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REVIEW ARTICLE

Supporting Mathematical Discussions: the Roles of Comparison and Cognitive Load

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Abstract Mathematical discussions in which students compare alternative solutions to a problem can be powerful modes for students to engage and refine their misconceptions into conceptual understanding, as well as to develop understanding of the mathematics underlying common algorithms. At the same time, these discussions are challenging to lead effectively, in part because they involve complex cognitive acts of identifying structural relationships within multiple solutions and comparing between these sets of relationships. While many of the considerations in leading such discussions have been described elsewhere, we highlight the cognitive challenges for students and the core role of relational reasoning that underpins student learning from these interactions. We review the literature on children's development of relational reasoning and learning from comparisons to highlight particular challenges for students. We also review literature that suggests pedagogical practices for maximizing the likelihood that children will notice the intended relationships among solutions while minimizing overload to their cognitive resources. These practices include providing explicit comparison cues and labels, sequencing comparison before explicit instruction, using spatially aligned visual representations, and capitalizing on teacher gestures.

Keywords Relational reasoning · Cognitive load · Executive function · Mathematical discussions

Cognitive Insights into Supporting Mathematical Discussions

Mr. Silas gave his students the problem, 1 ÷ 1/3 = ?, and then asked his students to discuss and compare their solutions. His aim was to elicit several specific solution

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strategies and then lead a mathematical discussion in which students identified commonalities and contrasts between these solutions, with the goal to illuminate why the algorithm to perform rational number division by multiplying by a reciprocal is meaningful. Mr. Silas knew that if given the opportunity to grapple with the problem, students would generate a range of solutions to this problem, from the algorithmic "invert and multiply" to the alternative symbolic strategy of finding common denominators to drawing representations of wholes and one thirds. Some strategies might invoke partitive thinking (see how many times a 1/3 of a shape may fit into a whole 1 of that shape) or a measurement model (what is the full distance if 1/3 of a full distance is 1). Revealing the links between the invert and multiply algorithm, partitive and measurement representations of fraction division would allow students to understand the logic behind the algorithm and better connect integer and rational number division. Comprehending, arguing about, and becoming facile with the different solutions would also strengthen students' skills in areas focal to the Common Core State Standards for Mathematical Practices (CCSS MP) (2010).

Leading mathematical discussions of this type is a powerful pedagogical practice, yet it is challenging on many levels for teachers to conduct effectively (Ball 1993, 2008; Carpenter et al. 1999; Kazemi and Hintz 2014; Smith et al. 2009; Sherin 2002). Other research has laid out many of the reasons that this deceptively simple activity requires careful attention (see Kazemi and Hintz 2014; Humphreys and Parker 2015; Smith and Stein 2011), so we will not review them here. Rather, we provide a cognitive analysis of one aspect of effective mathematical discussions—the mental exercise of comparing and making inferences about the relationships between multiple mathematical representations. A large literature on relational reasoning has provided insights into the cognitive underpinnings of comparing and contrasting, also described as relational reasoning (see Alfieri et al. 2013; Gentner 1983; Gentner et al. 2001). This body of literature in psychology and education is not typically reviewed in the context of mathematical discussions, but we believe that considering the cognitive demands inherent in this classroom activity yields novel insights into the complexities of this act, as well as into supportive instructional practices.

In particular, we aim to highlight the cognitive demands on individual students when asked to compare multiple solutions. By cognitive demand, we mean something different from the categorization of cognitive demand often used in a task analysis (e.g., Tekkumru Kisa and Stein 2015), in which cognitive demand is used to describe the aim of the interaction, such as distinguishing between a task that requires memorization versus guided practice. Rather, we define cognitive demand by drawing on psychological models of relational reasoning, working memory, and executive function to highlight the cognitive resources required for students to build deep conceptual insights through comparing multiple student solutions or other representations (also see Paas et al. 2003).

We review the literature to explain how considering students' cognitive load is fruitful for understanding why these mathematical discussions are so difficult to execute effectively. We also review several pedagogical practices that have been shown to reduce the cognitive load and improve the likelihood that students engage in meaningful relational reasoning and show increased conceptual understanding. We focus on practices including the selection of representations to compare, sequence of instruction and comparison, effective use of visual representations, spatial structure for supporting students' attention and inferences about similarities and differences, and teachers' use of gesture to link key representations.

