ORIT ZASLAVSKY

SEIZING THE OPPORTUNITY TO CREATE UNCERTAINTY IN LEARNING MATHEMATICS

ABSTRACT. The paper is a reflective account of the design and implementation of mathematical tasks that evoke uncertainty for the learner. Three types of uncertainty associated with mathematical tasks are discussed and illustrated: *competing claims, unknown path or questionable conclusion*, and *non-readily verifiable outcomes*. One task is presented in depth, pointing to the dynamic nature of task design, and the added value stimulated by the uncertainty component entailed in the task in terms of mathematical and pedagogical musing.

KEY WORDS: cognitive conflict, counterexample, mathematics teacher education, proof, uncertainty, task design

1. Introduction

In this paper I elaborate on the nature and potential of mathematical tasks that evoke to some extent a state of uncertainty for the learner, and point to the cognitive demands that such tasks put forth.

This paper is a reflective account of longitudinal work attempting to enhance learning and teaching processes of diverse populations – students, teachers, and teacher educators – using tasks that are intended to bring about uncertainty. Most of the tasks served a dual purpose: (a) for learning – as a vehicle to elicit meaningful learning and foster mathematical understanding, and (b) for research – as a means to identify dispositions towards, conceptions of and beliefs about mathematics, and mathematical concepts, principles and problem solving (the role of tasks in eliciting such cognitions is discussed by Goldin, 2000). The learning processes evoked by mathematical tasks were documented via journal keeping, in which I included detailed descriptions of numerous workshops with secondary in-service and pre-service mathematics teachers. In addition, for a number of tasks there were sessions that were videotaped and some of the participants were interviewed. Several tasks were tried out by these teachers in their own classes. The teachers, in return, provided me with detailed accounts of their experiences, which helped us gain more insights into the learning potential of the tasks.

With respect to mathematics educators' learning – these tasks served also as means to enhance both their *mathematical and pedagogical power*

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(Cooney, 1994, 2001), by engaging them in genuine learning experiences through which they became aware of their own learning processes and drew connections to their students' needs and difficulties. The vast majority of the tasks that were designed and tried out were challenging for teachers as well as for students, thus a teacher who dealt with these tasks as a learner would need to make very minor modifications, if any at all, in order to begin using them with his or her students.

Interestingly, the tasks that were found particularly worthwhile in terms of meaningful learning were all associated with elements of uncertainty. In the following sections, I discuss theoretical perspectives that support the use of tasks that elicit uncertainty, provide examples of different kinds of mathematical tasks that actually created much uncertainty, and distinguish between three types of uncertainty evoked by mathematical tasks. I elaborate on one example of a task, pointing to the iterative nature of and possible sources for the design of such tasks. Through this task I illustrate the kinds of mathematical and pedagogical learning enhanced by uncertainty.

2. Why deal with tasks?

The significant role that tasks play in the interplay between teaching and learning has been widely recognized (e.g., Leinhardt et al., 1990; Krainer, 1993; Sullivan and Mousley, 2001). Sierpinska (2004) maintains that the "design, analysis and empirical testing of mathematical tasks" is "one of the most important responsibilities of mathematics education" (p. 10, ibid). Kilpatrick et al. (2001, p. 9) claim that "The quality of instruction depends, for example, on whether teachers select cognitively demanding tasks, plan the lesson by elaborating the mathematics that the students are to learn through those tasks, and allocate sufficient time for the students to engage in and spend time on the tasks." Stein et al. (2000) developed a three phase framework of mathematical tasks used in classrooms for analyzing mathematics lessons. Their framework provided a tool for describing how tasks unfold during classroom instruction as well as for highlighting the significant influences tasks have on what students actually learn (Henningsen and Stein, 1997). In Jaworski's teaching triad (Jaworski, 1994) the important role of the task is expressed in the demand for mathematical challenge. Zaslavsky and Leikin (1999, 2004) take the teaching triad a step further and discuss the role and nature of tasks for professional development of mathematics educators (teachers and teacher educators), to whom they refer as facilitators of learning. Zaslavsky and Leikin (ibid) point to the dual nature of tasks for mathematics educators' learning: On the one hand, tasks are the content by which direct learning is facilitated. On the other hand, through a reflective process of designing and implementing tasks, they turn into means for indirect learning of the facilitator.

In light of the important function tasks have in facilitating both mathematical and pedagogical understandings, in general, and the special potential contribution of tasks entailing uncertainty, in particular, this paper is an attempt to conceptualize the latter kind of mathematical tasks.

3. Uncertainty, cognitive conflict, and learning

The grounds for creating learning situations that involve elements of uncertainty and doubt are rooted in Dewey's (1933) notion of reflective thinking. According to Dewey, "the origin of thinking is some perplexity, confusion, or doubt" (p. 15, ibid). Furthermore, reflective thinking entails: "1. a state of doubt, hesitation, perplexity, mental difficulty, in which thinking originates, and 2. an act of searching, hunting, inquiring, to find material that will resolve the doubt, settle and dispose of the perplexity" (p. 12, ibid). Fischbein (1987) asserts that the need for certitude is a strong driving force for learning. This need for certitude is commonly expressed in the development and persistence of intuitions.

Perplexity, confusion and doubt are often associated with and evoked by cognitive conflict. Although some psychologists have considered conflict as more disruptive than beneficial (e.g., Miller, 1944), there are many who consider conflict a positive contributing factor in rational thinking and the genesis of knowledge (e.g., Berlyne, 1960; Dewey, 1933; Festinger, 1957; Piaget, 1985).

Most research on cognitive conflict has been inspired directly or indirectly by Piaget's equilibration theory. Piaget viewed equilibration as a process that was stimulated by conflict or "disequilibrium", either between the individual and the environment or among the individual's own activities. It could be between an individual's action schemes and external realities or among different schemes within an individual. The source of disequilibrium could be either interpersonal conflict or intrapersonal conflicts that arise by social interactions. Piaget (ibid) and Vygotsky (1978) shared to some extent the idea that certain aspects of individual cognition originate in the "internalization" of social interactions. According to Vygotsky, all higher cognitive processes occur first as social relations before being internalized by the individual.

