Introduction

PME SPECIAL ISSUE: BODILY ACTIVITY AND IMAGINATION IN MATHEMATICS LEARNING

The use of manipulatives in mathematics education is part of a long tradition enriched by noted educators such as Maria Montessori, Georges Cuisenaire, Caleb Gattegno, and Zoltan Dienes. Like many teachers, these educators have observed that numerous students get engaged with materials that they manipulate with their hands and move them physically around, with an intensity and insight that do not seem to be present when they just observe a visual display on a blackboard, a screen, or a textbook. While one should not expect that students' experimentation with manipulatives and devices would automatically cause them to learn mathematics,[1] there must be something valuable that sustains their use even at the present age when it is simple to simulate them on a computer. It is not the same experience, for instance, to watch a movie displaying a geometrical object and to touch or walk around a plastic model of the same object. Clearly both experiences can be useful, but even if one could argue that they both reflect the same mathematical principle, they are not mere repetitions. One difference is that the use of materials and devices facilitates the inclusion of touch, proprioception (perception of our own bodies), and kinesthesia (self-initiated body motion) in mathematics learning. A key reason for the frequent dismissal of this difference is that visual perception is ordinarily conceived as self-contained and passive. Given that mathematical ideas get expressed mostly in visual form – strings of symbols, graphs, diagrams, etc. – other types of action and perception appear to be of little significance. For example, following this assumption, whether one is shown a cube turning or one walks around a cube, the cube's retinal images, except for the background being static in the former case, are the same and therefore one would see the same.

However, there is an emerging perspective, sometimes called "Exploratory Vision," which describes vision as fully integrated with all

[1]This thesis has been extensively criticized by many researchers; see, for example, Ball (1992); Cobb et al. (1992); Lesh et al. (1987); Meira (1998) and Teasley (1993). Meira (1998), for instance, asserts: "the transparency of devices follows from the very process of using them. That is, the transparency of a device emerges anew in every specific context and is created during activity through specific forms of using the device," (p. 138).

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the body senses and actions. Our eyes are constantly moving in irregular ways, momentarily fixing our gaze on a part of the environment and then jumping to another one. It is as if we are constantly posing questions to the visual environment and making bodily adjustments that might answer them.

> On this view, no end-product of perception, no inner picture or description is ever created. No thing in the brain is the percept or image. Rather, perceptual experience consists in the ongoing activity of schema-guided perceptual exploration of the environment. (Thomas, 1999, p. 218)

The notion of Exploratory Vision is not new in itself; for instance, it was described in 1905 by Poincaré, who wrote, "when it is said that we 'localize' such an object in such a point in space, what does it mean? It simply means that we represent to ourselves the movements that must take place to reach that object" (Poincaré, 1905/1952). Poincaré argued that one conceives of a localization in space by means of the "muscular sensations" that accompany our real or imaginary movements around or toward such location.

Another reason drawn on to set aside touch, kinesthesia, etc. in mathematics learning is that mathematical entities cannot be "materialized", one cannot touch, say, an infinite series or the set of even numbers. While true, the fact that these entities are imaginable is profoundly connected to perception and bodily action. It is increasingly becoming evident that there is a major overlap between perception and imagination (Decety, 1996a, b). If, for example, we imagine a house, the inner bodily processes are strikingly similar to what would happen if one would actually see the inexistent house. To imagine, for instance, a limit process, one extends perceivable aspects to physically impossible circumstances and conditions. In this regard, touch and kinesthesia can be instrumental to imagining. It is not unusual that to imagine inexistent objects and events one gestures shapes and motions or takes hold of an object, say a cardboard box, to help see them from different sides.

