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Implementation of inference-based diagnosis: computing delay bound and ambiguity levels

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Abstract Inference-based decentralized diagnosis is a framework introduced in the authors' former work, where inferencing over the ambiguities of the self and the others is used to issue diagnosis decisions. The implementation of the framework requires the online computation of the ambiguity levels by each of the local decision makers, following each of their local observations. This in turn requires knowing the delay bound of diagnosis, which needs to be computed offline, prior to the online monitoring for fault detection. The paper presents the offline computation of the delay bound of diagnosis, along with a certain set of languages, which together aid the online computation of the ambiguity levels.

Keywords Discrete event system · Decentralized diagnosis · Inference-based ambiguity management · Ambiguity level · Delay bound

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1 Introduction

There exists a long history of research on fault diagnosis of discrete event systems (DESs) (see for example Sampath et al. 1995; Zaytoon and Lafortune 2013). The notion of diagnosability requires detection of any fault within a uniformly bounded delay, which in turn requires that within that bounded delay, the post fault-behaviors generate observations that are distinguished from the pre-fault ones (Sampath et al. 1995).

For large, physically distributed systems, decentralized diagnosis is employed, where multiple local diagnosers that rely on their own subsets of accessible sensors make local diagnosis decisions that are pooled together to deduce a global one. See for example (Debouk et al. 2000; Su and Wonham 2005; Qiu and Kumar 2006, 2008; Wang et al. 2007, 2010, 2011; Qiu et al. 2009; Kumar and Takai 2009; Schmidt 2010; Takai and Ushio 2012; Cassez 2012; Chakib and Khoumsi 2012; Yamamoto and Takai 2014, 2015; Yin and Lafortune 2015; Yokota et al. 2017). The notion of codiagnosability that captures the property that a fault can be detected by at least one of the local diagnosers within a uniformly bounded delay was formally introduced in Qiu and Kumar (2006). This scheme, where at least one diagnoser issues a failure decision unambiguously, is "disjunctive" in nature. In contrast, a dual "conjunctive" scheme, which is incomparable to the disjunctive one, was later proposed in Wang et al. (2007), where a nonfailure decision is issued by a diagnoser when it is unambiguous about it and a fault is detected when none of the diagnosers issue a nonfailure decision.

In the disjunctive and conjunctive schemes mentioned above, each diagnoser makes a local diagnosis decision on the basis of the own knowledge. The process of utilizing the own knowledge as well as the inferred others' knowledge for the sake of decision-making was referred to as "inferencing" (where local diagnosers know each other's observation masks). A general framework for *inference-based decentralized decision-making* was introduced by the authors of the present paper in Kumar and Takai (2007) and adopted to the cases of diagnosis in Kumar and Takai (2009) and prognosis in Takai and Kumar (2011). In the inference-based setting, each diagnoser uses not only its own knowledge of the system behaviors, but also the inference about the possible knowledge of the system behaviors of other diagnosers to arrive at its own local decision. The "winning" local decision (namely, the one needing the least levels of inferencing) is set as the global decision. While the general framework of Kumar and Takai (2009) supports arbitrary levels of inferencing, the work of Kumar and Takai (2009) employs only the disjunctive scheme. Our recent work Takai and Kumar (2017) supports both the disjunctive and conjunctive schemes, along with multiple levels of inferencing. To achieve such generality over (Kumar and Takai 2009), the main new insight lies in identifying the *seed pair of failure and nonfailure behaviors* that must be disambiguated through inferencing. In Kumar and Takai (2009), this seed pair was simply taken to be all failure versus all nonfailure behaviors. However, this is unnecessarily strong, and instead, only those failure behaviors that have allowed the execution of post-fault behaviors to a certain minimum number of steps (equaling the uniformly bounded delay of detection) must be disambiguated from the nonfailure behaviors to capture both disjunctive and conjunctive decision-making. The inference-based framework of Takai and Kumar (2017) is general enough to subsume the disjunctive and conjunctive frameworks of Qiu and Kumar (2006) and Wang et al. (2007), respectively, which do not involve inferencing, and the conditional disjunctive and conditional conjunctive frameworks of Wang et al. (2007), which involve a single-level of inferencing. Furthermore, as the levels of inferencing are increased, a larger class of diagnosable systems are obtained (Takai and Kumar 2017).

Local supervisors (respectively, diagnosers) with conditional decisions involving a single-level inferencing were developed in Yoo and Lafortune (2007) (respectively, Yokota et al. 2017). The current paper presents the online implementation of the inference-based decentralized diagnosers in the generalized framework of Takai and Kumar (2017) supporting both conjunctive and disjunctive decision-making (different from the earlier version (Kumar and Takai 2009), which used only disjunctive decision-making, omitting the conjunctive one). The implementation requires the online computation of the diagnosis decisions and the associated ambiguity levels by each of the local decision makers, following each of their local observations. This in turn requires knowing the delay bound of diagnosis, which must be computed offline, prior to the online monitoring for fault detection. The paper presents the computation of the delay bound of diagnosis, and also a certain set of languages, which together aid the recursive online computation of the diagnosis decisions. Complexity analysis for the required offline as well as online computations is provided. The paper also shows that as the number of inferencing levels increases, the delay bound of diagnosis decreases and a larger class of systems become diagnosable. So there exists a tradeoff between the complexity versus the ability and delay of diagnosis.

Note that knowing the delay bound is also important to execute mitigation actions in a timely manner, and is a figure of merit of a diagnosis scheme. Algorithms for computing the delay bound are reported in the literature for various earlier schemes: disjunctive (Qiu and Kumar 2006), conjunctive (Yamamoto and Takai 2014), and conditional disjunctive and conjunctive (Yokota et al. 2017). The results on the computation of the delay bound were first reported at the authors' conference papers (Takai and Kumar 2016) but without proofs. This paper provides additional results on the online computation of the ambiguity levels and additionally includes all the correctness proofs, and new examples.

2 Notation and preliminaries

A deterministic automaton is a five-tuple $G = (Q, \Sigma, \delta, q_0, Q_{mark})$, where Q is the set of states, Σ is the finite set of events, $\delta : Q \times \Sigma \to Q$ is the partial transition function, $q_0 \in Q$ is the initial state, and $Q_{mark} \subseteq Q$ is the set of marked states.[1] Let Σ^* be the set of all finite traces of elements of Σ, including the empty trace ε. The function δ can be extended to $\delta : Q \times \Sigma^* \to Q$ in the usual manner. The generated and marked languages of G, denoted by $L(G)$ and $L_m(G)$, respectively, are defined as $L(G) = \{s \in \Sigma^* \mid \delta(q_0, s)!\}$ and $L_m(G) = \{s \in \Sigma^* \mid \delta(q_0, s) \in Q_{mark}\}$, where $\delta(q, s)!$ denotes that $\delta(q, s)$ is defined for each $q \in Q$ and each $s \in \Sigma^*$.

Let $K \subseteq \Sigma^*$ be a language. The set of all prefixes of traces in K is denoted by $pr(K)$. If $K = pr(K)$, then K is said to be (prefix-)closed. A closed language K is said to be *deadlock-free* if, for any $s \in K$, $\{s\}\Sigma \cap K \neq \emptyset$. For each trace $s \in \Sigma^*$, $|s|$ denotes its length. For any $m \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all nonnegative integers, let $\Sigma^{\geq m} := \{s \in \Sigma^* \mid |s| \geq m\}$ and $\Sigma^{\leq m} := \{s \in \Sigma^* \mid |s| \leq m\}$.