The role of conflict in cognition is expressed in Festinger's (1957) theory of "cognitive dissonance." According to this theory, the discomfort caused by logical inconsistency or contradiction motivates the individual to modify his or her beliefs in order to bring them into closer correspondence with reality. It should be noted that studies in mathematics education document reverse cases, where rather than modifying beliefs to come closer to "reality", beliefs are so overwhelming that reality is modified instead to fit beliefs.

Cognitive conflict also plays a considerable role in the acquisition of knowledge in Berlyne's (1960) theory of "conceptual conflict". In addition to logical contradiction Berlyne includes other types of conceptual conflict such as doubt – a conflict between the opposing tendencies to believe or not to believe a given statement; and perplexity – as a conflict among tendencies to accept mutually exclusive beliefs. According to Berlyne, the various types of conflict may result in a search for new knowledge.

In the work presented in this paper I use the word *uncertainty* as a unifying term to encompass the various constructs mentioned above, such as conflict, doubt, and perplexity. Additionally, inspired by Vygotsky's theory, my work builds on social interactions and communication as critical features for learning and constructing shared meaning. More specifically, as shown later on, social interactions may play a central role in creating uncertainty surrounding certain mathematical tasks as well as in leading to resolving uncertainties.

4. Tasks, uncertainty, and learning mathematics

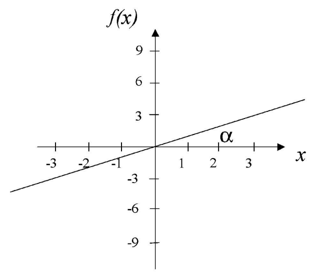
In the following section, I describe three interrelated types of uncertainty entailed in certain mathematical tasks and relate them to the dynamic process of learning mathematics.

4.1. *Competing claims*

One type of uncertainty can be characterized as involving two (or more) compelling *competing claims*. In this context, claims are viewed in a broad sense, to include outcomes, definitions, beliefs, *a priori* expectations, assumptions, and assertions offering different points of view of the same object. These competing claims may be contradicting statements with which the learner is confronted or an outcome that contradicts a well known (to the learner) mathematical truth (e.g., most tasks in Movshovitz-Hadar and Webb, 1998). For example, the question of how to define $(-8)^{1/3}$ raises competing assertions (Tirosh and Even, 1997; Movshovitz-Hadar and Webb, 1998). The first assertion could be that $(-8)^{1/3}$ is defined as (-2), because $(-8)^{1/3} = \sqrt[3]{(-8)} = -2$; The second assertion could be that $(-8)^{1/3}$ is defined as 2, because $(-8)^{1/3} = (-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$; A third possible assertion is that $(-8)^{1/3}$ is indefinable. These options can be seen as competing claims that contradict each other. In the case of choice of definition, uncertainty raised by competing claims need not be connected to correctness. It could be related to personal preferences, beliefs, values or the theoretical framework or context to which one refers. For example, in the case of $(-8)^{1/3}$, different theoretical frameworks suggest different approaches for dealing with rational exponents. One approach is defining rational exponents only for rational numbers r, $r = \frac{m}{n}$, that are reduced to their lowest terms. Accordingly, $(-8)^{1/3} = -2$ and $(-8)^{2/6}$ is undefined. Another approach is limiting the definition of rational exponents only to positive bases. The latter approach implies that $(-8)^{1/3}$ is undefined.

Uncertainty of competing claims may also occur when one of the "claims" is a (mis)conception or belief held by the learner which is in conflict with another claim. The other claim may be one with which he or she is confronted or one that is obtained by the learner in the course of the task. It could also be a competing belief or conception of the learner.

Such an example appears in Zaslavsky et al. (2002), who designed a special task that challenged a common implicitly held assumption regarding the representation of graphs in a coordinate system (Figure 1 is a variation of their task). The task poses questions, for which there are two competing answers to each of them. The slope task elicited much confusion concerning algebraic and geometric aspects of slope, scale, and angle, leading to a deep mathematical musing regarding properties of graphical representations of functions that remain invariant under non-homogenous change of scale. The degree of uncertainty people encountered (students, teachers, and teacher educators) varied considerably, depending on the assumptions and conceptions they held.



The graph represents the function f, $f\!: x \to x$.

- Is the slope of the function 1 or $\frac{1}{3}$?
- Does the graph bisect the angle between the axes?
- Does $\tan(\alpha)$ equal 1 or $\arctan\left(\frac{1}{3}\right)$?

Figure 1. The slope task.

The uncertainty evoked by competing claims concurs to a large extent with the kinds of conflicts described in the previous section. However, in the context of learning mathematics we consider another two types of uncertainty.

4.2. *Unknown path or questionable conclusion*

The second kind of uncertainty is associated with inquiry, exploration tasks and open-ended problems. The very nature of exploration and inquiry is a search for findings (e.g., patterns, connections, relationships) that are unknown to the learner, for which he or she may or may not have an intuitive feeling regarding what to expect. I refer to this case as an *unknown path or questionable conclusion* type of uncertainty. Exploration tasks that create this type of uncertainty are particularly common in technologically enhanced learning environments (e.g., Hadas and Hershkowitz, 2002). Typical paths that students go through in tackling such exploration tasks are described by Hadas and Hershkowitz (ibid). These paths demonstrate movements from an (interim) conjecture, through experimentation aimed at checking several instances of the conjecture, leading to either (1) a refutation of the conjecture followed by a shift to an alternative one, or to (2) reinforcement of the conjecture, followed by the need to support it by explanation and proof. Each shift from one conjecture to another is a result of a state of uncertainty about reaching a valid conclusion.

Another kind of unknown conclusion is associated with existence tasks. Many exploration tasks are closely related to existence tasks, since they present the need to check if certain cases are at all possible. Consider, for example, the task in Figure 2 (taken from Zaslavsky and Leikin, in progress).