This special issue includes four videopapers, each examining different aspects of how bodily activity and imagination participate in mathematics teaching and learning. They are:

1. Approaching Functions through Motion Experiments
   Ferdinando Arzarello and Ornella Robutti
2. Incorporating Experiences of Motion into a Calculus Classroom
   Marty Schnepp and Daniel Chazan
3. On Forms of Knowing: The Role of Bodily Activity and Tools in Mathematical Learning

Chris Rasmussen, Ricardo Nemirovsky, Jennifer Olszewski, Kevin Dost, and James L. Johnson
4. Coordination of Multiple Representations and Body Awareness
Marcelo Borba and Nilce Scheffer

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APPROACHING FUNCTIONS THROUGH MOTION EXPERIMENTS

According to current research, some of students' most serious difficulties in conceptualizing the function concept are related to the interpretation of graphs, particularly those in which a variable is time-dependent: for example, space–time or velocity–time graphs. This paper presents a part of an ongoing project conducted from the first years of secondary school (9th grade up), where Calculus is approached early within different learning contexts. Our videopaper is based on a teaching experiment which we designed to approach the function concept. For the experiment, we used a motion sensor and a symbolic-graphic calculator, which students (14–15 years old) use to produce and interpret graphs and number tables aimed at describing different kinds of motion (either of their own bodies or of other objects) (for more details about the experiment, see Ferrara and Robutti, 2002). The didactic aim of the teaching experiment was the construction of the concept of function (e.g. linear or quadratic functions), as a tool for modeling motion (uniform and accelerated, respectively). The aim of the research was to analyze students' cognitive processes involved in the construction of meanings of mathematical objects.

The analysis was developed using several theoretical frameworks, in order to capture the different dimensions of the students' activity. The social dimension, representing the mathematical discussion orchestrated by the teacher (Bartolini Bussi, 1996) was captured using a general Vygotskian framework, whose emphasis is on the social construction of knowledge and the mediation by cultural artifacts. The role of artifacts was then studied using the concepts of instrumental analysis proposed by Rabardel

(1995). Theoretical elements for taking into account the role of the body in cognition were provided by the embodied cognition approach (Lakoff and Nuñez, 2000). Analysis of students' discussions was helped by the semiotic-cultural analysis of language (Radford et al., 2003), and an analysis of gesture, that illuminates the cognitive processes of subjects (e.g. McNeill, 1992; Edwards, 2003). Gestures, together with language, help constitute thought and they reflect the imagistic mental representation that is activated at the moment of speaking ("gesture plays a role in thinking" Alibali et al., 2000).

In this paper, we will go over the elements of the framework (embodied cognition, instrumental analysis, cultural-semiotic analysis), seeking an integration between the cognitive aspects and the instrumental one, through the gesture and linguistic analysis. We shall describe how students' conceptualization consists in a complex process that, following Radford et al. (2003) we have called objectification of knowledge: starting from their perceptions and interacting with cultural artifacts through gestures and language, students can successfully build up mathematical concepts.

The conceptual genesis of function is considered in our experiment from different points of view: as covariance, namely when one variable is changing with respect to another (Slavit, 1997), and as descriptions of time-depending variables, through the visualization of graphs (Monk and Nemirovsky, 1994). A working hypothesis is the following: the meaning of function is deeply featured in the mediation of the artifact used in the learning process.

The classroom activities described in the present paper involved working groups, class discussions, and final remarks made by the teacher. The students of the class were divided in small groups of three to four students, each group with one calculator. Each group has carried out an experiment connecting its calculator with the motion sensor.

The first set of protocols shows the graph and table interpretations made by students working in small groups. It can be seen that the students go back and forth between the graph on the screen and the description of the motion experiment, trying to connect them. Concentrating their attention on the first horizontal line in a space–time graph (corresponding to an absence of motion), they begin the objectification of the knowledge, evidenced by many deictic words, accompanied by similar gestures. This step culminates later in the conceptualization of the relationship between space and time, namely velocity, marked by an intensification of generative action terms, related with iconic gestures. The successive interpretation of the space–time data table related to the motion graph takes place with another use of the artifact calculator: scrolling the two columns both horizontally and vertically.

This approach to the instrument is very similar to the one experienced with the graph. The interpretation begins from the part of the table corresponding to stillness and is later extended to the other parts, corresponding to uniform motions; as they considered the latter ones, the students built a deeper and clearer meaning of the velocity concept. Scrolling has made visible the two variables, space and time, not just that the only variable that had been explicit so far, namely the distance. This scheme of use (the scrolling) conveys reasoning along the covariational way of interpreting a function. The motion experience and the graph representation alone had not been enough to foreground the two covariational variables in this phenomenon. Scrolling is used in different ways: first with an explorative attitude to find a pattern in the data and later to verify a conjecture and to explain it to the classmates. This marks a relevant aspect of signs in accordance with Vygotsky's theory: the evolution from the immediate intellectual processes (typical of the graph interpretation) to the operations mediated by sign (e.g. the number table with the scrolling modality). Scrolling might have transformed the calculator into a psychological tool (in the sense of Vygotsky), by which the students realized the objectification of knowledge.

