, in each theme, of the mathematical questions to be studied by the students and, although teachers can group the questions in themes, they cannot have an influence on the higher levels of the hierarchy in any relevant way. Thus, a deep gap appears between, on the one hand, the levels of themes and questions (within the teacher's reach) and, on the other hand, those of discipline, areas and sectors (within the limits of the mathematical scope in contrast to the pedagogical environment):

SOCIETY → SCHOOL → PEDAGOGY → DISCIPLINE → AREA → SECTOR
Limits of the mathematical scope
THEME → QUESTION
Field of the teacher's activity

This gap affects the teaching of all mathematical areas.[8] In the case considered here, the teacher's confinement at the thematic level causes the disappearance of *the motivation for studying limits of functions* in secondary schools. Indeed, it becomes impossible to justify why this theme must be studied without going beyond the thematic level. As has already been stated, curricular techniques for computing limits (and, in particular, 'for solving indeterminations') make 'sense' only in a larger regional MO that includes both MO_1 (answering the problem of how to compute limits of functions) and MO_2 (answering the problem of the existence of the limit of functions). Nonetheless, the study of a regional MO would introduce more restrictions, for instance, the two following conditions:

a) To take into consideration a *much more extensive family of functions* than the one usually considered in high schools.
b) To carry out a new 'linkage' (in sectors and in themes into each sector) of the punctual and local mathematical organisations constituting the differential calculus studied at school, so that every new theme contains enough elements to carry out a relatively complete study (in terms of the moments or dimensions of the mathematical activity) of a local MO.
c) Among other outcomes, this new linkage can lead to the disappearance of the 'limits of functions' as a theme in the curriculum of secondary schools and may also result in a *global change of the curricular contents of elementary differential calculus*.

To go beyond the mathematical level, teachers need some means and legitimacy to rebuild the themes of the curricular sector (and, furthermore, the sectors of the corresponding area, for example, all school differential calculus). It is also very difficult for them to extend the family of functions commonly studied in high schools to include those functions that could 'give sense' to the problem of their differentiability – a possible way to bring together the problem of the existence and the computation of the limit of a function at a point. Once the limits of functions in secondary schools have lost their reason of being, the teachers' field of activity is restricted to the level of the specific mathematical questions and the isolated mathematical techniques which can then only appear in a quite opportunistic form.

The outcome is clear. The impossibility of 'giving sense' to the calculation of limits without going beyond the theme, together with a lack of technology to interpret and justify the mathematical techniques used by the students, strongly restrict the field of didactic tasks and techniques that the teacher can use, that is, his spontaneous *didactic practical block*. And, once confined to the thematic level, the teacher is at the mercy of the *common pedagogical ideology* that, concerning the teaching of limits of functions, can be expressed for instance in the following terms: 'the most important thing is that students "understand" the limit concept'; 'algebraic manipulations do not have enough sense by themselves'; 'comprehension of limits requires some kind of geometric or graphical interpretation'; etc. The confluence of both the 'thematic confinement' and the structure of the MO to be taught (the 'two-sided' organisation) can explain the failed attempts of the observed teacher to take back or to give new sense to the didactic process.

6.2. *Restrictions coming from the higher levels of determination*

The study of any kind of mathematical question is restricted by the higher levels of co-determination. In fact, the chain of levels needed to allow any specific mathematical question to be studied starts at the most general levels of the hierarchy: the school and society. Restrictions at these levels are related to the way the question is considered by Society and the kind of educational role conferred on the School. The outcomes and scope of these restrictions are not analysed here. We will only describe some restrictions commencing at the pedagogical and discipline levels, with some incursions into the intermediate ones, and it will be shown that these restrictions are very much related to the spontaneous didactic technology that generates the possible ways of organising the study of limits at school.

The *pedagogical level* is the origin of restrictions affecting the study of any kind of questions and, thus, the different disciplines that are taught in order to answer these questions. Most of these restrictions come from certain conceptions of teaching and learning that are taken for granted. One of them is the belief that there exists a pedagogical domain which is independent of the mathematical one, in the sense that decisions taken in the former domain (grouping students, time organisation, distinctions and links between disciplines, learning assessment, etc.) would not affect the nature of the concrete mathematical tasks that are being carried out in the class (Chevallard, 2000). This distinction would permit discussions about teaching and learning independently of the nature (mathematical, economical or linguistic) of what is being studied. Consequently, the responsibility of this common organisation cannot be left to the mathematics teacher, who is subsequently even more confined to his or her 'thematic level'.