This task lends itself rather naturally to explorations in a dynamic geometry environment, as in fact occurred in several workshops conducted with pre-service and in-service teachers (ibid). It turned out that for such a task the use of the technologically enhanced environment presented some

For a given triangle $\triangle ABC$, is there a point D **in** the triangle such that the areas of triangles $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$ are equal?

Does your answer depend on the type of the triangle? If it does, how does it?

If there is a triangle for which such point D exists, are there any other points in the triangle that fulfill this condition? Explain your answer.

Figure 2. An existence task.

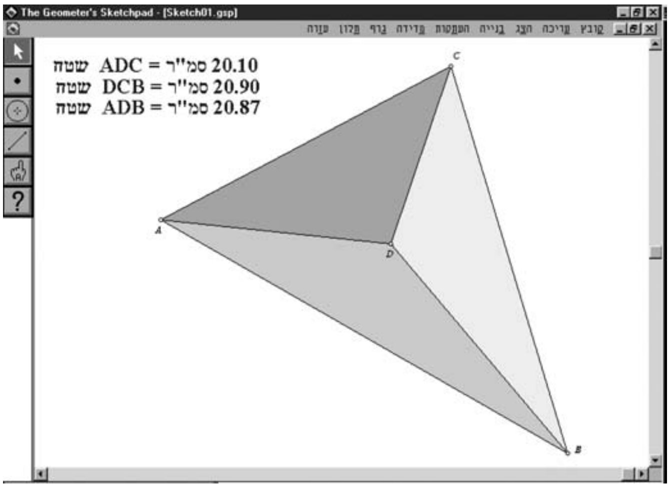


Figure 3. Uncertainty evoked by a "free search" strategy.

unforeseen obstacles adding new dimensions of uncertainty to the task. Most teachers began this task by employing "free search" strategies. One strategy that they used was "moving a random point (in the triangle) to fit the condition". However, this strategy did not yield a point on the screen that precisely fulfilled the condition; thus, they remained with a "questionable conclusion." Figure 3 depicts the position of a very "close" but not "exact" point. Thus, they remained uncertain whether the anticipated point really existed. (Note that the areas of the three triangles are almost equal: $S_{\triangle ADC} = 20.10$, $S_{\triangle DCB} = 20.90$, $S_{\triangle ADB} = 20.87$.)

Another, more structured "free search" approach, which some teachers employed, was based on their analysis of the ratio between the distances of the anticipated point from two sides of the triangle. They tried to find a point position for which the ratio of its distances from two of the given triangle sides was inverse to the ratio of the lengths of the respective sides. Although they were able to prove that for such a point the areas of the two relevant triangles were equal, on the screen this still did not yield the intended point. Those who were confronted with this result were perplexed by the fact that the ratios appeared to be of inverse values, but still the areas measures produced by the computer were close but not exactly equal (see Figure 4, in which $S_{\triangle AKC} = 22.44$ and $S_{\triangle AKB} = 22.35$).

On top of leading to alternative problem solving strategies, the uncertainty involved in this existence task enhanced teachers' awareness of the limitations of the media with respect to "free search" strategies, opposed to its potential with respect to more structured approaches (ibid).

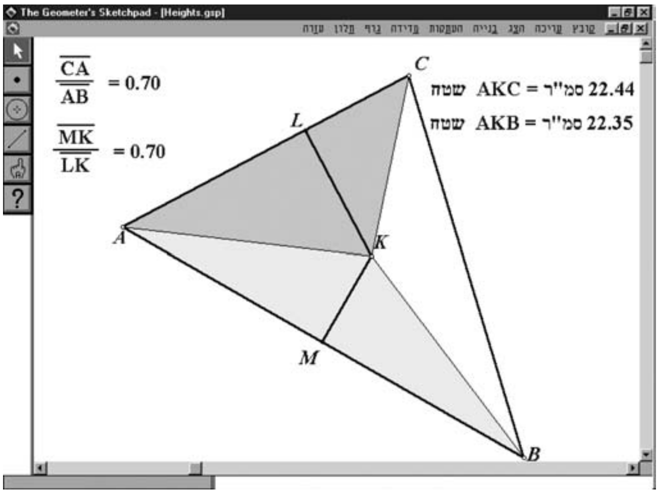


Figure 4. Uncertainty evoked by another “free search” strategy.

4.3. *Non-readily verifiable outcomes*

A third type of uncertainty has to do with the lack of confidence one may have regarding the correctness or validity of an outcome (e.g., solution to a problem) which requires verification. When one lacks readily available verification methods for the specific outcome in question, I consider it a case of *non-readily verifiable outcomes*. This type of uncertainty is particularly common to some specific domains of mathematics, such as the areas of probability and combinatorics. In the case of combinatorial tasks uncertainty is associated with the difficulty to verify the result and to estimate the number of cases under question (Fischbein and Grossman, 1997; Mashiach-Eizenberg and Zaslavsky, 2004). Although there are ways to verify solutions to combinatorial problems, seldom are they readily available to the problem solver. Thus, in such cases, people usually feel as if the result they got is non-verifiable. This difficulty is expressed in a quote from an interview with Joel – an undergraduate student – in response to the interviewer’s prompt concerning how Joel checks his solutions to combinatorial problems:

This is the problem in Combinatorics that you can’t check. I have no tools, or at least I never learned how to check what I do. That is, I use formulas that were proven, and with them I try to solve all kinds of problems. But I don’t have any indication for checking myself. Even now, if you ask me, I would have no indication except your expression. (Mashiach-Eizenberg, 2001).

This type of uncertainty, which is associated with a subjective lack of available verification tools, is not restricted to certain topics. It could be a personal feeling, even in cases for which there are well established methods of verification (e.g., in the case of solving linear equations); if one does not know how to verify a solution to a problem, he or she may feel a sense of uncertainty, even if there are no competing solutions at question.