The second set of protocols shows a part of a classroom discussion, in which a student describes the interpretation of the graph and the table, as it had been shared within her group. This description is very rich in gestures accompanying words. Her words and gestures can be divided into different types according to what they are referring to. They can refer to the motion or to its graphical representation. A new element of our research is the distinction among three different ways to communicate an idea relative to a body motion activity.

(1) Both language and gesture refer to a physical situation.
(2) Both language and gesture refer to a representation of the physical situation.
(3) Language refers to the physical situation, and gesture to its representation.

A gesture which refers to a physical situation (case 1), and thus simulates it, is called iconic gesture (McNeill, 1992; Radford et al., 2003) or, more precisely, iconic-physical gesture (Edwards, 2003). In the other two cases, the gesture simulates a representation of a phenomenon: so it can be considered at another cognitive level, and it can be called iconic-symbolic (Edwards, 2003). In order to distinguish the levels of symbolization referring to a graphical environment (as in this case) or to an algebraic environment (for instance through a formula), we introduce the notions of iconic-representational gesture (a gesture refers to a graphical representation of a phenomenon), and iconic-symbolic gesture (gesture refers to a symbolic representation).

The third set of protocols shows similar experiments with a sonar and a calculator, taken from another class, in which the students first use their body motion to reproduce some graphs sketched by the teacher on the blackboard, and then discuss what happened. The main element examined is the way in which gestures (which are iconic-representational, according to the previous definition) incorporate in a compressed way the features of the time law. In fact, when the speed is increasing, the hand moves faster, and when the speed decreases, the hand moves slower. Two features are thus compressed in the same gesture: the first (namely the trajectory made by the hand) expresses how the function varies (the space–time graph); the second (the speed of the hand) incorporates the velocity of the moving body. This double embodiment of information seems to be a 'natural' representation of the movement: it is a mediating tool, in order to grasp the situation in a more viable way.

Our claim is that the scientific concepts can be grasped by the students in a deep way, provided students can live and share their conceptual genesis from experiences in contexts suitably designed by the teacher. Hence our tasks are conceived so that the students:

- can have meaningful sensory-motor experiences;
- are supported in interpreting/interacting with suitable representations of the phenomena they perform;
- are encouraged to communicate with each other the meaning of the representations in group discussions.

Anticipating, predicting, preparing, and selecting are the main processes through which students can start constructing the meaning of scientific concepts, as they work with and on their perceptions and bodily motion. The roots of scientific concepts are blends, metaphors, and gestures insofar as these bodily experiences are permeated by the practices of our culture. They can accumulate and concentrate in clusters of experiences, into which students enter with their bodies (their actions), which are described through their language and gestures, and through suitable representations. These clusters of experiences allow students to deepen their different experiences and to make connections with others, in an interactive and reflective attitude. In different ways, such concrete interpreting clusters can evolve and compress into abstract scientific concepts.

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INCORPORATING EXPERIENCES OF MOTION INTO A CALCULUS CLASSROOM

In an exchange with a fellow mathematician, Jean Dieudonné, about the teaching of secondary mathematics, the mathematician René Thom (1973) asserts, "Whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (p. 204). We find support for Thom's assertion in Brousseau's (1997) analysis of mathematics teaching. Brousseau's notion of the didactical contract attempts to explain how the role of mathematics teacher is shaped by its institutional context. In his view, the role of mathematics teacher is shaped by the responsibility of teaching mathematics; justification of activity in mathematics classrooms must include an explanation for how the activity is mathematical. In that sense, pedagogy of the mathematics classroom rests on a philosophy of mathematics.