Let $I = \{1, 2, \ldots, n\}$ denote the index set of local diagnosers that perform the task of diagnosis. We assume that the limited sensing capabilities of the ith local diagnoser D_i ($i \in I$) can be represented by the local observation mask, $M_i : \Sigma \to \Delta_i \cup \{\varepsilon\}$, where Δ_i is the set of locally observed symbols. An event $\sigma \in \Sigma$ with $M_i(\sigma) = \varepsilon$ is said to

[1] In this paper, an automaton is deterministic, unless otherwise stated.

be unobservable under M_i. The local observation mask M_i is extended to $M_i : \Sigma^* \to \Delta_i^*$ in the usual manner. Two traces $s, s' \in \Sigma^*$ with $M_i(s) = M_i(s')$ are said to be M_i-indistinguishable. In addition, the inverse map of M_i, denoted by $M_i^{-1} : \Delta_i^* \to 2^{\Sigma^*}$, is defined as $M_i^{-1}(t) = \{s \in \Sigma^* \mid M_i(s) = t\}$ for each $t \in \Delta_i^*$. For any languages $L \subseteq \Sigma^*$ and $L' \subseteq \Delta_i^*$, $M_i(L) \subseteq \Delta_i^*$ and $M_i^{-1}(L') \subseteq \Sigma^*$ are defined as $M_i(L) = \{M_i(s) \in \Delta_i^* \mid s \in L\}$ and $M_i^{-1}(L') = \{s \in \Sigma^* \mid M_i(s) \in L'\}$, respectively. For any subset $Q' \subseteq Q$ of the state set Q of G, the *unobservable reach* set $UR_{G,i}(Q') \in 2^Q$ is defined as

$$UR_{G,i}(Q') = \{q' \in Q \mid \exists q \in Q', \exists s \in M_i^{-1}(\varepsilon) \cap L(G) : \delta(q, s) = q'\}. \tag{1}$$

Let $L \neq \emptyset$ be a closed language that represents the generated language of a plant (system to be diagnosed) modeled as a finite automaton $G = (Q, \Sigma, \delta, q_0, Q)$, and $K \subseteq L$ be a nonempty closed language that represents a nonfailure specification. Traces in $L - K$ are considered as failure traces and the task of diagnosis is to determine the execution of any trace in $L - K$ within an additional bounded number of system executions. Without loss of generality, the plant language L can be taken to be deadlock-free (Kumar and Takai 2009).

In what follows, we need a finite acceptor of the post-fault traces in which a fault occurred at least m steps in the past, i.e., traces in $F_0(m) := L \cap (L - K)\Sigma^{\geq m}$. When m corresponds to the delay bound of diagnosis, $F_0(m)$ corresponds to the set of traces for which a failure decision can be issued. Also, when $m = 0$, $F_0(m)$ simply corresponds to the set of all failure traces.

To construct a finite acceptor of the language $F_0(m) = L \cap (L - K)\Sigma^{\geq m}$ for any $m \in \mathbb{N}$, we augment a finite generator $G_K = (Q_K, \Sigma, \delta_K, q_{K,0}, Q_K)$ of the nonfailure specification language $K \subseteq L$ by adding $m + 1$ dump states $d_0, d_1, \ldots, d_m \notin Q_K$. Formally, the augmented automaton is defined as

$$\tilde{G}_{K_m} = (\tilde{Q}_{K_m}, \Sigma, \tilde{\delta}_{K_m}, q_{K,0}, \{d_m\}),$$

where $\tilde{Q}_{K_m} = Q_K \cup \{d_j \mid j \in \{0, 1, \ldots, m\}\}$, and the state transition function $\tilde{\delta}_{K_m} : \tilde{Q}_{K_m} \times \Sigma \to \tilde{Q}_{K_m}$ is defined as follows (Yamamoto and Takai 2015): For each $\tilde{q}_{K_m} \in \tilde{Q}_{K_m}$ and each $\sigma \in \Sigma$,

$$\tilde{\delta}_{K_m}(\tilde{q}_{K_m}, \sigma) = \begin{cases} \delta_K(\tilde{q}_{K_m}, \sigma), & \text{if } \tilde{q}_{K_m} \in Q_K \wedge \delta_K(\tilde{q}_{K_m}, \sigma)! \\ d_0, & \text{if } \tilde{q}_{K_m} \in Q_K \wedge \neg\delta_K(\tilde{q}_{K_m}, \sigma)! \\ d_{l+1}, & \text{if } \tilde{q}_K = d_l (m \geq 1 \wedge 0 \leq l \leq m - 1) \\ d_m, & \text{if } \tilde{q}_{K_m} = d_m, \end{cases}$$

where $\neg\delta_K(\tilde{q}_{K_m}, \sigma)!$ denotes the negation of $\delta_K(\tilde{q}_{K_m}, \sigma)!$ It follows from the definition of $\tilde{G}_{K_m}$ that $L(\tilde{G}_{K_m}) = \Sigma^*$ and $L_m(\tilde{G}_{K_m}) = \Sigma^* - K\Sigma^{\leq m}$.

We construct the synchronous product $G \parallel \tilde{G}_{K_m}$ (Kumar and Garg 1995) of the finite plant model $G = (Q, \Sigma, \delta, q_0, Q)$ and $\tilde{G}_{K_m}$, which is denoted by

$$G \parallel \tilde{G}_{K_m} = (Q \times \tilde{Q}_{K_m}, \Sigma, \xi_m, (q_0, q_{K,0}), Q \times \{d_m\}).$$

Then we have $L(G \parallel \tilde{G}_{K_m}) = L(G) \cap L(\tilde{G}_{K_m}) = L$ and $L_m(G \parallel \tilde{G}_{K_m}) = L_m(G) \cap L_m(\tilde{G}_{K_m}) = L \cap (L - K)\Sigma^{\geq m} = F_0(m)$, i.e., $G \parallel \tilde{G}_{K_m}$ can be used as a finite accepter of $F_0(m)$. For simplicity of notation, in the case of $m = 0$, we drop the subscript m in the notation, i.e., $\tilde{G}_{K_0} = (\tilde{Q}_{K_0}, \Sigma, \tilde{\delta}_{K_0}, q_{K,0}, \{d_0\})$ and $G \parallel \tilde{G}_{K_0} = (Q \times \tilde{Q}_{K_0}, \Sigma, \xi_0, (q_0, q_{K,0}), Q \times \{d_0\})$ are also denoted by $\tilde{G}_K = (\tilde{Q}_K, \Sigma, \tilde{\delta}_K, q_{K,0}, \{d\})$ and $G \parallel \tilde{G}_K = (Q \times \tilde{Q}_K, \Sigma, \xi, (q_0, q_{K,0}), Q \times \{d\})$, respectively.

3 Inference-based diagnosis framework

The material in this section summarizes our earlier work reported in Takai and Kumar (2017) that introduced the inference-based diagnosis framework, along with the notion of N-inference diagnosability and its verification test.