This way of thinking is so generalised and accepted that there is no discussion about whether or not it constitutes an important element of the spontaneous didactic technology. Evidence of this was obtained from the interview we held with the observed teacher (Espinoza, 1998) who used 'generic technological elements' to describe, interpret and justify his practice. He referred, for instance, to the necessity of developing the link between curricular contents and daily life as a tool to increasing students' motivation, and to the use of technology for improving both learning capacities and motivation. It is clear that the pressure of these principles can affect the lowest levels of the mathematical activity carried out in the classroom to compute the limit of the function at a point: the types of tasks that are chosen, the corresponding techniques, the way we describe and justify the performed work, etc.

The *discipline* level – here, mathematics – introduces some restrictions related to the way mathematics is interpreted in educational institutions. In particular, according to Brousseau (1997), 'common teaching models' can be considered as supported by 'naive epistemological models'. In secondary schools, the prevailing epistemological model of mathematics is very eclectic. Most of its main characteristics come from what is called the 'quasi-empirical' epistemology or 'quasi-empiricism' (Lakatos, 1978) identifying mathematical activity with the exploration of open problems. The other characteristics can be related to the 'constructive epistemology' that considers mathematical concepts as the result of human actions and operations (Gascón, 2001).

In the case presented here, the strategy of the observed teacher can be explained by the first kind of characteristics. What has been observed is an attempt from the teacher to regularly guide his students to the exploratory moment, attaching great importance to the exploration of the different problems in a free and creative way. It can be noted that this kind of didactic organisation assigns a very reduced amount of responsibility to the students who merely have to carry out the exploration of the specific problems proposed by the teacher.

It is clear that the possible ways of teaching about questions related to the computation of the limits of functions are also affected by restrictions starting in the *area* and in the *sector* to which those mathematical questions are referred to by the curriculum. For example, the *epistemological model of calculus* prevailing in secondary schools has an important role in specifying the possible ways of organising the study of the limits of functions. Some authors are beginning to produce empirical evidence to support this thesis and are starting to suggest the interdependence between the epistemological model specific to a given school mathematical area and the didactic organisations that can be used to study it (Artigue, 1998; Bloch, 1999; Schneider, 2001).

6.3. *Conclusions*

Considering the dimensions or moments of the mathematical activity that prevail in the observed spontaneous teaching process, we can state that the didactic organisation carried out by the considered teacher is relatively incomplete and biased as it concentrates the didactic process on the *exploratory moment* and the first steps of the *technical work*. In the research presented herein, a second teaching process was also considered and was used as a contrast to the first one. The didactic organisation carried out by this second teacher, even if it appeared very different to the former, was also only centred around two didactic moments. In fact, this second didactic strategy can be considered as a more 'classical' one, based on exhibiting the main technological elements (definition of limit, properties, etc.) of MO_2 and on 'presenting' the principal techniques to compute limits from MO_1. This didactic organisation left room for the technological–theoretical moment only and the moment of the technical work, in which the students 'applied' and 'practised' the techniques the teacher had just showed them through some typical examples on the blackboard (Espinoza, 1998).

In general terms, we can postulate that if the knowledge to be taught is made of a collection of punctual mathematical organisations that are not linked to each other through an operative technological-theoretical discourse, then the possible corresponding spontaneous didactic organisations that the teacher can use will not be able to really integrate the six different moments of the didactic process. Reciprocally, when the didactic technologies available in the teaching institutions are based on naive epistemological models (Euclideanism, naive constructivism or 'quasi-empiricism') and on general pedagogical slogans, then the possible didactic organisations tend to favour only a few of the didactic moments to the detriment of the others. It can be foreseen – but this requires, of course, further empirical research – that these spontaneous didactic organisations will have difficulties in overcoming the problem of the atomisation of the curriculum: it will not favour the integration of mathematical contents previously learned into the new ones, nor the links between different types of problems of the same mathematical organisation, nor, even less, the connections between different mathematical areas (algebra and analysis, for instance).

The concrete teaching process presented here highlights some didactic restrictions coming from different levels of specificity and affecting to a different extent both the mathematical knowledge actually taught and the possible ways of teaching it. These restrictions come from the different institutions involved in the teaching and learning process (society, mathematical community, educational system, school and classroom) and cannot be explained without taking into account the global process of didactic transposition. Problems of great social concern such as the loss of motivation towards scientific activities, the absence of meaning of school mathematical problems, the imposition of 'scholarly' mathematical contents that loses its rationale as it is brought into schools, the atomisation of mathematical curricula, etc., need a deeper understanding of the set of institutional restrictions that regulate teaching and learning processes. Without knowing their functioning and their extent, we will not be able to act on them in a controlled and well-founded manner to ensure progress in mathematics education.

ACKNOWLEDGEMENTS

We would like to thank Christer Bergsten and Yves Chevallard for their helpful reviews of this English version. This work was supported by DGI BSO2003-04000 (Spain), FONDECYT 1020342 and DICYT 9933ES (Chile).