This paper is designed to stimulate reflection on relationships between teachers' beliefs about mathematics and the nature of the instruction that seems justifiable to them. In particular, we are interested in the question of relationships between teachers' beliefs about mathematics and the introduction of bodily experience and imagining into the mathematics classroom. If mathematical abstractions grow, to a large extent, out of bodily activities, and mathematical understanding and thinking are perceptuo-motor activities, what sorts of implications are there for the learning of mathematics in classrooms? We examine these implications in the context of Calculus instruction at the high school level. We are interested in interrelationships between a teacher's conceptualization of instructional goals (for example, what it means to understand motion and the role of such understandings in a Calculus course) and issues (for example, challenges in learning and teaching the mathematics of motion) and the instructional possibilities the teacher entertains as plausible or reasonable ways to reach these goals.

Our paper is centered on the examination of the teaching of one teacher, the lead author of the paper, Marty Schnepp. Marty's teaching of Calculus is not standard (for more details about his Calculus instruction, see Schnepp and Nemirovsky, 2001; Chazan and Schnepp, 2003); his teaching involves discussion of shared experiences with motion into a high school Calculus class. Rather than describe his teaching as the teaching of Calculus, Marty conceptualizes his Calculus instruction as designed to teach the mathematics of motion. In this instruction, he uses line becomes motion (LBM) technology to bring experiences of motion into the mathematical conversations he has with his students. Marty uses this device (and others) to teach his students to develop their understandings of motion, to learn to associate mathematical calculations with aspects of motion, and to see and understand velocity graphs in a disciplined (mathematical) way. In our view, this is one attempt to develop mathematics instruction that takes bodily activity seriously as a source of mathematical understandings and insights.

A central claim of our paper is that Marty could not in good conscience take this approach to the teaching of Calculus, if he felt that he was teaching students about motion, rather than the mathematics of motion. If he felt that his work with students could be divided into two components, contextual matters related to motion and the mathematics of the Calculus, then, as a mathematics teacher, he would not be able to teach as he does; he would have to leave the matters of motion to the physics teacher.

With this examination of one teacher's use of activities with motion in his Calculus class, we hope both to illustrate Thom's quote and to suggest that the introduction of bodily experience into the classroom requires much more than an argument for its potential in improving student learning. We caution that widespread use of bodily experience in classrooms will depend on teachers being able to articulate how such activity is mathematical activity that is legitimate for the mathematics classroom. Our suggestion is that bodily experience cannot be brought into the mathematics classroom without addressing teachers' views of what Schoenfeld (1990) calls "formal and informal mathematics" or what Kaput (1993) calls "the math/experience linkage." As such, it is important that curriculum developers, teacher educators, professional developers, and technologists articulate the views of mathematics and of learning and teaching which inform their commitment to integrate bodily experience into the mathematics classroom. And, beyond articulating such views, it is important to find a language to communicate with others, most crucially teachers, who may not share those points of view.

Our paper is organized in three parts. The first two sections of the first part of the paper outline a conceptualization of the challenges of learning and teaching the mathematics of motion. These two sections concretize what Marty Schnepp means by the mathematics of motion, how he sees understandings of motion as inextricably linked with mathematical analyzes of motion. Using material from textbooks, we highlight the kinds of issues he hopes to address with his students.

For example, the task of teaching students to read velocity–time graphs is in part helping students learn to imagine the types of motion that particular graphs may describe, consistent with Noble et al.'s (2004) argument that "interpreting a graph or a table entails perceiving a range of possibilities distributed across its spatial layout" (p. 2). This is a part of what they view as learning a "disciplined" way of seeing, mathematical vision. In this way of thinking, learning to interpret a velocity over time graph is inextricably linked with understanding what velocity is. One cannot read such a graph simply as the graph of a function. One must read it as a graph of a velocity function and understand the field of possible motions that such a graph might describe. As they argue:

> Such gradual mastering of visual interpretations is not achieved by the performance of isolated and self-contained sequence of steps, but by interpretive efforts that encompass ways of doing things and domains of familiarity. Experimenting with partial interpretations based on familiar contexts leads, not to a 'blind' set of procedures, but instead, to a complex way of seeing that summons explicit and tacit expectations. (Noble et al., in press).

Noble et al.'s point of view suggests that there is much more to learning to interpret velocity graphs than simply understanding that negative velocities suggest speed in an opposite direction. Such an understanding is important. But, those who read velocity graphs effectively also appreciate the ways in which such negative velocities interact quantitatively with positive velocities. And, they also appreciate that a velocity graph does not imply a particular starting place. This is a view of the learning of mathematics where mathematics and lived experience are always in contact, and not just at the beginning and the end of problem solving as is suggested by the words "applying mathematics."