3.1 Existence condition of N-inferring diagnosers

Let $C = \{0, 1, \phi\}$ be the set of diagnosis decisions, where "0" represents a nonfailure decision, "1" represents a failure decision, and "ϕ" represents an unsure decision. Each inference-based local diagnoser D_i is defined as a map $D_i : M_i(L) \to C \times \mathbb{N}$ (Kumar and Takai 2009), where for each $s \in L$, $D_i(M_i(s)) = (c_i(M_i(s)), n_i(M_i(s)))$. Here $c_i(M_i(s)) \in C$ denotes the diagnosis decision of D_i following an observation $M_i(s) \in M_i(L)$, and $n_i(M_i(s)) \in \mathbb{N}$ denotes the ambiguity level of the diagnosis decision of D_i. Let $n(s)$ be the minimum ambiguity level of local decisions (Kumar and Takai 2009), i.e., $n(s) := \min_{i \in I} n_i(M_i(s))$.

The decentralized diagnoser $\{D_i\}_{i \in I}$ that consists of local diagnosers D_i $(i \in I)$ issues the global diagnosis decision. Formally, $\{D_i\}_{i \in I}$ is defined as a map $\{D_i\}_{i \in I} : L \to C$. For each $s \in L$, the diagnosis decision $\{D_i\}_{i \in I}(s)$ is given as follows (Kumar and Takai 2009):

$$\{D_i\}_{i \in I}(s) = \begin{cases} 1, & \text{if } \forall i \in I : n_i(M_i(s)) = n(s) \Rightarrow c_i(M_i(s)) = 1 \\ 0, & \text{if } \forall i \in I : n_i(M_i(s)) = n(s) \Rightarrow c_i(M_i(s)) = 0 \\ \phi, & \text{otherwise.} \end{cases} \tag{2}$$

The global diagnosis decision is taken to be the same as a local diagnosis decision possessing the minimum level of ambiguity.

A useful notion of a decentralized diagnoser is the boundedness of the ambiguity level of its decisions. Let $N \in \mathbb{N}$ be a given nonnegative integer. A decentralized diagnoser $\{D_i\}_{i \in I} : L \to C$ is said to be *N-inferring* (Takai and Kumar 2017) if the following two conditions hold:

1. Either

$$\forall s \in L - K : \{D_i\}_{i \in I}(s) = 1 \Rightarrow n(s) \le N, \tag{3}$$

or

$$\forall s \in K : \{D_i\}_{i \in I}(s) \neq 1 \Rightarrow n(s) \le N. \tag{4}$$

2. There exists $m \in \mathbb{N}$ such that

$$\forall s \in (L \cap (L - K)\Sigma^{\ge m}) \cup K : n(s) \le N \Rightarrow \{D_i\}_{i \in I}(s) \neq \phi. \tag{5}$$

Given a plant language L, a nonfailure specification language $K \subseteq L$, and a nonnegative integer $m \in \mathbb{N}$, we inductively define a monotonically decreasing sequence $\{(F_k(m), H_k(m))\}_{k \ge 0}$ of language pairs as follows (Takai and Kumar 2017):

- Base step:

$$F_0(m) := L \cap (L - K)\Sigma^{\ge m}, \; H_0(m) := K.$$

- Induction step:

$$F_{k+1}(m) := F_k(m) \cap \left(\bigcap_{i \in I} M_i^{-1} M_i(H_k(m)) \right),$$

$$H_{k+1}(m) := H_k(m) \cap \left(\bigcap_{i \in I} M_i^{-1} M_i(F_k(m)) \right).$$

In the base step, $F_0(m) = L \cap (L - K)\Sigma^{\geq m}$ is the set of failure traces for which at least m events occurred after the occurrence of the failure, and $H_0(m) = K$ is the set of nonfailure traces. In the induction step, $F_{k+1}(m)$ (respectively, $H_{k+1}(m)$) is a sublanguage of $F_k(m)$ (respectively, $H_k(m)$) consisting of traces for which there exists an M_i-indistinguishable trace in $H_k(m)$ (respectively, $F_k(m)$) for each $i \in I$.

Then we have the following definition of N-inference diagnosability.

Definition 1 (Takai and Kumar 2017) The pair (L, K) of regular languages is said to be *N-inference diagnosable* if there exists $m \in \mathbb{N}$ such that $F_{N+1}(m) = \emptyset$ or $H_{N+1}(m) = \emptyset$.

Remark 1 A relation among various notions of diagnosability for decentralized diagnosis is shown in Fig. 1, and shows that the framework analyzed here is most general.

Example 1 We consider a plant modeled by the finite automaton G shown in Fig. 2a. Let $\Delta_1 = \{a, a', c, d, e\}$, $\Delta_2 = \{b, b', c, d, e\}$, and

$$M_1(\sigma) = \begin{cases} a, & \text{if } \sigma \in \{a_1, a_2\} \\ a', & \text{if } \sigma \in \{a'_1, a'_2\} \\ \sigma, & \text{if } \sigma \in \{c, d, e\} \\ \varepsilon, & \text{otherwise,} \end{cases}$$

$$M_2(\sigma) = \begin{cases} b, & \text{if } \sigma \in \{b_1, b_2\} \\ b', & \text{if } \sigma \in \{b'_1, b'_2\} \\ \sigma, & \text{if } \sigma \in \{c, d, e\} \\ \varepsilon, & \text{otherwise.} \end{cases}$$

In addition, let $K \subseteq L$ be a closed regular language generated by the finite automaton G_K shown in Fig. 2b. In this example, the failure is modeled by the occurrence of the event f.

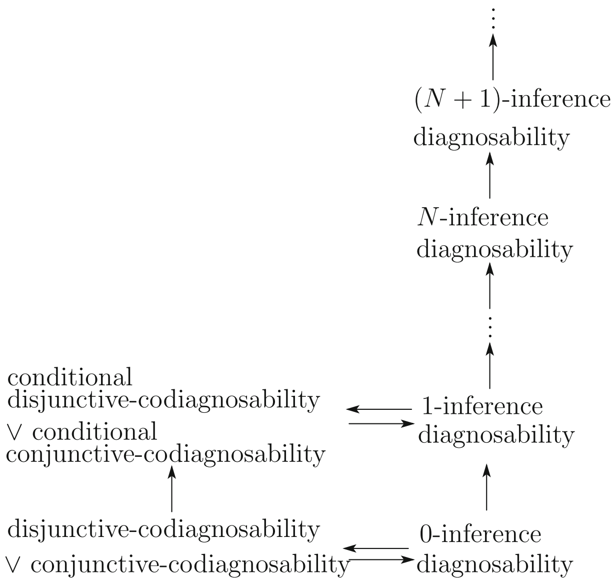

Fig. 1 A relation among notions of diagnosability for decentralized diagnosis

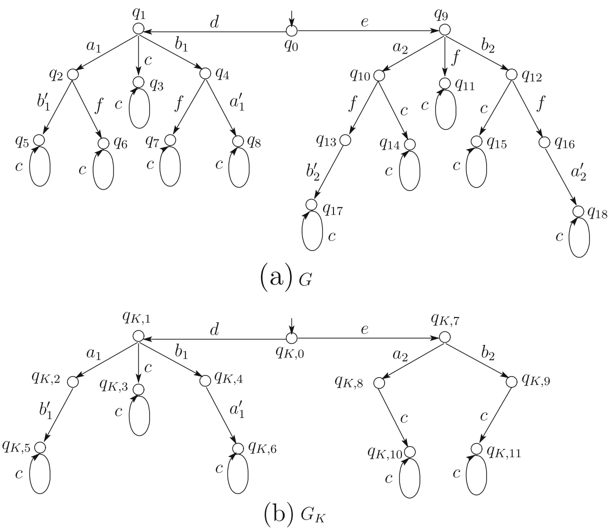

Fig. 2 Automata G and G_K of Example 1

We show that (L, K) is 2-inference diagnosable but not 1-inference diagnosable. We consider any $m \in \mathbb{N}$ such that $m \geq 1$. Initially, we have