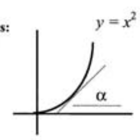
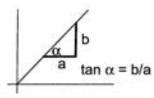
NOTES

1. Brousseau (1997) presents a compilation of his works published between 1970 and 1990.
2. A more complete version of this work can be found in Bosch et al. (2003) and Espinoza (1998).
3. Usually, since limits of functions are studied before their continuity, the 'regularity' of some functions (polynomials, for instance) is used to justify that the limit of a function at a point of its domain equals the value of the function at this point. This kind of argument is clearly a circular one and constitutes, as we will see, one of the weaknesses of the 'mathematical knowledge to be taught'.
4. Here is an example of an *institutional relativity* of the functions that mathematical objects can assume in a MO. While in MO1 the 'rules' play the role of the *technological discourse*, in MO2 they are an integral part of the *mathematical tasks.*
5. Artigue (2003) presents an analysis of the evolution of the teaching of calculus that helps to explain the current situation.
6. The functions $f(x)$ used in the process were essentially rational or simple irrational ones.
7. Didactic phenomena, like social, economic or linguistic ones, are independent of the will, the formation and the capacity of the individual subjects of the institution. As a result, taking the thematic confinement as such a didactic phenomenon is to consider it as a phenomenon the teacher does not create voluntarily and can only influence locally and to a relatively insignificant level.

8. Gascón (2003a) proposes an analysis of the effect of the thematic confinement on the teaching of geometry in Spanish secondary schools.

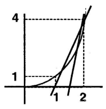
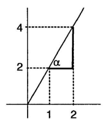
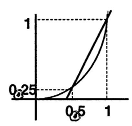
APPENDIX

TABLE AI
First description of the didactic process.

EPISODE	Didactic Moment	Main Player	Present mathematical objects (N : New)	Observed didactic activities
Introduction to the problem of calculating the slope of the tangent to the curve $y = f(x)$ at a given point.	FIRST ENCOUNTER with Π_1	Teacher	*Function* *Point* *Slope (N)* *Limit of a function at a given point (N)* *Curve*	**The teacher (T) starts the lesson saying to the students (S):** **T:** Limit of a function at a given point. Our purpose now is to find the straight of a curve in a point. **The teacher draws on the blackboard:** 
	Short Institution-nalisation	Student → T T → S	*Curve*	**S:** ¿What is a curve? **T:** A curve is a line. **The teacher continues his introduction.**
		T	*Tangent* *Circumference* *Point of a circumference* *Parabola*	**T:** Do you remember when we look for the tangent of a circumference at one of its points? **T:** Now we want find the slope of any curve, for instance a parabola, at a given point. **T draws:** 
Definition of tan α and of the slope of a curve at a given point.	Institution-nalisation To remember an old problem and an old technique	T	*Angle α* *Tangent of an angle* *tan α = b/a* *slope = tangent* **(N)** *Tangent of a curve* **(N)** *Inclination angle α* **(N)**	**T:** The definition of the tangent is: **The teacher draws on the blackboard:**  **T:** Thus the slope of a parabola at any point will be the tangent of the inclination angle α. **The teacher writes on the blackboard:** slope = tan α **T:** Now our goal is to calculate tan α.

(*Continued on next page*)

TABLE AI
(*Continued*)

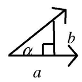
Calculate the slope of the straight line $y = 2x$		T→S S	*Straight line* *Slope of a straight line*	**The teacher proposes a problem to the students.** **T:** Calculate the slope of the straight line $y = 2x$. Come on, do it! **After 3-5 minutes, he asks a student to solve the problem at the blackboard** **S** ***(writes, without speaking)*:**  $\tan\alpha = \frac{4-2}{2-1} = 2$
Calculate the tangent of the angle of the secant line passing through two given points of the curve.	Technical Work of Π_0	T→S S	*Slope* *Straight line* *tan α*	**The teacher proposes a problem that cannot be solved using only the mathematical objects known by the students.** **T:** Find the slope of the straight line tangent to the parabola $y = x^2$ at the point $x = 2$.  x | y 1 | 2 2 | 4 **T:** We have difficulties to calculate the tangent line; We need another point of the curve to find it.
The limit object is needed to solve the problem.		T	*Slope* **(N)** $y = x^2$ **(N)** $\tan\alpha$ **(N)**	**The teacher proposes another problem: to calculate the slope of the parabola at a point, drawing a sequence of secant straight lines** **T:** Determinate the slope of the parabola $y = x^2$ at a point $x = 1$. **The teacher solves the problem on the blackboard. At the same time he asks some questions to the students:** **T:** For x = 0'5 we calculated it, and we know that the result is....1'5. Now, if x value is 0'75, how much is the slope? Come on, do it.  $\tan\alpha = \frac{1-0{,}25}{1-0{,}5} = \frac{0{,}75}{0{,}5} = 1{,}5$ **The teacher gives two minutes for the students to calculate it.**