In order to incorporate activity predicated on such views into the mathematics classroom, however, Brousseau's theory suggests that an argument must be made for the mathematicalness of such activity. Fortunately, for the adherents of such classroom activity, there is a range of views of mathematics, including ones that conceptualize a role for experienced motion in the development of mathematics. For this reason, in the third section of the first part of the paper, we move back from the teaching of Calculus and, using Kitcher's (1983) perspectives on mathematics as an idealizing theory, explore philosophical support for viewing motion and mathematics as inseparable. Kitcher offers an evolutionary theory of mathematical knowledge. He suggests that the origins of mathematics lie in sensory perception and the world around us. He then suggests that built on this substratum of experience mathematics grows as an idealizing theory of the world. Mathematics consists in idealized theories of ways in which we can operate on the world. To produce an idealized theory is to make some stipulations – but they are stipulations which must be appropriately related to the phenomena one is trying to idealize (Kitcher, 1983, p. 161).

Such a theory describes the world not as it is, but as it would be if accidental or complicating features were removed. "Thus we can conceive of idealization as a process in which we abandon the attempt to describe our world exactly in favor of describing a close possible world that lends itself to much simpler description" (p. 120). An important aspect of the development of such simpler descriptions is, in Kitcher's view, a desire to make such descriptions internally consistent.

> . . .It would be futile to deny that observation is one source of scientific change. The burden of the last paragraph is that observation is not the only such source. There are always "internal stresses" in scientific theory, and these provide a spur to modification of the corpus of [scientific] beliefs. . .. To oversimplify, we can think of mathematical change as a skewed case of scientific change: all the relevant observations are easily collected at the beginning of inquiry; mathematical theories develop in response to these and all the subsequent problems and modifications are theoretical. . . (p. 153).

From this point of view, rather than being surprising or inexplicable, the effectiveness of mathematics in the natural sciences is support for the idealizing nature of mathematical theory and for its origins in the world of our senses.

Returning to the classroom, such a perspective on mathematics suggests that if a Calculus teacher spends time on students' conceptions of motion, by watching or physically experiencing it in other ways, he has not abandoned mathematics for physics. Instead, by doing so, the teacher is allowing students to built an important proto-mathematical (Kitcher's word!) substratum of experience and vocabulary upon which the mathematics of motion can be built. Similarly, when the use of LBM software reverses the arrow of representation, and examines the degree to which the world of motion represents idealized mathematical theories, the idealized theory is being made accountable to the world it is meant to idealize.

The second part of our video paper is comparatively short. It gives background on Marty's instruction during the first month of a Calculus class and, in preparation for the rest of the paper, indicates the task relevant to the video clips analyzed in the third part of the paper.

The third part of the paper analyzes two clips from one session during the third week of instruction in one of Marty's Calculus classes. In the first clip, students attempt to verify a conjecture about the average velocity of one car whose displacement in time should match that of one traveling at a variable rate. While struggling with the terms "position," "distance traveled," and "displacement," they give evidence of understanding that the same velocity graph can describe motions with different starting points. In the second clip, at the instigation of a student who can imagine a trip with a velocity at each point, but an average velocity of zero, the class works on clarifying usages of the terms "speed" and "velocity." With each clip, we first connect the clip with challenges of learning the mathematics of motion as articulated early in the paper and then with challenges of the teaching of the mathematics of motion.

Our analysis focuses on students' interaction with the LBM devices, their capacity to imagine alternative motions to those that they see in front of them, and the ways in which they hold the world accountable to their mathematical theories. Throughout this analysis, there are many instances of issues related to language as a vehicle for capturing individual intuitions related to a common demonstration; there are conflicts that arise among student usages and the teacher also plans instruction purposefully to raise issues that he hopes will lead to shared understandings of accepted usages.

To reiterate, we believe that in order to teach in this way, he must be able to argue that the understandings of motion he seeks to have students develop are mathematical, not extra-mathematical. Otherwise, these issues and the mathematical insights they generate could not have a place in the mathematics classroom.