$$\begin{aligned} F_0(m) &= d(a_1 f c^m c^* + b_1 f c^m c^*) + e(f c^m c^* + a_2 f b_2' c^{m-1} c^* + b_2 f a_2' c^{m-1} c^*), \\ H_0(m) &= pr(d(c^+ + a_1 b_1' c^+ + b_1 a_1' c^+) + e(a_2 c^+ + b_2 c^+)). \end{aligned}$$

Since

$$\begin{aligned} M_1(F_0(m)) &= d(ac^m c^* + c^m c^*) + e(c^m c^* + ac^{m-1} c^* + a' c^{m-1} c^*), \\ M_2(F_0(m)) &= d(c^m c^* + bc^m c^*) + e(c^m c^* + b' c^{m-1} c^* + bc^{m-1} c^*), \\ M_1(H_0(m)) &= pr(d(c^+ + ac^+ + a' c^+) + e(ac^+ + c^+)), \\ M_2(H_0(m)) &= pr(d(c^+ + b' c^+ + bc^+) + e(c^+ + bc^+)), \end{aligned}$$

we have

$$\begin{aligned} F_1(m) &= F_0(m) \cap \left(\bigcap_{i \in I} M_i^{-1} M_i(H_0(m)) \right) \\ &= d(a_1 f c^m c^* + b_1 f c^m c^*) + ef c^m c^*, \\ H_1(m) &= H_0(m) \cap \left(\bigcap_{i \in I} M_i^{-1} M_i(F_0(m)) \right) \\ &= dc^m c^* + e(a_2 c^m c^* + b_2 c^m c^*). \end{aligned}$$

Moreover, by the iterative computation, we obtain

$$\begin{aligned} F_2(m) &= ef c^m c^*, \\ H_2(m) &= dc^m c^*, \end{aligned}$$

and finally, we have $F_3(m) = H_3(m) = \emptyset$, which implies that (L, K) is 2-inference diagnosable. However, since $F_2(m) \neq \emptyset$ and $H_2(m) \neq \emptyset$ for any $m \in \mathbb{N}$, it is not 1-inference diagnosable.

The following theorem shows that N-inference diagnosability is a necessary and sufficient condition for the existence of an N-inferring decentralized diagnoser with no missed and incorrect detections.

Theorem 1 (Takai and Kumar 2017) *There exists an N-inferring decentralized diagnoser $\{D_i\}_{i\in I} : L \to C$ that satisfies*

$$\exists m \in \mathbb{N}, \forall s \in L \cap (L-K)\Sigma^{\geq m} : \{D_i\}_{i\in I}(s) = 1, \tag{6}$$

$$\forall s \in K : \{D_i\}_{i\in I}(s) \neq 1 \tag{7}$$

if and only if the pair (L, K) of regular languages is N-inference diagnosable.

3.2 Online computation of local diagnosis decisions and ambiguity levels

For the pair (L, K) of regular languages that is N-inference diagnosable (so that there exists $m \in \mathbb{N}$ such that $F_{N+1}(m) = \emptyset$ or $H_{N+1}(m) = \emptyset$), a local diagnoser can compute its diagnosis decision and associate a level of ambiguity as follows: For each $s \in L$, the ith local diagnoser D_i computes

$$n_i^f(M_i(s)) := \min\{k \in \mathbb{N} \mid M_i(s) \notin M_i(H_k(m))\}, \tag{8}$$

$$n_i^h(M_i(s)) := \min\{k \in \mathbb{N} \mid M_i(s) \notin M_i(F_k(m))\}. \tag{9}$$

Here $n_i^f(M_i(s))$ represents the ambiguity level of a failure decision contemplated by the ith diagnoser following the observation $M_i(s)$. Similarly, $n_i^h(M_i(s))$ represents the ambiguity level of a nonfailure decision contemplated by the ith diagnoser following the observation $M_i(s)$. Since $F_{N+1}(m) = \emptyset$ and $H_{N+1}(m) = \emptyset$ imply $H_{N+2}(m) = \emptyset$ and $F_{N+2}(m) = \emptyset$, respectively, both $n_i^f(M_i(s))$ and $n_i^h(M_i(s))$ are bounded above by $N+2$.

For a local diagnoser $D_i : M_i(L) \to C \times \mathbb{N}$, its diagnosis decision and ambiguity level following an observation $M_i(s) \in M_i(L)$, i.e.,

$$D_i(M_i(s)) = (c_i(M_i(s)), n_i(M_i(s))),$$

is determined as follows (Kumar and Takai 2009):

$$c_i(M_i(s)) = \begin{cases} 1, & \text{if } n_i^f(M_i(s)) < n_i^h(M_i(s)) \\ 0, & \text{if } n_i^h(M_i(s)) < n_i^f(M_i(s)) \\ \phi, & \text{if } n_i^f(M_i(s)) = n_i^h(M_i(s)), \end{cases} \tag{10}$$

$$n_i(M_i(s)) = \min\{n_i^f(M_i(s)), n_i^h(M_i(s))\}. \tag{11}$$

It was shown in Takai and Kumar (2017) that the decentralized diagnoser $\{D_i\}_{i\in I} : L \to C$ for which the local diagnosers are given by Eqs. 8–11 is N-inferring and satisfies

$$\forall s \in L \cap (L-K)\Sigma^{\geq m} : \{D_i\}_{i\in I}(s) = 1 \tag{12}$$

and Eq. 7, i.e., any failure can be correctly detected by the decentralized diagnoser $\{D_i\}_{i\in I} : L \to C$ within m steps.

Remark 2 In summary, the decentralized diagnosis scheme for an N-inference diagnosable pair (L, K) of regular languages can be implemented as follows. When the plant executes a trace $s \in L$, it is observed as the trace $M_i(s)$ at the ith local site. Using Eqs. 8 and 9, the ith local diagnoser computes the values $n_i^f(M_i(s))$ and $n_i^h(M_i(s))$. When $n_i^f(M_i(s))$ (respectively, $n_i^h(M_i(s))$) is smaller, the ith local diagnoser issues a failure (respectively, nonfailure) decision with the ambiguity level $n_i^f(M_i(s))$ (respectively, $n_i^h(M_i(s))$), whereas when the two values are the same, the unsure decision with the ambiguity level $n_i^f(M_i(s)) = n_i^h(M_i(s))$ is issued, as shown in Eqs. 10 and 11. All local decisions are collected at a central decision fusion unit, where, according to Eq. 2, a global decision is always taken to be a winning local decision, i.e., a local decision possessing the minimum ambiguity level.