The Role of Cognitive Load in Comparative Thinking

Let us return to Mr. Silas' planned classroom discussion to consider the role of cognitive load in students' participation. Mr. Silas asks three students to describe their solution strategies for the class, and then, he asks the class to decide which solution they would like to try on a new problem. In order for students to benefit from this activity, they must meaningfully compare and contrast the solution strategies. Drawing on the literature on relational reasoning, this process involves several steps (see Holyoak et al. 1994). First reasoners must understand each of the representations being compared as not a static object but as a system of relationships. This means that beyond simply hearing and accepting a student's solution as stated, they must mentally represent the solution as a meaningful system—including understanding the relationship between each step in the solution or understanding how operations on one side of an equation are related to those on the other side of the equation. They must do this also for additional solutions being presented. Then, these relational systems must be aligned and key correspondences mapped together. So, rather than just watching a set of different solutions being presented, students who gain conceptually from these interactions will do cognitive work to understand how the same elements of the problem are treated in different ways by each solution method. They must also purposefully ignore irrelevant similarities or differences about the solution strategies (e.g., that one solution is written horizontally and the other vertically). Then after mapping these correspondences and noticing key similarities and differences, reasoners will draw inferences about the underlying principles of the mathematics (e.g., about division operations).

These steps to relational reasoning have been well theorized, with experimental and neuroscientific support (see Vendetti et al. 2015), yet the detailed process by which students must engage cognitively in a comparison is not often considered in classroom contexts. Much of this seems to happen easily or on the fly for students, but in part, that is because their comparative thinking is not well assessed. Students may participate in mathematical discussions in ways that only requires talk about one or another of the strategies, rather than requiring grappling with the commonalities or differences (Richland et al. 2004). Also, the commonalities or differences between the strategies may seem so clear to more knowledgeable teachers that they do not recognize the importance of clarifying these to students. At the same time, students may instead be attentive to irrelevant aspects of the solutions or just engaging with the solutions as a list of different methods without trying to cognitively participate in comparison.

Better consideration of the steps of making relational comparisons allows us to further identify the cognitive resources necessary to complete them effectively. Two aspects of reasoners' cognitive architecture seem to be particularly important. The first is working memory (WM; Baddeley and Hitch 1974; Engle et al. 1999). WM is the cognitive resource that enables humans to hold information in mind and to manipulate that information without losing it. This is information that is the subject of explicit attention and can only be retained for short periods of time while it remains the focus of attention. For example, if one student describes his solution verbally, other students in the class will need to use explicit attention to hold that information actively in mind to think about it. If they do not exert that work, it will "go in one ear and out the other," and students will not be able to retrieve that information for later consideration.

Executive function (EF) resources are shared with working memory and are also an important requirement for relational reasoning (see Diamond 2013; Miyake et al. 2000). Reasoners must exert attentional resources to control what information should go into working memory, such as switch between tasks, inhibit attention to irrelevant information, and update their working memory with new information (Miyake et al. 2000). For example, if one of the students begins describing their strategy, then realizes she is describing something different from what she actually did, and then restarts her explanation, the student listening must take that initial description out of their working memory by inhibiting any reflection they had begun based on that statement, switch to the revised explanation, and update their working memory with it. The cognitive tasks of switching, inhibiting, and updating are limited, and reasoners need to utilize these abilities effectively in order to not overload WM and EF resources.

When reasoners have to grapple with a high WM or EF load, they are less likely to draw relational comparisons and benefit from these opportunities (Waltz et al. 2000). They may make errors based on attention to distracting information (Cho et al. 2007; Richland et al. 2006), or when the load is too high, reasoners may be unable to integrate relationships between representations (Bunge et al. 2009). In classrooms, this might translate to just listening to the sequence of solutions described without having the resources available to perform the cognitive work of aligning and mapping correspondences between them, meaning the student might commit one of the described strategies to memory but would not have conducted the relational reasoning. Also due to a lack of inhibitory control, s/he might be susceptible to retaining irrelevant information or even incorrect information such as a misconception solution method.