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ON FORMS OF KNOWING: THE ROLE OF BODILY ACTIVITY AND TOOLS IN MATHEMATICAL LEARNING

In this paper we characterize how bodily activity and emerging tool fluency combine in mathematical learning and how this combination suggests an alternative view on the nature of knowing. In particular, we develop the idea of knowing-with, which characterizes aspects of meaning making as it relates to developing expertise with tools.

Our analysis examines a total of eight, 90–120-minute open-ended individual interviews with three undergraduate students, Jake, Monica, and Kenny. Each student had completed three semesters of calculus and had taken or was taking differential equations. In the interviews students worked with a physical device called the "water wheel" that was first used in an earlier study with high school students to investigate their intuitive ideas of chaotic behavior (Nemirovsky, 1993; Nemirovsky and Tinker, 1993).

Our ideas in this paper are closely related to the work of Polanyi (1958) and his way of conceptualizing the use of tools. For Polanyi, an important aspect of tool use is that the user typically does not distinguish between the tool and herself. This subsidiary awareness is an act of making the tool form a part of one's own body. The significance of Polanyi's perspective for our analysis lies in the centrality of "dwelling" in the tool and the interplay between subsidiary and focal awareness. Certainly there are times when we analyze tools to decide, for example, which tool best suits a current need, but the point that Polanyi made is that when a person uses a tool, like a cane or a water wheel, she "pours herself" into the tool, becoming the tool.

We propose that the idea of knowing-with a tool is not a different type of knowing to be contrasted with the well-known distinction between knowing-how and knowing-that (Ryle, 1949), but rather a distinction independent of these other forms of knowing. The opposite of knowing-with is knowing-without. We all have had experiences of knowing-without embedded in feelings of something being alien, foreign, and belonging to others. If, say, we travel to a country where people speak a language we do not understand, our listening to their talk will elicit some perceptions in us, such as how smooth or sharp are their sounds, how profusely they gesture, or how incongruent their body language might be with the tone of their utterances. The difference between knowing-with and without is not absolute but contextual. A native speaker of, say, English, may experience a partial knowing-without English when he encounters users of a certain slang or speakers with different accents. Similarly, when one is using a tool one is familiar with – a tool that participates in our subsidiary awareness and therefore in who we are – one may drift into a context in which the tool comes to be alien and behaving strangely.

For example, in the analysis of Jake's interview we discuss how the encounter with an unexpected result prompted a fundamental shift in Jake's focal awareness. From using the water wheel as a tool, and therefore as part of his subsidiary awareness to generate graphical shapes conforming to his assumptions on basic properties of circular functions and derivatives, the water wheel ceased to be tool and became an object of observation and study, which called for a new background of ideas and properties, such as weight, forces, vibrations and so forth.

Our analysis of these students' interaction with the water wheel also highlights how bodily interaction with a tool affords a certain way of knowing a mathematical idea such as acceleration that is different from knowing that acceleration is the slope of the tangent line at a point. For example, when sensing and feeling the water wheel, Jake's way of knowing acceleration grows in him as he physically moved the wheel with his hand. By carefully attending to variations in the force he had to apply, Jake distinguishes when and why the wheel would experience a maximum acceleration. Much as knowing, say, humor or poetry with a foreign language, entails developing particular cultural subtleties and nuances that enable one to grasp humor or poetry, Jake's knowing-with the water wheel enables him to develop certain views and sensitivities to force and angular position that enable him to grasp acceleration. As dwelling in the wheel and emerging tool fluency combine, Jake knows acceleration with the wheel.

In the final section of the paper we reflect on our analysis to underscore how knowing-with (1) engages multiple and different combinations of dwelling in the tool, (2) invokes the emergence of insights and feelings that are unlikely to be fully experienced in other ways, and (3) is in the moment. We conclude this introduction by highlighting these three characteristics of knowing-with.

Dwelling in the Tool. Illustrated in the analysis are multiple and different forms of dwelling in the tool, including touching and sensing the water wheel, imagining what it is like to be the water wheel, and personifying the water wheel. These forms of dwelling in the tool recruit a variety of different perceptuo-motor, linguistic, and imaginative resources and often play out together in combination as students construct new insights into the ideas being talked about. All of these forms of dwelling in the tool connect with Polanyi's insight that when using a tool, "We pour ourselves out into them and assimilate them as parts of our own existence. We accept them existentially by dwelling in them" (Polanyi, 1958, p. 59).