To compute the diagnosis decision $c_i(M_i(s))$ and the ambiguity level $n_i(M_i(s))$ using Eqs. 8–11, we first need to compute the set $\{(F_k(m), H_k(m))\}_{0\le k\le N+1}$ of language pairs. However in order to do that, we need to know $m \in \mathbb{N}$ (a delay bound) for which it holds that $F_{N+1}(m) = \emptyset \vee H_{N+1}(m) = \emptyset$. These computations *rely on the various constructions and the theoretical results used for verifying N-inference diagnosability*, which we summarize in the subsection below.

3.3 Verification of N-inference diagnosability

For simplicity of presentation, we consider the case of two local diagnosers, i.e., $I = \{1, 2\}$, as in Takai and Kumar (2017). The results continue to hold for an arbitrary number of local diagnosers. We also consider the case of $N \ge 1$ since inferencing is not involved in the case of $N = 0$.

Violation of N-inference diagnosability of (L, K) requires that, for any $m \in \mathbb{N}$, $F_{N+1}(m) \neq \emptyset$ and $H_{N+1}(m) \neq \emptyset$. As shown in the following proposition, which can be proved in the same way as Proposition 1 of Takai and Kumar (2017), the nonemptiness of $F_k(m)$ and $H_k(m)$ $(1 \le k \le N+1)$ can be characterized as the existence of certain $2k+1$ traces in L.

Proposition 1 *Consider the pair (L, K) of regular languages and any $m \in \mathbb{N}$.*

1. *For any $s_0 \in L$ and any $k \in \mathbb{N}$ such that $1 \le k \le N+1$, $s_0 \in F_k(m)$ if and only if $s_0 \in F_0(m)$ and there exist $2k$ traces $s_{10}, s_{11}, \ldots, s_{1(k-1)}, s_{20}, s_{21}, \ldots, s_{2(k-1)} \in L$ such that*
   - $\forall i \in I, \forall i' \in \{0, 1, \ldots, k-1\}$:
   $$s_{ii'} \in \begin{cases} H_0(m), & \text{if } i' \text{ is an even number} \\ F_0(m), & \text{if } i' \text{ is an odd number,} \end{cases}$$
   - $\forall i \in I : M_i(s_0) = M_i(s_{i0})$,
   - $k \ge 2 \Rightarrow \forall i' \in \{0, 1, \ldots, k-2\}$:
   $$\begin{cases} M_2(s_{1i'}) = M_2(s_{1(i'+1)}), & \text{if } i' \text{ is an even number} \\ M_1(s_{1i'}) = M_1(s_{1(i'+1)}), & \text{if } i' \text{ is an odd number,} \end{cases}$$
   - $k \ge 2 \Rightarrow \forall i' \in \{0, 1, \ldots, k-2\}$:
   $$\begin{cases} M_1(s_{2i'}) = M_1(s_{2(i'+1)}), & \text{if } i' \text{ is an even number} \\ M_2(s_{2i'}) = M_2(s_{2(i'+1)}), & \text{if } i' \text{ is an odd number.} \end{cases}$$
2. *For any $s_0 \in L$ and any $k \in \mathbb{N}$ such that $1 \le k \le N+1$, $s_0 \in H_k(m)$ if and only if $s_0 \in H_0(m)$ and there exist $2k$ traces $s_{10}, s_{11}, \ldots, s_{1(k-1)}, s_{20}, s_{21}, \ldots, s_{2(k-1)} \in L$ such that*

– $\forall i \in I, \forall i' \in \{0, 1, \ldots, k-1\}$:

$$s_{ii'} \in \begin{cases} F_0(m), & \textit{if } i' \textit{ is an even number} \\ H_0(m), & \textit{if } i' \textit{ is an odd number}, \end{cases}$$

– $\forall i \in I : M_i(s_0) = M_i(s_{i0})$,
– $k \geq 2 \Rightarrow \forall i' \in \{0, 1, \ldots, k-2\}$:

$$\begin{cases} M_2(s_{1i'}) = M_2(s_{1(i'+1)}), & \textit{if } i' \textit{ is an even number} \\ M_1(s_{1i'}) = M_1(s_{1(i'+1)}), & \textit{if } i' \textit{ is an odd number}, \end{cases}$$

– $k \geq 2 \Rightarrow \forall i' \in \{0, 1, \ldots, k-2\}$:

$$\begin{cases} M_1(s_{2i'}) = M_1(s_{2(i'+1)}), & \textit{if } i' \textit{ is an even number} \\ M_2(s_{2i'}) = M_2(s_{2(i'+1)}), & \textit{if } i' \textit{ is an odd number}. \end{cases}$$

Employing Proposition 1, we first provide a construction to test the nonemptiness of $F_{N+1}(m)$ for any $m \in \mathbb{N}$. To begin, for any $m \in \mathbb{N}$ and any $k \in \{1, 2, \ldots, N+1\}$, we build a generator T_{Fk_m} of all traces $s_0, s_{10}, s_{11}, \ldots, s_{1(k-1)}, s_{20}, s_{21}, \ldots, s_{2(k-1)} \in L$ that satisfy the last three conditions of the first part of Proposition 1. For this, in the case of $k = N+1$, we define the index set I_N of $(2N+3)$ traces $s_0, s_{10}, s_{11}, \ldots, s_{1N}, s_{20}, s_{21}, \ldots, s_{2N} \in L$ of Proposition 1 and I_{FN} of among those that are failure traces $s_0, s_{11}, s_{13}, \ldots, s_{1l}, s_{21}, s_{23}, \ldots, s_{2l} \in L - K$, where

$$l = \begin{cases} N-1, & \text{if } N \text{ is an even number} \\ N, & \text{if } N \text{ is an odd number}, \end{cases}$$

as follows:

$$\begin{aligned} I_N &= \{0, 10, 11, \ldots, 1N, 20, 21, \ldots, 2N\}, \\ I_{FN} &= \{0, 11, 13, \ldots, 1l, 21, 23, \ldots, 2l\}. \end{aligned}$$

Using the notation I_{FN}, we define a finite automaton

$$T_{Fk_m} = (R_{Fk_m}, \Sigma_{Fk}, \alpha_{Fk_m}, r_{Fk_m,0}, R_{Fk_m,mark})$$

as follows:

- The state set R_{Fk_m} is defined as

  $$\begin{aligned} &R_{Fk_m} \\ &= (Q \times \tilde{Q}_{K_m}) \times Q_{10} \times Q_{11} \times \cdots \times Q_{1(k-1)} \times Q_{20} \times Q_{21} \times \cdots \times Q_{2(k-1)}, \end{aligned}$$

  where

  $$Q_{ii'} = \begin{cases} Q_K, & \text{if } i' \text{ is an even number} \\ Q \times \tilde{Q}_{K_m}, & \text{if } i' \text{ is an odd number} \end{cases}$$

  for any $i \in I$ and any $i' \in \{0, 1, \ldots, k-1\}$.
- The initial state $r_{Fk_m,0} \in R_{Fk_m}$ is given as

  $$r_{Fk_m,0} = ((q_0, q_{K,0}), q_{10,0}, q_{11,0}, \ldots, q_{1(k-1),0}, q_{20,0}, q_{21,0}, \ldots, q_{2(k-1),0}),$$

  where

  $$q_{ii',0} = \begin{cases} q_{K,0}, & \text{if } i' \text{ is an even number} \\ (q_0, q_{K,0}), & \text{if } i' \text{ is an odd number} \end{cases}$$

  for any $i \in I$ and any $i' \in \{0, 1, \ldots, k-1\}$.