Thus, to ensure that students engage cognitively in the comparison that lies at the heart of a mathematical discussion of solution strategies, the conditions of the interaction must both aid the reasoner to notice the necessity of finding correspondences between the solutions, as well as ensure that their cognitive resources are not overloaded to effectively make the comparison.

Development of Relational Reasoning

As children age, they move from a tendency to make comparisons by noticing similarities between appearances, or object properties, to noticing similarities between relationships—often more abstract aspects of the comparisons. For example, when presented with an incomplete analogy like *loaf of bread: slice of bread :: lemon: ?*, younger children are more likely than older children to complete the problem with a perceptual match (e.g., a yellow football). At the same time, older children are more likely than younger children to select the correct causal relational match (slice of lemon) (Goswami and Brown 1989; Rattermann and Gentner 1998).

This progression, known as the relational shift (Gentner 1983), develops in part with increased knowledge of the domain, in which they are doing the relational reasoning. Thus, when a novice is reasoning about a new domain, they will tend to attend to appearance and less to relations. This is true even in adults (Chi et al. 1981), so this aspect of development is not specific to maturation but in knowledge development (for review, see Gentner and Rattermann 1991). Thus, in a classroom context, when learners know less, they will be particularly susceptible to missing the key relationships and instead attending to appearances.

In addition, with maturation, children develop the capacity to increasingly exert EF and WM control over their attention, which allows them to become better able to attend to relations over objects, and thus make more abstract comparisons even when there are distracting appearances to mislead them (Richland et al. 2006). Children's early executive function skills predict reasoning capacity gains through adolescence (Richland and Burchinal 2013), which means that over the long term, those with higher early skills gain more over time. Again, this suggests that teachers must be cognizant of the variability in their students' needs for support during comparisons.

Children's need for support for noticing relations and disattending to distracting and irrelevant information during comparisons will be highest when children are younger and when their knowledge about a domain is lower. Teachers may not realize that while some students will notice key relationships and make the comparisons the teacher intends, others may systematically fail to benefit from these opportunities.

We next review principles that provide insight into modes for optimizing these interactions with the aim of reducing cognitive load and drawing attention to the key elements to be compared.

Sequencing Instruction and Selecting Analogs

Mr. Silas must first make decisions about the instructional sequence, such as whether to provide instruction about a key procedure and then have students solve a problem and discuss their solutions or have students compare solutions before providing a solution. A meta-analysis of studies on comparison revealed that having students compare representations before being taught a rule can be more powerful for promoting generalization and deep understanding than providing the rule first (Alfieri et al. 2013). This is quite informative and provides insight into design of how mathematical discussions may be situated within the broader instructional context. At the same time, if students do not have adequate background knowledge, they may not be able to represent the relationships within the compared solutions adequately in order to find and map the correspondences. Thus, students must have adequate knowledge to benefit from a comparison between solution strategies (Rittle-Johnson et al. 2009).

Teacher knowledge is also important in leading these discussions successfully, not only in determining an optimal order for sequencing comparisons and direct instruction but also for determining key mathematical objects to compare. Returning to the discussion led by Mr. Silas, he must select which of the class's solutions to highlight (e.g., three of the same to emphasize a solution method, one or two incorrect strategies to contrast common mistakes with a correct solution, or three alternative correct solutions to practice argumentation and improve understanding of the concept). These decisions are very much tied to the instructional aims for the interaction, whether it be to promote mathematical talk for the sake of improving flexibility in representations of mathematics or to improve mathematical mindset (Boaler and Dweck 2016) or to provide deeper insight into the mathematical content. The goal might also be to reveal the relationships between more and less efficient strategies (Carpenter et al. 2014). Teachers must have adequately deep pedagogical content knowledge in order to identify key solutions or representations to compare.

As is evident from the example of dividing by a fraction, developing deep mathematical knowledge of the mathematics curriculum is not trivial, and as Ma (1999) identified through interviews with US and Chinese elementary teachers, even the ability to flawlessly solve problems in a specific content area (e.g., division of fractions) may not index deep content knowledge expertise. In the case of dividing fractions, for example, most (though not all) interviewed US teachers were able to reach a solution when asked to divide fractions, yet few were able to describe their rationale for algorithm usage or generate meaningful examples from non-mathematical contexts to explain the underlying concept.