For example, we characterize one of the forms of dwelling in the tool as animating or personifying the tool. The most prominent example of this form of dwelling in the water wheel is evident in the following excerpt in which Monica gives the water wheel a voice, complete with likes and dislikes. "Man, I'm no longer being pushed down. Now you want me to go back up? I don't want to go up. So I'm unhappy." Monica's dwelling in the tool in this way fostered a subsequent exchange between the water wheel being at her focal awareness and the water wheel being at her subsidiary awareness as she developed a certain view or sensitivity to the role of gravity on the motion of the water wheel's heavy spot.

Emergence of Insights. Similar to the way in which humor or poetry is difficult or impossible to translate, knowing an idea with a tool invokes the emergence of insights and feelings that are hard or difficult to fully sense in other ways. Students in this study knew much about derivatives, forces, and graphing, for example, but these competencies were not immersed in their sense-making efforts with the water wheel. For example, Monica knew that the acceleration is zero when there is a local maximum or minimum on the velocity graph, but she had to recognize anew this relationship in the motion of the wheel.

In the Moment. The third characteristic of knowing-with is that it tends to be in the moment or in the circumstances of a particular experience.

By dwelling bodily and imaginatively in these circumstances one develops sensitivities rooted in the nuances of momentary events. These sensitivities open up new possibilities for understanding in subsequent situations where the circumstances are different. For example, Jake's analysis of certain terms in the differential equations included critical aspects that he imaginatively re-constituted from his past experiences of dwelling in the water wheel. Since knowing-with tends to be attached to evanescent circumstances, this may account for why, although essential, it has often gone unnoticed in mathematics education.

There is a widespread tendency to assume that, once a concept has been formally articulated and students have at one time proven fluent with the corresponding notation, the learning of this concept has been accomplished and a degree of readiness has been achieved for more advanced ones. Through the episodes with Kenny, Monica, and Jake we illustrate that often these assumptions do not hold. Ideas such as rate of change or linearity are not encapsulated by definitions or formal derivations. Each time we grapple with a new context we need to re-encounter them in a different light or in relation to unfamiliar circumstances. We are never "done" with them. The fact that Kenny, Monica, and Jake struggled with ideas that had presumably been taught to them years earlier does not speak of a teaching failure per se, but points to a pervasive neglect of bodily and imaginative dimensions of mathematics learning.

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COORDINATION OF MULTIPLE REPRESENTATIONS AND BODY AWARENESS

The main objective of this paper is to bring back the discussion about multiple representations into the landscape of research related to function.

Much research was developed during the 1980s and early 1990s regarding ways to integrate the coordination of graphs, tables and equations into the teaching and learning of topics related to the conceptual field of function. It is our intention to relate this approach to the discussion regarding different interfaces linked to computer technologies, as well as the discussion regarding the role of the body in mathematics education. We believe that new discussion regarding body, interface and theoretical constructs such as humans-with-media shed some light over the previous discussion regarding multiple representations.

To ground the discussion, we describe three teaching experiments, in which 8th graders used CBR, a motion detector linked to a graphing calculator, and LBM, a device developed at TERC that also has a motion detector, to generate mathematical ideas related to the motion concepts associated with their movements. The second author conducted the teaching experiments with pairs of students who had been introduced to the calculators in their classroom beforehand. The students connected their body expression to the Cartesian graphs generated by the motion detector. Discussions related to geometry, kinesthetic action, graphs and functions emerged from the students' narratives. Data are presented based on the videotaping conducted throughout the teaching experiments and the analysis developed by the authors with the help of GPIMEM, the research group to which they belong. Results suggest that the use of the sensor can expand what has been labeled the epistemology of multiple representations. Framing the analysis of this episode there is a theoretical view based on the notion of humans-with-media, which emphasizes the role of media in constructing knowledge, thus breaking away from the polemical internal-external dichotomy in the epistemological debate regarding knowledge production. We present a perspective on the nature of the relationship between technology and humans which is basically opposed to the view of technology and humans as a dichotomy. Although distinctions between humans and technology will be made, we hold that the relationship between them is much closer, from their epistemological perspective, than previously proposed. Secondly, we show how the discussion about multiple representations is transformed within this perspective, and how the role of body is paramount in such a view.