- The set $R_{Fk_m,mark}$ of marked states is defined as

$$\begin{aligned} R_{Fk_m,mark} = \{r_{Fk_m} \in R_{Fk_m} \mid & [r_{Fk_m}(0) \in Q \times \{d_m\}] \\ & \wedge[\forall i \in I, \forall i' \in \{0, 1, \dots, k-1\} : \\ & ii' \in I_{FN} \Rightarrow r_{Fk_m}(ii') \in Q \times \{d_m\}]\}, \end{aligned}$$

  where, for each $r_{Fk_m} = ((q, \tilde{q}_{K_m}), q_{10}, q_{11}, \dots, q_{1(k-1)}, q_{20}, q_{21}, \dots, q_{2(k-1)}) \in R_{Fk_m}$, we let $r_{Fk_m}(0) = (q, \tilde{q}_{K_m})$ and $r_{Fk_m}(ii') = q_{ii'}$ for each $i \in I$ and each $i' \in \{0, 1, \dots, k-1\}$.
- The event set Σ_{Fk} is defined as

$$\Sigma_{Fk} = \underbrace{(\Sigma \cup \{\varepsilon\}) \times (\Sigma \cup \{\varepsilon\}) \times \cdots \times (\Sigma \cup \{\varepsilon\})}_{(2k+1) \text{ times}} - \{(\varepsilon, \varepsilon, \dots, \varepsilon)\}.$$

- For each

$$r_{Fk_m} = ((q, \tilde{q}_{K_m}), q_{10}, q_{11}, \dots, q_{1(k-1)}, q_{20}, q_{21}, \dots, q_{2(k-1)}) \in R_{Fk_m}$$

  and each

$$\sigma_{Fk} = (\sigma, \sigma_{10}, \sigma_{11}, \dots, \sigma_{1(k-1)}, \sigma_{20}, \sigma_{21}, \dots, \sigma_{2(k-1)}) \in \Sigma_{Fk},$$

  $\alpha_{Fk_m}(r_{Fk_m}, \sigma_{Fk})!$ if the following five conditions are satisfied:

  - $\sigma \neq \varepsilon \Rightarrow \xi_m((q, \tilde{q}_{K_m}), \sigma)!$,
  - $\forall i \in I, \forall i' \in \{0, 1, \dots, k-1\}$:

$$\sigma_{ii'} \neq \varepsilon \Rightarrow \begin{cases} \delta_K(q_{ii'}, \sigma_{ii'})!, & \text{if } i' \text{ is an even number} \\ \xi_m(q_{ii'}, \sigma_{ii'})!, & \text{if } i' \text{ is an odd number,} \end{cases}$$

  - $\forall i \in I : M_i(\sigma) = M_i(\sigma_{i0})$,
  - $k \geq 2 \Rightarrow \forall i' \in \{0, 1, \dots, k-2\}$:

$$\begin{cases} M_2(\sigma_{1i'}) = M_2(\sigma_{1(i'+1)}), & \text{if } i' \text{ is an even number} \\ M_1(\sigma_{1i'}) = M_1(\sigma_{1(i'+1)}), & \text{if } i' \text{ is an odd number,} \end{cases}$$

  - $k \geq 2 \Rightarrow \forall i' \in \{0, 1, \dots, k-2\}$:

$$\begin{cases} M_1(\sigma_{2i'}) = M_1(\sigma_{2(i'+1)}), & \text{if } i' \text{ is an even number} \\ M_2(\sigma_{2i'}) = M_2(\sigma_{2(i'+1)}), & \text{if } i' \text{ is an odd number.} \end{cases}$$

  If $\alpha_{Fk_m}(r_{Fk_m}, \sigma_{Fk})!$, then

$$\alpha_{Fk_m}(r_{Fk_m}, \sigma_{Fk}) = ((q', \tilde{q}'_{K_m}), q'_{10}, q'_{11}, \dots, q'_{1(k-1)}, q'_{20}, q'_{21}, \dots, q'_{2(k-1)}),$$

  where

$$(q', \tilde{q}'_{K_m}) = \begin{cases} \xi_m((q, \tilde{q}_{K_m}), \sigma), & \text{if } \sigma \neq \varepsilon \\ (q, \tilde{q}_{K_m}), & \text{otherwise} \end{cases}$$

  and

$$q'_{ii'} = \begin{cases} \delta_K(q_{ii'}, \sigma_{ii'}), & \text{if } \sigma_{ii'} \neq \varepsilon \wedge [i' \text{ is an even number}] \\ \xi_m(q_{ii'}, \sigma_{ii'}), & \text{if } \sigma_{ii'} \neq \varepsilon \wedge [i' \text{ is an odd number}] \\ q_{ii'}, & \text{otherwise.} \end{cases}$$

For each $\sigma_{Fk} = (\sigma, \sigma_{10}, \sigma_{11}, \dots, \sigma_{1(k-1)}, \sigma_{20}, \sigma_{21}, \dots, \sigma_{2(k-1)})$ in Σ_{Fk}, we let $\sigma_{Fk}(0) = \sigma$ and $\sigma_{Fk}(ii') = \sigma_{ii'}$ for each $i \in I$ and each $i' \in \{0, 1, \dots, k-1\}$. In addition, for a nonempty trace $s_{Fk} = \sigma_{Fk,1}\sigma_{Fk,2} \cdots \sigma_{Fk,l} \in \Sigma_{Fk}^* - \{\varepsilon\}$, we let $s_{Fk}(0) = \sigma_{Fk,1}(0)\sigma_{Fk,2}(0) \cdots \sigma_{Fk,l}(0)$ and $s_{Fk}(ii') = \sigma_{Fk,1}(ii')\sigma_{Fk,2}(ii') \cdots \sigma_{Fk,l}(ii')$ for each $i \in I$ and each $i' \in \{0, 1, \dots, k-1\}$. For the empty trace $\varepsilon \in \Sigma_{Fk}^*$, we let $\varepsilon(0) = \varepsilon$ and $\varepsilon(ii') = \varepsilon$ for each $i \in I$ and each $i' \in \{0, 1, \dots, k-1\}$.

From the construction of T_{Fk_m}, the following proposition is obtained in the same way as Proposition 2 of Takai and Kumar (2017).

Proposition 2 *For any* $m \in \mathcal{N}$ *and any* $k \in \{1, 2, \ldots, N+1\}$, *consider any* $(2k+1)$ *traces* $s_0, s_{10}, s_{11}, \ldots, s_{1(k-1)}, s_{20}, s_{21}, \ldots, s_{2(k-1)} \in L$ *such that*

$$\forall i \in I, \forall i' \in \{0, 1, \ldots, k-1\} : ii' \in I_N - I_{FN} \Rightarrow s_{ii'} \in K.$$

Then these $(2k+1)$ *traces satisfy the last three of the four conditions of the first part of Proposition 1 if and only if there exists* $s_{Fk} \in L(T_{Fk_m})$ *such that* $s_{Fk}(0) = s_0$ *and* $s_{Fk}(ii') = s_{ii'}$ *for any* $i \in I$ *and any* $i' \in \{0, 1, \ldots, k-1\}$.