Thus, to lead a classroom discussion comparing solutions or multiple ways of understanding a mathematical concept requires awareness of the representations one intends to compare. As highlighted in this example, this is not a small concern, yet it is important to note that division of fractions was the area in which US teachers exhibited the lowest level of skill within Ma's examination, and experienced teachers are often able to predict the solutions or misconceptions their students may produce. In addition, the teacher must be able to anticipate the solution strategies his or her students will produce, as well as common misconceptions that may be integral to the concept.

Many teachers could benefit from resources that provide advice in regards to which analogs to present and in what order. This could include teacher networks or professional development, textbooks, or other curriculum-related materials. Textbooks that provide not only correct solutions to instructed problems but also key misconceptions or alternative strategies to elicit from students, as has historically been done in Japanese textbooks, could provide a rich resource to supplement teacher knowledge (Stigler and Hiebert 1999). Some popular commercial curricula already provide multiple solutions to problems (e.g., Everyday Mathematics) or in well-established pedagogical approaches such a Cognitively Guided Instruction (Carpenter et al. 2014). In addition, a promising intervention model could be providing productive comparisons to be made at key moments in a curriculum (see Star et al. 2014).

Misconceptions and Comparison

Perhaps the main challenge for teachers attempting to conduct mathematical discussions that treat misconceptions is to successfully incorporate students' misconceptions into a more complete understanding of the content. A famous example from Ball (1993) exemplifies this difficulty, when students discuss different answers to the problem, 1/6 + 1/6, comparing a solution with tiles that led to 2/6 and a symbol manipulation strategy with 1/12 as the solution, eventually deciding that the answer will be different depending on whether one is adding symbolic numbers or manipulatives. While such misconceptions are pervasive throughout the mathematics curriculum, leading a discussion in ways that highlight the reasons students believe the misconception while also leading them to understand why it is not accurate in this context, are extremely challenging.

Recent studies have elucidated that certain conditions can improve the likelihood that learners learn from misconceptions, including high prior knowledge (Große and Renkl 2006; Booth et al. 2013), explicit cues (Booth et al. 2013), visual alignment (Begolli and Richland 2016), and adequate available WM/EF resources (Begolli et al. 2015).

Many of these strategies converge on the notion of making key relational comparisons very explicit. For example, Große and Renkl (2006) found that only students with high prior knowledge were able to benefit from incorrect examples. However, studies by Booth and colleagues (Booth et al. 2015a,b; Booth et al. 2013) show that low-performing students benefit the most from incorrect solutions. A potential explanation for these disparate results is that in Große and Renkl (2006), the differences between high and low performers disappeared when the misconceptions were explicitly highlighted. In contrast, in the studies administered by Booth and colleagues, the misconceptions were always explicitly highlighted. Thus, while prior knowledge may interact with the use of misconceptions in instruction, a cognitive support, such as providing explicit cues to identify the incorrect strategy, may lead to benefits for all students (see Durkin and Rittle-Johnson 2012) and perhaps especially those with low prior knowledge (Booth and Cooper et al. 2015; Booth and Oyer et al. 2015; Booth et al. 2013).

Provide Explicit Cues to Compare

While a teacher may plan a discussion carefully, selecting ideal analogs and spending valuable class time going through two or three in sequence, students may not be aware that they are intended to compare these problems. As the scores of studies over the decades (see Alfieri et al. 2013; Bransford et al. 2000; Gick and Holyoak 1980, 1983), as well as teachers themselves (e.g., survey of algebra teachers reported by the National Mathematics Advisory Panel 2008) have reported, students regularly fail to notice that they would benefit from comparing a better known case or context with a new problem to solve. This leads to failures of transfer and lack of schema formation from making these broad comparisons.

The challenge of noticing the relevance of a comparison, and successfully making the key structure mapping, is that learners often require high amounts of scaffolding to attend to the crucial relationships, determine which relations align across contexts, and make the relevant inferences. They require explicit support for all of these elements, such that simply providing the cue to compare by either making that statement, or even simply discussing them in turn, is not adequate.