(*Continued on next page*)

TABLE AI
(*Continued*)

	First Encounter with Π_1	T		**T:** What is the slope value? **S:** 1. **T:** Impossible. How much is it? **S:** 1'75. **T:** 1'75. O.K. **The teacher writes a table on the blackboard.** **T:** 0'5 $\rightarrow$ 1'5 0'75 $\rightarrow$ 1'75 **T:** How much is the image for the x value? **S:** 1'x **T:** What are you saying! **The teacher is surprised with this answer. He answers the question writing in the table:**
		T	*Limit* **(N)** $\lim_{x\to 1}(1+x) =$ **(N)**	0'5 $\rightarrow$ 1'5 0'75 $\rightarrow$ 1'75 $x \rightarrow 1 + x$ **T:** The slope for the x value is $1 + x$, and the slope of the curve at $x = 1$ is the limit of $1 + x$ as x comes near 1. **The teacher writes on the blackboard** **T:** $\lim_{x\to 1} (1 + x)$
End of the lesson				

TABLE A2
Analysis of the first session as reported in Table A1.

<table>
<tr><th>SESSION</th><th>Type of Mathematical Problems</th><th>Mathematical Techniques</th><th>Explicit Technological-Theoretical Elements</th><th>Dominant Moment & Sub-moments</th><th>Elements of the Didactical Techniques</th></tr>
<tr><td rowspan="2">1</td><td>Π_1: Calculate the slope of the straight line tangent tangent to the curve $y = f(x)$ at a given point a.</td><td></td><td>ϑ_1 : Informal definition of a curve like a 'line' in the plane.
ϑ_2: Definition of the tangent of an angle:
$\tan \alpha = b/a$.

ϑ_3 : Definition of the 'slope of a curve at a given point' as a generalisation of the slope of a straight line.</td><td>First Encounter with Π_1</td><td rowspan="2">* Considering the initial problem 'Calculate the slope of a curve at a given point' as the generalisation of some kind of previous solved problems: 'Calculate the slope of straight lines'.
* Identify two mathematical objects (the slope of a straight line and the slope of a curve in a given point).
* Put in action at the same time four mathematical objects without a clear differentiation between them (straight line slope, curve slope at a given point, straight tangent line and tangent of an angle).
* Remember an old MO around Π_0 and modify it progressively to get the new type of problems Π_1. After that, introduce successive variations of the used technique in order to solve the new considered problems. That allows to considering the new problem as a variation of an old one.

In this session appears a specific management of the old/new dialectic, supported by a technological ambiguity (the identification of the slope of an angle with the slope of a straight line and the slope of a curve at a given point).

It is an institutionalisation sub-moment that redefines old knowledge and builds the new mathematical environment of the lesson very quickly.</td></tr>
<tr><td>Π_0: Calculate the slope of the straight line $y = ax + b$

π_{01}: Calculate the slope of $y = 2x$

Π_0': Calculate the slope of straight lines that intersect a curve $y = f(x)$ at a given fixed point x_0 and a point close to it.
π_{01}': Calculate the slope of the secant straight line that intersects the curve $y = x^2$ at the points (1, 1) and (0'5, 0'25).
π_{02}': Calculate the slope of the secant straight line that intersects the curve $y = x^2$ at the points (1, 1) and (0'75, 0'5625).
π_{03}': : Calculate the slope of the secant straight line that intersects $y = x^2$ at the points (1,1) and $(x, f(x))$.
π_{11}: Calculate the slope of the straight line tangent to the curve $y = x^2$ at $x = 2$.
π_{12}: Calculate the slope of the straight line tangent to the curve $y = x^2$ at $x = 1$.</td><td>τ_0: Choose two points of the straight line, (x_0, y_0) and (x_1, y_1), draw the characteristic triangle and calculate:
$tan\alpha = \frac{y_1 - y_0}{x_1 - x_0}$
τ_0': Calculate the slope of a general straight line using the formula:
$tan\alpha = \frac{y_1 - y_0}{x_1 - x_0}$
τ_0'': Generalise the results obtained in a sequence of secant straight lines at a given curve in order to calculate the slope of the secant straight line that passes through a given point and a general point x of the curve.</td><td>ϑ_4: A straight line is determined by two points or by one point and its slope.

Θ_1: Generalise a result obtained by an iterative sequence using the induction.

ϑ_5: Use of the verbal expression 'limit as x comes near x_0' and of the notation
$\lim_{x \to 1} (x+1)$</td><td>Institution-nalisation (remembering) of the previous work done with the tangent of angles

First Encounter with Π_1</td></tr>
</table>

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