Note that the three types of uncertainty discussed above are interrelated. Thus, for example, uncertainty regarding a questionable conclusion (second type) could (though need not) be enhanced by the lack of verification tools (third type); or uncertainty regarding non-readily verifiable outcomes may be triggered by two competing claims.

4.4. *The dynamic and subjective nature of uncertainty*

Figure 5 encapsulates the three types of uncertainty that could be entailed in mathematical tasks, as discussed and illustrated earlier. Within a particular task there could be movement from one type of uncertainty to another. This kind of movement, which is exemplified in the following section, indicates the dynamic nature of uncertainty. A resolution of one state of uncertainty could lead to another state of uncertainty.

It should be noted that the state of uncertainty or doubt is entirely subjective. The same task may stimulate doubt for some people but none for others. What may be an unknown path or questionable conclusion for one

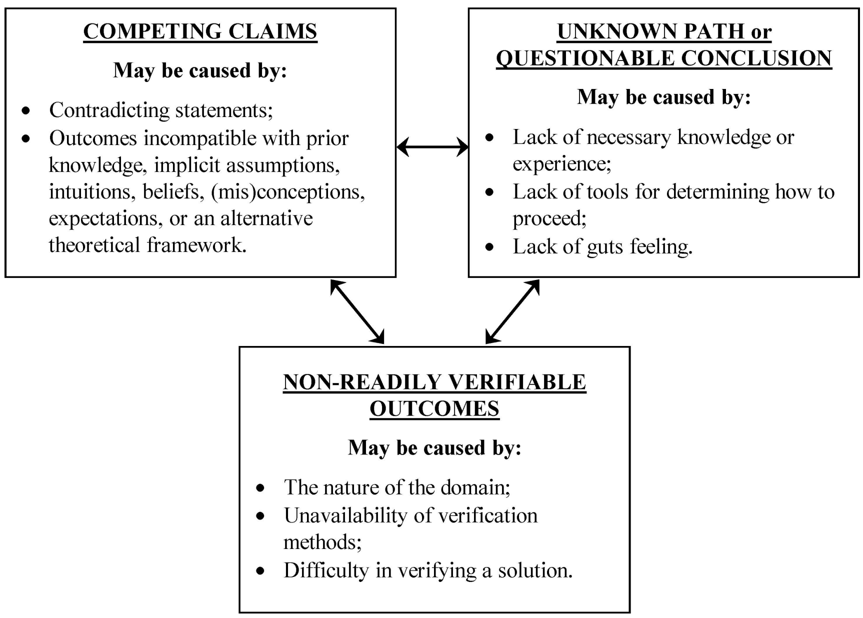


Figure 5. Types of uncertainty evoked by mathematical tasks.

person may be obvious to another. A result may seem non-verifiable at one point and turn verifiable at a different stage. A person can be very doubtful regarding a provable conjecture that he or she proposed, until he or she are able to fully support it by logical inference. Moreover, uncertainty may lead a person to abandon a (correct or incorrect) conjecture, in the course of dealing with a task. For those who do not encounter a state of doubt when dealing with the task alone, doubt may occur as a result of social interactions with another person who reached a different conclusion or encountered confusion over the same task.

In the next section I elaborate on one task that repeatedly evoked uncertainty which led to meaningful learning. It illustrates the dynamic nature of uncertainty as well as the iterative aspect of task evolution. I focus on mathematical subtleties and cognitive demands that underlie the task and present its added value compared to a more commonly used version.

5. THE ADDED VALUE OF TASKS EVOKING UNCERTAINTY – AN EXAMPLE

The task on which I focus in this section, to which I refer as the square task, originated from a regular textbook task that stated what the measure of the angle α in Figure 6 is supposed to be, and requested the proof for that assertion.[1] As I further illustrate, a small change in the task led to an unexpected cognitive conflict encountered by a group of pre-service secondary mathematics teachers within the framework of an undergraduate geometry methods course. Based on the conflict that arose and the conflict resolution encountered by this group, the task was modified once again, to recreate

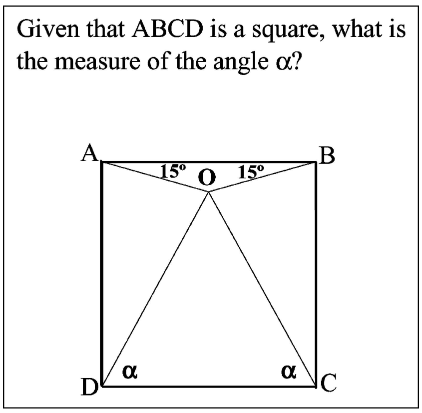


Figure 6. The first modification of the square task.

such conflict in future occasions. Consequently, the square task turned into a source of uncertainty that led to understandings that would not have been reached in its original form. In this section I present some anecdotes of actual experiences with the square task and point to the iterative process of task development as an outcome of my own (indirect reflective) learning through practice as facilitator of leaning.

5.1. *The square task-first iteration*

In the first iteration, the modification of the textbook task removed the disclosure of the measure of the angle α (Figure 6), in order to create some degree of uncertainty of an *unknown path*. A group of pre-service students were asked to solve the problem in Figure 6 as part of a homework assignment within the framework of an undergraduate geometry methods course. On the day when they were supposed to hand in their assignment, an argument developed spontaneously between a student and a teacher namely Bob and Ruth during the lesson. (All names in this paper are pseudonyms.)

Bob claimed that he was able to prove that $\alpha = 60°$. However, Ruth argued that his proof must have a flaw, since she could produce a counterexample in which $\alpha = 55°$. In order to convince Bob, she sketched on the board the case in Figure 7, and showed how all the angle measures in the square fit well with each other.

At this point, Bob was not able to refute Ruth's claim although he was convinced that she was wrong. Similarly, Ruth was not able to find a flaw in Bob's proof, yet she was convinced that she had generated a counterexample that disproved his proof. This became an extremely perplexing situation for all the student-teachers in the class, for now they were confronted with two *competing claims* and could not come up with compelling arguments to settle their state of uncertainty. Both cases in Figure 7 and in Figure 8 seemed equally possible.