For the case of $m = 0$, we drop the subscript m in T_{Fk_m} for simplicity of notation, i.e., $T_{Fk} := T_{Fk_0}$. It then follows from the definition of the state transition function $\alpha_{F(N+1)}$ of $T_{F(N+1)}$ that $T_{F(N+1)}$ generates all $(2N+3)$ traces that satisfy the last three of the four conditions of the first part of Proposition 1.

To establish non-N-inference diagnosability, we must check $F_{N+1}(m) \neq \emptyset$ for *any* $m \in \mathbb{N}$, which represents the number of steps of post-fault executions. To allow an arbitrary number of post-fault executions (equivalently, an arbitrary value of m), the post-fault extensions in $T_{F(N+1)}$ must visit cycles leading to an arbitrary growth in the number m of post-fault executions. Note that a single cycle may not elongate all $(2l+1)$ failure traces $s_{i_{FN}}$ ($i_{FN} = 0, 11, 13, \ldots, 1l, 21, 23, \ldots, 2l$) because some of these trace elements may witness only ε-transitions along that cycle. Hence a multitude of cycles may need to be executed sequentially to elongate all the $(2l+1)$ failure trace elements. To keep track of which cycles elongate which of the trace elements (by executing at least one non-ε-transition), we collapse all the maximal strongly connected components (max-SCCs) of $T_{F(N+1)}$ into individual nodes, labeling those nodes with the trace elements that witness at least one non-ε-transition in the corresponding max-SCCs, and build a nondeterministic acyclic automaton over the node set of max-SCCs as follows (Takai and Kumar 2017):

$$\mathcal{T}_{F(N+1)} = (V_{F(N+1)}, \Sigma_{F(N+1)}, \beta_{F(N+1)}, V_{F(N+1),0}, V_{F(N+1),mark}),$$

where

- The state set $V_{F(N+1)}$ is defined as

  $$V_{F(N+1)} = \{V_{F(N+1),0}, V_{F(N+1),1}, \ldots, V_{F(N+1),|V_{F(N+1)}|-1}\},$$

  where, for any $k \in \{0, 1, \ldots, |V_{F(N+1)}| - 1\}$, $V_{F(N+1),k}$ is a max-SCC of $T_{F(N+1)}$. Without loss of generality, we assume that $r_{F(N+1),0} \in V_{F(N+1),0}$.
- The set $V_{F(N+1),mark}$ of marked states is defined as

  $$V_{F(N+1),mark} = \{V_{F(N+1),k} \in V_{F(N+1)} \mid V_{F(N+1),k} \cap R_{F(N+1),mark} \neq \emptyset\}.$$

- The nondeterministic state transition function $\beta_{F(N+1)} : V_{F(N+1)} \times \Sigma_{F(N+1)} \to 2^{V_{F(N+1)}}$ is defined as

  $$\begin{aligned}&\beta_{F(N+1)}(V_{F(N+1),k}, \sigma_{F(N+1)})\\ &\quad = \{V_{F(N+1),k'} \in V_{F(N+1)} \mid k \neq k'\\ &\qquad \wedge [\exists r_{F(N+1)} \in V_{F(N+1),k}, \exists r'_{F(N+1)} \in V_{F(N+1),k'} :\\ &\qquad\quad \alpha_{F(N+1)}(r_{F(N+1)}, \sigma_{F(N+1)}) = r'_{F(N+1)}]\}\end{aligned}$$

  for each $V_{F(N+1),k} \in V_{F(N+1)}$ and each $\sigma_{F(N+1)} \in \Sigma_{F(N+1)}$.

A *labeling* function $J_{FN} : V_{F(N+1)} \to 2^{I_{FN}}$ is defined as

$$\begin{aligned} &J_{FN}(V_{F(N+1),k}) \\ &= \{i_{FN} \in I_{FN} \mid [\exists r_{F(N+1)} \in V_{F(N+1),k} : r_{F(N+1)}(i_{FN}) \in Q \times \{d\}] \\ &\quad \wedge [\exists r_{F(N+1)}, r'_{F(N+1)} \in V_{F(N+1),k}, \exists \sigma_{F(N+1)} \in \Sigma_{F(N+1)} : \\ &\quad\quad \alpha_{F(N+1)}(r_{F(N+1)}, \sigma_{F(N+1)}) = r'_{F(N+1)} \wedge \sigma_{F(N+1)}(i_{FN}) \neq \varepsilon]\} \end{aligned}$$

for each $V_{F(N+1),k} \in V_{F(N+1)}$. Then, by the construction, $i_{FN} \in J_{FN}(V_{F(N+1),k})$ means that $s_{i_{FN}}$ can be extended to an arbitrarily long failure trace in a max-SCC $V_{F(N+1),k}$. Therefore, we can test whether $F_{N+1}(m) \neq \emptyset$ for any $m \in \mathbb{N}$ as shown in the following proposition.

Proposition 3 (Takai and Kumar 2017, Proposition 3) *Consider the pair (L, K) of regular languages. Then, $F_{N+1}(m) \neq \emptyset$ for any $m \in \mathbb{N}$ if and only if there exists a path $V_{F(N+1),0} = V_{F(N+1),k_0} \xrightarrow{\sigma^{(k_0)}_{F(N+1)}} V_{F(N+1),k_1} \xrightarrow{\sigma^{(k_1)}_{F(N+1)}} \cdots \xrightarrow{\sigma^{(k_{h-1})}_{F(N+1)}} V_{F(N+1),k_h} \in V_{F(N+1),mark}$ in the acyclic automaton $\mathcal{T}_{F(N+1)}$ such that*

$$\bigcup_{k \in \{k_0, k_1, \ldots, k_h\}} J_{FN}(V_{F(N+1),k}) = I_{FN}. \tag{13}$$

Dually, to check the nonemptiness of $H_{N+1}(m)$ for any $m \in \mathbb{N}$, we construct $T_{H(N+1)}$ and $\mathcal{T}_{H(N+1)}$ that are dual to $T_{F(N+1)}$ and $\mathcal{T}_{F(N+1)}$, respectively. Among the $(2N + 3)$ traces of the second part of Proposition 1 in the case of $k = N + 1$, $s_{10}, s_{12}, \ldots, s_{1l}, s_{20}, s_{22}, \ldots, s_{2l} \in L - K$ are failure traces, where

$$l = \begin{cases} N, & \text{if } N \text{ is an even number} \\ N - 1, & \text{if } N \text{ is an odd number,} \end{cases}$$

and their index set is denoted by

$$I_{HN} = \{10, 12, \ldots, 1l, 20, 22, \ldots, 2l\}.$$

For any $m \in \mathbb{N}$ and any $k \in \{1, 2, \ldots, N + 1\}$, we define a finite automaton

$$T_{Hk_m} = (R_{Hk_m}, \Sigma_{Hk}, \alpha_{Hk_m}, r_{Hk_m,0}, R_{Hk_m,mark})$$

as follows:

- The state set R_{Hk_m} is defined as

  $$\begin{aligned} &R_{Hk_m} \\ &= Q_K \times Q_{10} \times Q_{11} \times \cdots \times Q_{1(k-1)} \times Q_{20} \times Q_{21} \times \cdots \times Q_{2(k-1)}, \end{aligned}$$

  where

  $$Q_{ii'} = \begin{cases} Q \times \tilde{Q}_{K_m}, & \text{if } i' \text{ is an even number} \\ Q_K, & \text{if } i' \text{ is an odd number} \end{cases}$$

  for any $i \in I$ and any $i' \in \{0, 1, \ldots, k - 1\}$.
- The initial state $r_{Hk_m,0} \in R_{Hk_m}$ is given as

  $$r_{Hk_m,0} = (q_{K,0}, q_{10,0}, q_{11,0}, \ldots, q_{1(k-1),0}, q_{20,0}, q_{21,0}, \ldots, q_{2(k-1),0}),$$

  where

  $$q_{ii',0} = \begin{cases} (q_0, q_{K,0}), & \text{if } i' \text{ is an even number} \\ q_{K,0}, & \text{if } i' \text{ is an odd number} \end{cases}$$

  for any $i \in I$ and any $i' \in \{0, 1, \ldots, k - 1\}$.