Tikhomirov (1981), a Russian psychologist, proposed that computer technology will reorganize human thinking, as it extends memory and alters the class of problems that are posed for humans. He refutes views that the computer can simply be placed in juxtaposition to humans, or merely extends human capabilities; he proposes the notion of a system composed of a human and a computer. He argues that computers play a qualitatively different role than language plays in relation to humans, as is traditionally asserted in Soviet psychology. Tikhomirov, who developed his ideas prior to the advent of personal computers, was able to see ahead of his time, as he considered that the focus of our concern should not be on what was being lost as computers come into play, but rather on the nature of the problems which could be solved by this kind of system.

Within our research group, GPIMEM, we have expanded this notion of a human-computer system in two ways (Borba and Penteado, 2001). First, we propose that the notion of computer itself should be transformed to incorporate all of the different kinds of interfaces that are interconnected. Thus, devices such as calculators, graphing calculators, printers, modems, video, etc., should be incorporated into this broader notion of computer. These different interfaces, which are transforming our daily lives, no longer allow us to think in terms of a single computer as an isolated unit, as Tikhomirov did at the time, with good reason. We also believe that it is important to emphasize that, for the most part, computers have invited interaction among humans, and therefore we propose thinking in terms of several humans interacting with many different computer interfaces, as opposed to individual humans. Some members of our research group have been using the notion of humans-with-media to develop research. In such a perspective, it is the system as a unit that generates knowledge; neither humans alone, nor media alone. From this perspective, computers cannot produce knowledge by themselves, nor can humans; both medium and human must be present in a given system. In addition to the different types of technologies and humans, it is important to consider the socio-cultural-physical environment, since both humans and technology produce the environment and are shaped by it. Knowledge is not produced by "a lonely knower", nor by collectives of humans. It is produced by collectives of humans-with-media. Technology is not external or internal to us. Technology is full of humanity and humans are impregnated with technologies of intelligence.

Building on this theoretical perspective, we want to investigate how students coordinate their body motion with graphical representations. We believe that answers to this question may have implications for the teaching of function in late middle school and early high school, especially as interfaces like those used in this research become available to a larger public.

The case which is presented illustrates how technologies of information can create links between body activity and representations which are officially recognized by the mathematics academy. We want to claim that open-ended tasks with the use of sensors connected to calculators and mini-cars can add new dimensions to the discussion regarding multiple representations which was popular up to the mid 1990s. In this way, coordination of multiple representations would encompass more than just the academically recognized representations of mathematical objects such as tables, algebra and graphs. Such representations would also have to be coordinated with body actions allowing for the expression of the being. We claim that this new aspect of coordination expands the epistemology of multiple representations proposed by Confrey and Smith (1994). In our theoretical framework, knowledge is constructed by collectives that include humans and technologies of intelligence, such as orality, writing and computer technology. As mentioned before, knowledge is always produced by collectives of humans-with-media, and it is transformed as different media or humans join a given collective.

The analysis presented in the paper is focused on the theme of body movements articulated with the representations attributed to them, i.e., the graphs on the Cartesian plane represented by the software and the calculator, taking into consideration the significant contribution of the gestures, the oral communication, and the interpretation of the students' narratives regarding their experience with that activity.

The fieldwork involved teaching experiments composed of sessions carried out with six students, with the researcher interacting with one pair of students at a time, in a combination of interviews and teaching-learning situations based on several authors. The teaching experiments were conducted in a computer laboratory at UNESP – a university in Rio Claro, São Paulo, Brazil – in at least six sessions per pair during the year 1999. The sessions, lasting 60 minutes each, involved ten different activities related to the theme of movement which were carried out with the use of devices, such as CBR and LBM, which connect standard mathematics representations with movements developed by humans or things. The sessions were video-taped by a technician. The research subjects were 8th-grade students, between the ages of 13 and 15, from a public school in the city of Rio Claro, São Paulo, Brazil. Prior to the teaching experiments, they had participated in classroom activities involving calculators, computers and sensors.

We saw the format of a videopaper as particularly suitable to convey data regarding students' actions, body motion and its link to standards mathematical. We suggest that new media, such as the videopapers, not only transform knowledge, but also the way we can communicate with others who read a research paper or read-see-listen to a videopaper.

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