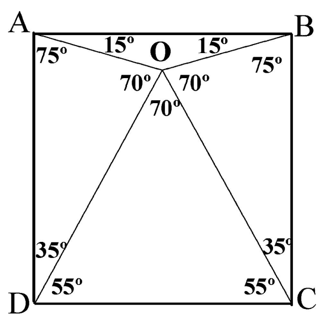


Figure 7. Ruth's "counter-example".

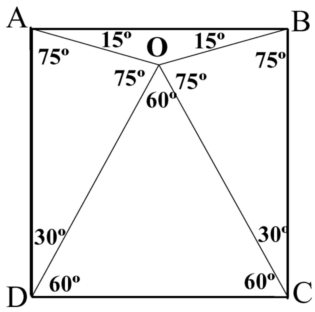


Figure 8. The case that Bob proved.

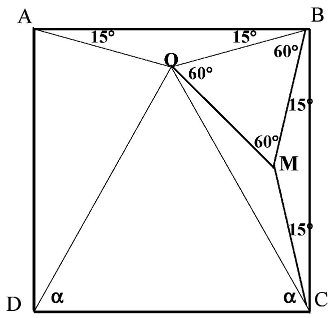


Figure 9. The auxiliary construction in Bob's proof.

The proof that Bob presented was a direct proof that he had found in a textbook, based on an auxiliary construction (see Figure 9). This did not satisfy Ruth. She felt that the construction was not a natural thing to do and could not imagine how one would think of this idea on his own. Actually, the direct proof that Bob presented was not convincing for her, since it did not offer her an explanatory reason why the measure of α must be 60°.

It took Ruth three days to resolve the conflict to her satisfaction. Part of the time was devoted to the question whether the two cases in Figures 7 and 8 were both possible. She encountered conflict regarding the two *competing claims* accompanied by uncertainty of *non-readily verifiable outcome* which was manifested by her feeling that she lacked the means by which to check whether α could really be 55°. She was determined to find a way to verify or refute this case. This persistence led to a moment that was particularly notable for her. It was when she realized why the case she had regarded as a counter-example actually did not exist. By constructing the square more accurately, she learned that it really looked very different (see Figure 10) – the points O and P did not coincide. That led her to try to sketch the case of $\alpha = 65°$. This again was an enlightening

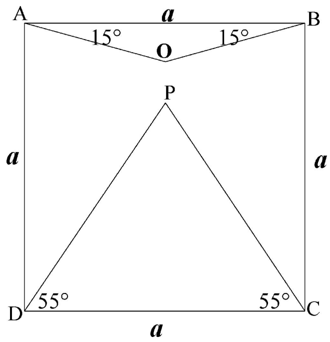


Figure 10. Ruth's resolution of her "counter-example".

discovery – if the points O and P coincide and the measure of α is 65°, then it must be a 3-dimensional body.

Ruth kept sharing with me some milestones along this path, and was grateful that she had to work it out for herself the hard way. At the end she reached a deep understanding of the problem and how to resolve it. For years later, even after she graduated and got her degree, Ruth would refer to this learning experience of hers as one of the most powerful that she had encountered.

5.2. *The square task – further iterations*

The above class event and what followed provoked much re-thinking about and appreciation of the task and the subtleties that it entails. These thoughts led to a second variation that aimed at recreating in other groups what occurred spontaneously in the first group, namely, uncertainty of *competing claims*. Thus, the second iteration was intended for a class activity (contrary to the first time when it was given as a home assignment). This time half the group got the task with 55° and the other half got it with 60° (see Figure 11). At the beginning the work was done individually and the participants were not aware of the different versions. Only after spending some time dealing with the task alone, they were grouped into pairs each having worked on a different version.

Indeed, the second version evoked much debate and conflict very similar to the authentic situation. This version of the task led to some additional observations. For example, in several subsequent occasions it was argued that if both 55° and 60° were possible then judging by angle calculations many other measures were possible too, as illustrated in Figure 12.

The latter observation led to the question of whether at all α is uniquely determined – which expresses uncertainty of an *unknown path*

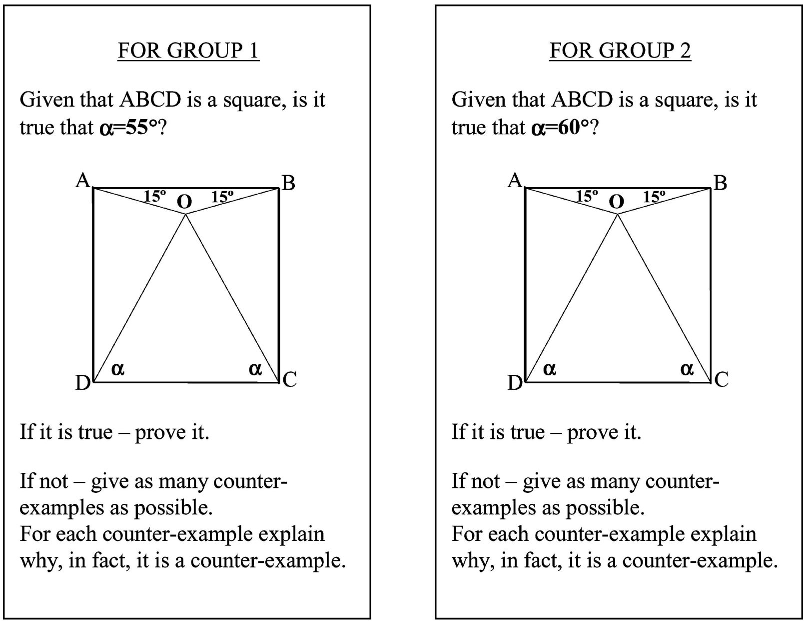


Figure 11. A second modification of the square task.