One way to facilitate an instructional comparison is by organizing problem solutions into hierarchies of smaller sub-goals, rather than linear steps. Doing so highlights the structure of the solutions and makes plain the correspondences across them. Students who learn from comparisons in which the sub-goal structure of the problem solutions are explicitly identified, for example by labeling the sub-goals (Catrambone 1996) or providing additional explanation for each sub-goal (Catrambone and Holyoak 1990), are better able to transfer their knowledge to new problems. Organizing instruction on problem solutions around sub-goals makes it easier for students to see the conceptual underpinnings of problem solutions (Atkinson et al. 2003) and reduces the cognitive load during instruction (Gerjets et al. 2004).

Labeling in general provides a powerful cue to compare examples and to convey relational structure. When two problems, solutions, sub-goals, etc., are labeled the same way, learners are implicitly invited to compare them (Namy and Gentner 2002), making common labels a straightforward way for students to know which examples should be compared to yield fruitful insights. Familiar labels can also highlight systems of relations and help learners identify and represent the relevant structure (Gentner and Loewenstein 2002; Loewenstein and Gentner 2005), making it easier to notice and carry out key comparisons.

Make Compared Representations Visible Simultaneously

Teachers leading a classroom discussion of solution strategies must decide whether to have their students present their solutions orally or visually, and also whether to have one solution visible to the class at a time, or whether to leave all the discussed solutions visible simultaneously. A growing literature suggests that the perceptual cues within a comparison can be important to maximizing the likelihood that reasoners make the intended comparison (Alfieri et al. 2013). In particular, making solutions visible, and leaving them visible throughout the discussion, has been shown to be most effective at promoting relational reasoning (Gadgil et al. 2012; Richland and McDonough 2010). Experiments with young children (Christie and Gentner 2010) and late elementary/middle school mathematics students show this pattern (see Begolli and Richland 2015; Rittle-Johnson and Star 2007). For example, Matlen and colleagues (2011) found that elementary-age students were more likely to learn and retain geoscience concepts when text passages describing the concepts were accompanied by visual representations of both the source and target than of just the target. Simultaneous presentation prompted students to compare the two domains and reduced cognitive effort of having to recall each representation. Similarly, mathematics studies have shown learning gains when source and target representations are displayed simultaneously versus sequentially, leading to gains in procedural knowledge, flexibility, and conceptual understanding (e.g., Christie and Gentner 2010; Begolli and Richland 2015; Rittle-Johnson and Star 2007).

The role of simultaneous visual representations may be particularly important when one of the solutions being compared is a misconception. Richland et al. (2004) provided students with two solutions, in which the first was a misconception and the second was a correct solution, and participants watched videotapes in which the verbal instruction was identical, but the solutions were either only heard orally, were seen one at a time sequentially, or were visible simultaneously. The least learning was produced when participants viewed the solutions sequentially, with errors revealing that these students used the misconception more often at an immediate and delayed posttest. In contrast, students in the visible simultaneous condition learned the most on both procedural and conceptual measures. Listening to the solutions verbally only was in the middle, suggesting that this was less effective than viewing the representations together, yet it led to less reification of the misconception than viewing it sequentially.

A cognitive interpretation of these results is that viewing the misconception made it a stronger representation that needed to be inhibited, raising the demands on the EF system. Making the two solutions visible simultaneously at the same time reduced WM demands, meaning that this should reduce the overall cognitive load and make it more possible for students to engage with the full comparison. Having no visual information would likely impose burden on the WM system, but less on the EF system, which would explain the performance between the two other conditions.

Use Spatial Organization to Highlight Key Relations

Another strategy for reducing the attentional demand on learners is to harness the opportunities available through spatial organization. Whenever two representations are compared, there are many similarities and differences that could be attended to. Learning is enhanced when the spatial organization of the representations highlights the alignments. For example, Kurtz and Gentner (2013) found that participants were faster and more accurate at detecting differences in skeletal structures when two skeletal images were presented in the same orientation relative to when they were presented in a symmetrical orientation. As shown in Fig. 1, when the figures being compared were aligning so that the commonalities and differences were in the same spatial position, learners were better able to notice them. Importantly, results were not simply for speed of noticing but also for accuracy. Matlen et al. (2014) found that placing images in direct spatial alignment, such that students could readily make the correct correspondences, optimized the speed and accuracy with which same/different relations were processed. This