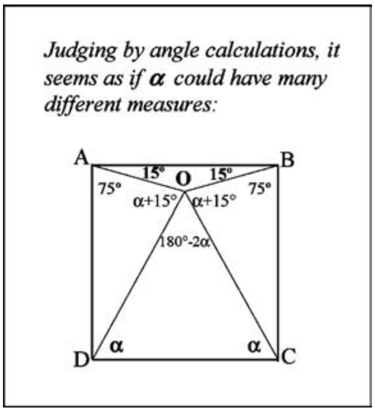


Figure 12. A common observation.

or questionable conclusion. To one group of student-teachers it seemed that the measure of α was not unique. Thus they tried to figure out the possible range of measures. They tried to examine extreme or special cases, such as 0°, 45° and 60°, and concluded that it could probably be any measure between 45° and 60°, yet they were somewhat uncertain about this speculation. For a while they felt that they had no accessible way to verify their supposition. Thus, at this point they encountered uncertainty of a *non-verifiable* nature. However, after a short period of time their doubts led some of them to turn to the computer for support. They sketched the given square in a dynamic geometry environment, and then examined many different squares, satisfying the same conditions, that were generated by dragging one of its vertexes. This move contributed to their belief that the measure of α was unique for all squares and that it would always be 60°. Yet they still remained puzzled as to the reason for this result – why 60°? Interestingly, it seemed as if the way the task was set up and the uncertainty that it evoked resulted in a need to better understand the problem and not to suffice with formal deductive arguments.

There were a number of student-teachers who turned to their knowledge of trigonometry to get an answer to the question what the measure of α must be. By means of trigonometry they reached a confirmation that α must be 60°. Yet, they questioned the right to use trigonometry in a geometry methods course. In their discussion of this dilemma they acknowledged the legitimacy to use any valid method to solve a mathematical problem. However, they also expressed a strong need, as prospective teachers, to be prepared to incorporate such a geometric problem in their future classroom before their students learn trigonometry. Most of them felt that the trigonometric solution did not provide any insight to the problem situation.

At an in-service workshop, after struggling over the task for a while, one teacher pointed to a contradiction stemming from the assumption that $\alpha = 55°$ (see Figure 13). She actually generated an indirect proof that could apply to any measure of α other than 60°. Another indirect proof was

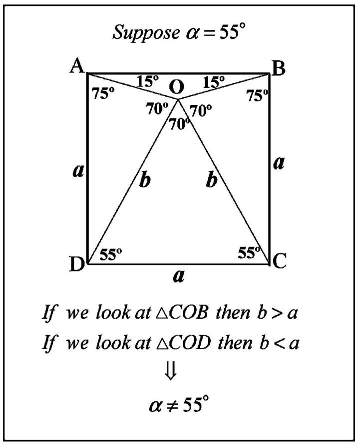
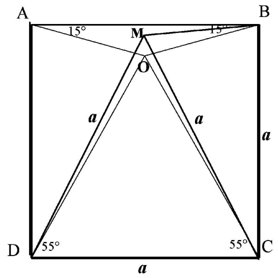


Figure 13. A proof by contradiction that $\alpha \neq 55°$.

Suppose $\alpha = 55^\circ$



Thus, we can construct an equilateral triangle $\triangle MDC$

$\Rightarrow \measuredangle MCD = 60^\circ \Rightarrow \measuredangle MCB = 30^\circ$

$\triangle CMB$ *is an isosceles triangle* $\Rightarrow \measuredangle CMB = \measuredangle CBM = \frac{180^\circ - 30^\circ}{2} = 75^\circ$

But $\measuredangle CBO = 90^\circ - 15^\circ = 75^\circ \Rightarrow \measuredangle CBM = \measuredangle CBO \Rightarrow$ *the whole equals its part*?!

$\Downarrow$

$\alpha \neq 55^\circ$

Figure 14. Another proof by contradiction that $\alpha \neq 55^\circ$.

suggested by a different teacher (see Figure 14). The latter used an auxiliary construction that seemed much more plausible to the participants than the one in Figure 9.

Both proofs by contradiction, that emerged for a specific measure (55°) and continued to the more general assumption that $\alpha \neq 60^\circ$, seemed very convincing. They provided explanatory grounds for resolving the earlier uncertainty regarding the two *competing claims* (55° or 60°?).

There was a group of teachers who were bothered by the uncertainty that the square task evoked and were not satisfied with “just” a formal proof that $\alpha \neq 55^\circ$. They searched for additional grasp of the problem by starting out with an isosceles trapezoid ABCD (see Figure 15), again in a dynamic geometry environment, applying a continuous translation of its side DC parallel to AB. That is, by dragging the side DC and monitoring the kinds of quadrilaterals that appear, it became clear that the family of quadrilaterals that was constructed in this way included many isosceles trapezoids, one rectangle, but no square. This approach provided a dynamic demonstration of the problem and of the fact that α could not be 55°. Instead of looking at the case of $\alpha = 55^\circ$ as Ruth did (see Figure 10), that is as a square in which points O and P do not coincide, they looked at it as a rectangle (that is not a square).

It is interesting to point out that the direct proof that Bob proposed (see Figure 9) was suggested in other incidents mostly by people who had seen

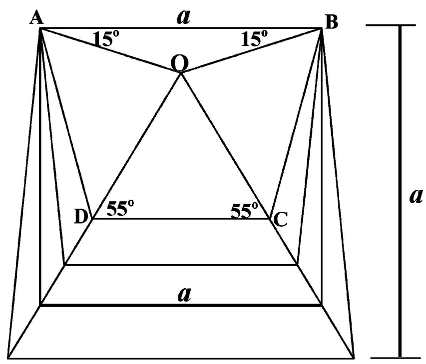


Figure 15. A dynamic explanatory demonstration why $\alpha \neq 55°$.

this proof in a textbook. In the in-service workshops some teachers pointed to what seemed to them a bit "shaky" in the direct proof – how does one know for sure that the constructed point M is located in □ *CBO*? Unlike the indirect proof that developed naturally in the course of dealing with the task, the direct proof hardly emerged. Moreover, although the direct proof gave a formal confirmation which could settle the case, it left people with an inner drive to search for additional explanation of the problem situation.

Following the evolving experience surrounding the square task another version of the task was designed, in which everyone was asked to consider both possibilities for α. The two modifications have been successfully used numerous times, always yielding most of the above considerations and questions.

Looking back at the original textbook version of the problem (see Figure 16), it seems rather obvious that the main difference between the two tasks is the degree of uncertainty associated with them. The three

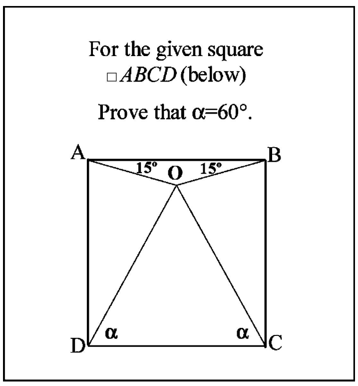


Figure 16. The original textbook problem.

modifications of the task emerged in response to the facilitator's desire to create uncertainty. The first iteration was designed to create an *unknown path and questionable conclusions* type of uncertainty. In the course of implementing it, a *competing claims* type of uncertainty arose accompanied by a *non-verifiable* type of uncertainty. Consequently, the second and third iterations were designed to evoke all three types of uncertainty.

Although the above anecdotes refer mainly to pre-service and in-service mathematics teachers, the square task was also tried out with secondary students in a number of classes of 9th and 12th grade top level students. Ron (1996) used the second modification as a homework assignment, but gave all the students the same part (Figure 11, for Group 2). The homework was used to facilitate in the following lesson whole group discussion surrounding the task. Apparently, the students encountered the same conflicts as the teachers. However, they needed more assistance in resolving the conflict, and expressed less inner motivation to get at the roots of the problem. The focus of their activity was primarily on their tendency to lean on supposed (counter-)examples that in fact do not exist due to excessive constraints (as discussed by Peled and Zaslavsky, 1997; Zaslavsky and Ron, 1998).

5.3. *The square task – teachers' reflections*

The above account provides an idea of the mathematical musing that took place and that may be stimulated by the uncertainty involved in the square task for diverse populations of students and mathematics educators. With respect to mathematics educators, there was another dimension to this learning experience – the pedagogical aspect in its broad sense. This experience seemed to contribute to their awareness of and sensitivity to ways in which students may learn. Following are a few excerpts illustrating teachers' reactions to the second modification of the task (Figure 11) from a learner's perspective, as came up at whole group reflective discussions.

Miriam: At first when I worked on the problem by myself, I felt like a student. I felt unconfident and said to myself: "this can also happen to me. It's not always possible to find the correct answer."

I observed my colleagues struggle with the problem, and that made me feel better about the possibility to hesitate or err. It became legitimate. I felt that I was in the process of learning. I wondered how I, as a teacher, could create such situations in my classroom and how I could change regular questions into questions that stimulate discussion.

Sara: I first checked the angle 55°, and didn't reach a contradiction. I also tried 60° and again didn't reach contradiction. So I concluded that the statement is false, that is, that the measure of α is not necessarily 55°.

Later on, during the discussion, I realized that $\triangle COD$ was uniquely determined and was angry at myself for not checking how two different angles fit the same triangle. As a teacher I always teach my students to be critical about what they do. When I realized that I did not act accordingly it bothered me.

Joseph: I enjoyed this a lot. It caused me to be much more involved by raising my curiosity. The activity called for integration between people and between domains and I liked it, because when someone faced an impasse there was hope that someone else would help move this process, and this it neat.

Rose: I left last week with a clear decision not to confront my students with uncertainty and not to leave them hanging in the air. But after giving it more thought I decided to dare and tried something similar with an algebraic problem. I was very surprised by how well it went and how enthusiastic and cooperative the children were. They came to see me every day and shared with me their progress.

The following excerpt is from an interview with Mike, an in-service teacher who worked on the square task with another teacher – Ronit. As Mike dealt with the problem and thought that he had solved it, Ronit kept raising a question to which he was not sure how to respond. He tried working on a system of equations that did not lead him to any result. It was a new experience for him not to be able to offer a convincing explanation. As expressed in the excerpt below, Mike compared this situation to his teaching experience.

Mike: . . . It's not because I'm not a good teacher, it's just that this "student" [Ronit] asked me a question that I am never asked in my class.

Interviewer: So maybe you should educate your students to ask such questions?

Mike: Maybe, but it can lead to questions with no answers.

Interviewer: Is that bad?

Mike: It's difficult. I can't recall being in such a situation in which I am asked a question that I can't answer and need to think about it at home. If she were a young student, I'd convince her. It's possible to convince by changing your voice and tone. That's how it goes in my classroom – you see that you are right and they [the students] don't understand, so you repeat your explanation over and over until they are convinced.

The excerpts above convey the potential of uncertainty in evoking teachers' reflections on their practice in light of their personal learning experiences. They also indicate the role of social interactions in making the most of the learning opportunity created by the task.

6. Concluding Remarks

In this paper I tried to point to two kinds of interrelated dynamics: The dynamic movements and interplay between the different types of uncertainty that may be evoked by a task (as depicted in Figure 5), and the dynamic iterations in task design as a response to uncertainty that occurs in the course of implementation of the task. These two kinds of dynamics are presented in Figure 17.

A key factor in identifying opportunities for creating uncertainty and translating these opportunities to task design is reflection. As mentioned earlier, the paper is an account of my own learning from a reflective practitioner's perspective (Schön, 1987), in and on action. For me, the process of planning and implementing the kind of tasks discussed earlier provided meaningful learning situations both mathematically and pedagogically. For example, Bob and Ruth's authentic debate which I had overheard created a moment of uncertainty for me. It led me to deep mathematical musing which I would not have encountered otherwise. In addition, as a mathematics teacher educator it drew my attention to the learning potential of the problem for many different learners, similarly to the experience discussed in Zaslavsky (1995).