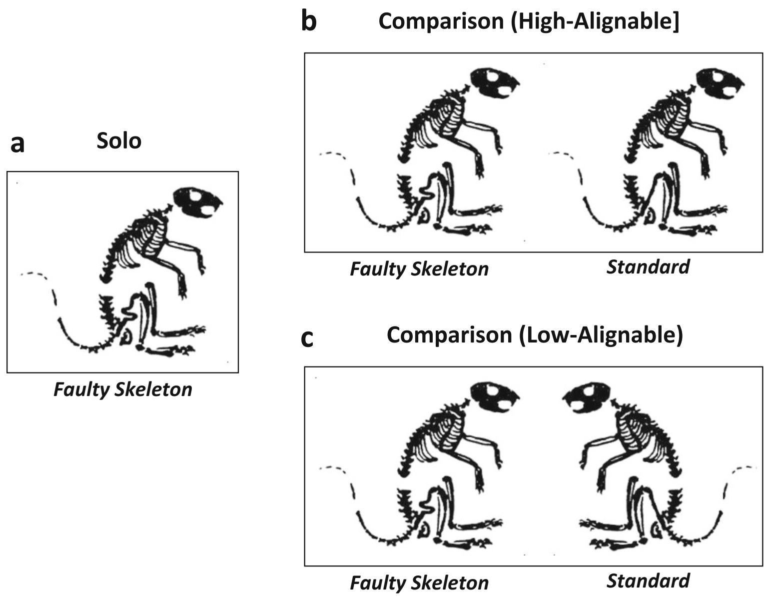

Fig. 1 **a** Solo faulty skeleton and **b**, **c** comparison of faulty skeleton and standard (high alignable (**b**) and low alignable (**c**))

suggests that teachers could benefit from not simply placing solutions on the board but from doing so in a planful way that could highlight the key correspondences or contrasting information.

Use Linking Gestures to Move Between Spatial Representations

Gestures are another modality for helping to make the importance of a comparison clear to learners. Linking gestures are hand movements that move between two (or more) representations that are being compared, sometimes highlighting the specific alignments between these representations and other times simply providing support for noticing the relevance of one representation to another (Alibali et al. 1997, 2014; Alibali and Nathan 2007; Richland 2015). For instance, Richland and McDonough (2010) provided undergraduates with examples of permutation and combination problems that incorporated visual cueing, such as gesturing back and forth between problems and allowing the examples to remain in full view, versus comparisons that did not incorporate visual cueing. Students who studied the problems with visual cueing were more likely to succeed on difficult transfer problems. Linking gesture use is correlated with high mathematics learning in students (Richland 2015), and teacher gesture is well known to improve learning outcomes (see Goldin-Meadow 2003).

Summary

Thus, overall, mathematical discussions provide a powerful opportunity for students to engage in relational reasoning and comparison between solution strategies. Mathematical discussions will have many aims, and some may be to simply demonstrate that students can solve solutions multiple ways or that they can participate in talk about mathematics. Other instructional goals, however, will aim to allow students to foster their understanding by drawing similarities or differences from multiple solutions. These goals will depend on teacher knowledge, which will also guide selection of solutions for comparison, although this can also be supported by curriculum materials.

When leading a discussion with the aim to produce deeper understanding of the key construct, such as division of fractions in the example introduced here, considering the cognition involved in relational reasoning provides new insights into strategies for best leading these mathematical discussions. Considering student's WM and EF resources while teachers invite students to discuss and compare their fellow students' solution strategies is an important component of ensuring that all students successfully make those comparisons. In fact, teachers may not consider mathematical discussions to be comparisons, although in fact, that lies at the heart of many of these interactions. The literature on relational reasoning is informative, suggesting particular strategies for reducing that cognitive load, such as using visual-spatial and interactional cues to support students in these efforts, as well as appropriately sequencing. This may include making solutions visible simultaneously to facilitate noticing and mapping correspondences, creating spatial alignments between these representations to further highlight these correspondences, and using linking gestures to highlight the relevant mappings.

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

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