Doing mathematics often involves establishing mathematical truths by checking conjectures, and deciding whether to reject or accept them. The process of establishing truth in mathematics entails an act of explaining

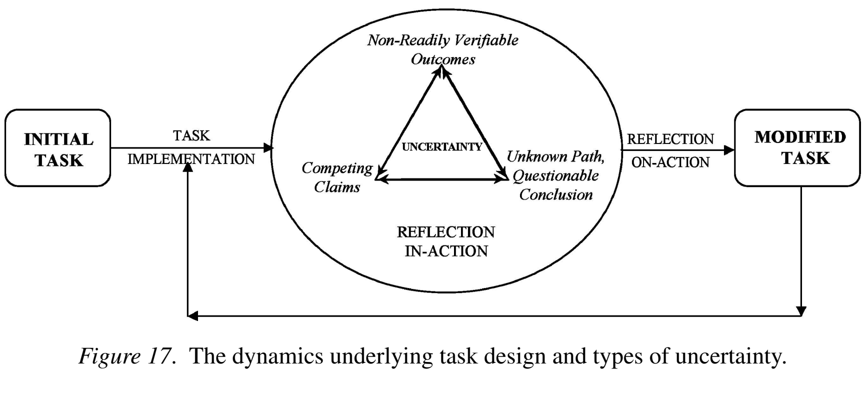


Figure 17. The dynamics underlying task design and types of uncertainty.

and convincing, an act that is central to mathematical thinking (Burton, 1984; Mason et al., 1982). The mathematical thinking that was fostered through just one doubt provoking task – the square task – encompassed issues related to validity and certainty, multiple proof methods, existence and uniqueness, examples and counterexamples, and connections, mostly initiated by the learners. Moreover, in this context the indirect method of proof, that is considered problematic and arose much debate among mathematicians (Courant and Robbins, 1996), emerged iteratively in a natural setting, as a manifestation that this method may be at times rather convincing. As shown in Figures 13 and 14, the two proofs by contradiction developed spontaneously, in contrast to the common direct proof that Bob suggested (Figure 9). It seems that this may be the case particularly when the direct proof is based on an auxiliary construction that does not come to mind naturally. Students are known to feel that such constructions are beyond their capability of thinking on their own, and their role in such cases is usually limited to accepting a construction that is suggested by an expert (a book, a teacher or an excelling student) and figuring out how to use it for proving. Especially interesting is how the state of uncertainty developed doubtfulness that led to a genuine need for proof and a search of multiple sources of explanations much beyond the "right answer." More specifically, in terms of Hanna (2000), it enhanced an inner need for proofs that explain. Overall, these are the kinds of mathematical understanding that secondary mathematics teachers should develop.

Tasks are usually constructed or adapted by mathematics educators in order to provide learning opportunities for others. It may seem like the dynamics of task development portrayed in this paper is particularly significant for mathematics teacher educators, since there are far more resources of learning materials for students than for mathematics pre-service and in-service teachers. Nonetheless, the ongoing process of seizing the opportunity to create uncertainty in learning mathematics may be in itself a powerful source for both teachers and teacher educators' learning and teaching. It follows that teacher education programs should aim at increasing mathematics teachers' awareness of the potential of learning situations involving uncertainty and enhancing their ability to design and adapt such tasks for their students.

Many worthwhile mathematical tasks are associated with elements of uncertainty. Once a teacher or teacher educator has the mindset to incorporate doubt provoking tasks, the sources for such tasks may vary, as exemplified earlier, including: (1) common textbook problems that could be slightly modified to create uncertainty; (2) widespread students' mathematical difficulties and misconceptions that could be confronted by the task; (3) mathematical subtleties throughout the curriculum that could form the basis to question implicit assumptions or beliefs; (4) authentic classroom events that entail debate and conflict that may be recreated by the task.

A critical factor in successfully implementing a task with the potential of generating unforeseen elements of uncertainty and doubt relies on appropriate classroom setting and climate. This paper provides supporting evidence of the significant role that social interactions and exchange of personal viewpoints and preferences play in fully exploiting the potential of such a task. Clearly, it is unlikely that the full scope of solutions, approaches, and additional concerns would have been reached by one individual alone.

In spite of the value and potential of uncertainty as a driving force for learning mathematics, a word of caution needs to be said - uncertainty should be treated with care. A state of conflict may easily lead to frustration and fragility (Movshovitz-Hadar, 1993). Additionally, Behr and Harel (1995) point to evidence that conflict resolution is often accomplished by application of erroneous procedures. They raise two important questions. The first relates to the need to create situations that involve an *appropriate* level of cognitive conflict for the individual learner. The second has to do with the need to guide the learner to "conflict resolutions through the construction of knowledge structures which are consistent with domain principles" (p. 83, ibid). An attempt to create uncertainty by confronting a learner with a mathematical contradiction may not necessarily lead to cognitive conflict as appears to be the case in many examples of people living at peace with mathematical inconsistencies (Tall, 1990; Tirosh, 1990; Vinner, 1990). Even when conflict is evoked, it may not be effective, as in the case of intuitive rules that to a large extent are stable and resistant to change (Tirosh et al., 1998).

In light of the above, the effect of the square task, in its wide scope, on learning should not be taken for granted. The dynamics described in section 5 illustrate how interweaving uncertainty in the square task hand in hand with promoting social interactions (in the spirit of Cobb et al., 1993), elicited an authentic need to prove and convince. The social context was critical for this to develop, concurring with Vygotsky's (1978) view that all higher cognitive processes occur first as social relations. In addition, the different kinds of uncertainty led to a refinement of the participants' understanding of what valid arguments and inferences are. They engaged in a comparison between different suggested proofs and evaluated each one's merits and limitations. In this sense, the task led to manifestations of higher order thinking and abstraction, which are not commonly exhibited.

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NOTE

1. At this point I prefer not to disclose the original textbook task in order to leave the reader who may not be familiar with this problem uncertain with respect to the measure of α.

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ORIT ZASLAVSKY
Department of Education in Technology and Science
Technion – Israel Institute of Technology
Haifa, Israel
E-mail: orit@tx.technion.ac.il