- The set $R_{Hk_m,mark}$ of marked states is defined as

$$R_{Hk_m,mark} = \{r_{Hk_m} \in R_{Hk_m} \mid \forall i \in I, \forall i' \in \{0,1,\ldots,k-1\} : \\ ii' \in I_{HN} \Rightarrow r_{Hk_m}(ii') \in Q \times \{d_m\}\},$$

where, for each $r_{Hk_m} \in R_{Hk_m}$, $r_{Hk_m}(0)$ and $r_{Hk_m}(ii')$ ($i \in I, i' \in \{0,1,\ldots,k-1\}$) are defined in the same way as $r_{Fk_m}(0)$ and $r_{Fk_m}(ii')$, respectively, where $r_{Fk_m} \in R_{Fk_m}$.
- The event set Σ_{Hk} is defined as $\Sigma_{Hk} = \Sigma_{Fk}$.
- For each

$$r_{Hk_m} = (q_K, q_{10}, q_{11}, \ldots, q_{1(k-1)}, q_{20}, q_{21}, \ldots, q_{2(k-1)}) \in R_{Hk_m}$$

and each

$$\sigma_{Hk} = (\sigma, \sigma_{10}, \sigma_{11}, \ldots, \sigma_{1(k-1)}, \sigma_{20}, \sigma_{21}, \ldots, \sigma_{2(k-1)}) \in \Sigma_{Hk},$$

$\alpha_{Hk_m}(r_{Hk_m}, \sigma_{Hk})!$ if the following five conditions are satisfied:

  - $\sigma \neq \varepsilon \Rightarrow \delta_K(q_K, \sigma)!$,
  - $\forall i \in I, \forall i' \in \{0,1,\ldots,k-1\}$:

$$\sigma_{ii'} \neq \varepsilon \Rightarrow \begin{cases} \xi_m(q_{ii'}, \sigma_{ii'})!, & \text{if } i' \text{ is an even number} \\ \delta_K(q_{ii'}, \sigma_{ii'})!, & \text{if } i' \text{ is an odd number,} \end{cases}$$

  - $\forall i \in I : M_i(\sigma) = M_i(\sigma_{i0})$,
  - $k \geq 2 \Rightarrow \forall i' \in \{0,1,\ldots,k-2\}$:

$$\begin{cases} M_2(\sigma_{1i'}) = M_2(\sigma_{1(i'+1)}), & \text{if } i' \text{ is an even number} \\ M_1(\sigma_{1i'}) = M_1(\sigma_{1(i'+1)}), & \text{if } i' \text{ is an odd number,} \end{cases}$$

  - $k \geq 2 \Rightarrow \forall i' \in \{0,1,\ldots,k-2\}$:

$$\begin{cases} M_1(\sigma_{2i'}) = M_1(\sigma_{2(i'+1)}), & \text{if } i' \text{ is an even number} \\ M_2(\sigma_{2i'}) = M_2(\sigma_{2(i'+1)}), & \text{if } i' \text{ is an odd number.} \end{cases}$$

If $\alpha_{Hk_m}(r_{Hk_m}, \sigma_{Hk})!$, then

$$\alpha_{Hk_m}(r_{Hk_m}, \sigma_{Hk}) = (q'_K, q'_{10}, q'_{11}, \ldots, q'_{1(k-1)}, q'_{20}, q'_{21}, \ldots, q'_{2(k-1)}),$$

where

$$q'_K = \begin{cases} \delta_K(q_k, \sigma), & \text{if } \sigma \neq \varepsilon \\ q_K, & \text{otherwise} \end{cases}$$

and

$$q'_{ii'} = \begin{cases} \xi_m(q_{ii'}, \sigma_{ii'}), & \text{if } \sigma_{ii'} \neq \varepsilon \wedge [i' \text{ is an even number}] \\ \delta_K(q_{ii'}, \sigma_{ii'}), & \text{if } \sigma_{ii'} \neq \varepsilon \wedge [i' \text{ is an odd number}] \\ q_{ii'}, & \text{otherwise.} \end{cases}$$

For each $\sigma_{Hk} \in \Sigma_{Hk}$ and each $s_{Hk} \in \Sigma^*_{Hk}$, $\sigma_{Hk}(0)$, $s_{Hk}(0)$, $\sigma_{Hk}(ii')$, and $s_{Hk}(ii')$ ($i \in I, i' \in \{1,2,\ldots,k-1\}$) are defined in the same way as $\sigma_{Fk}(0)$, $s_{Fk}(0)$, $\sigma_{Fk}(ii')$, and $s_{Fk}(ii')$, respectively, where $\sigma_{Fk} \in \Sigma_{Fk}$ and $s_{Fk} \in \Sigma^*_{Fk}$.

The following proposition can be obtained in a similar way to Proposition 2.

Proposition 4 *For any $m \in \mathcal{N}$ and any $k \in \{1,2,\ldots,N+1\}$, consider any $(2k+1)$ traces $s, s_{10}, s_{11}, \ldots, s_{1(k-1)}, s_{20}, s_{21}, \ldots, s_{2(k-1)} \in L$ such that $s_0 \in K$ and*

$$\forall i \in I, \forall i' \in \{0,1,\ldots,k-1\} : ii' \in I_N - I_{HN} \Rightarrow s_{ii'} \in K.$$

Then these $(2k+1)$ *traces satisfy the last three of the four conditions of the second part of Proposition 1 if and only if there exists* $s_{Hk} \in L(T_{Hk_m})$ *such that* $s_{Hk}(0) = s$ *and* $s_{Hk}(ii') = s_{ii'}$ *for any* $i \in I$ *and any* $i' \in \{0, 1, \ldots, k-1\}$.

For the case of $m = 0$ and $k = N+1$, we use the notation $T_{H(N+1)}$, dropping the subscript $m = 0$ for simplicity, and as with $T_{F(N+1)}$ versus $\mathcal{T}_{F(N+1)}$, using $T_{H(N+1)}$, we construct a nondeterministic acyclic automaton

$$\mathcal{T}_{H(N+1)} = (V_{H(N+1)}, \Sigma_{H(N+1)}, \beta_{H(N+1)}, V_{H(N+1),0}, V_{H(N+1),mark})$$

over the set of max-SCCs of $T_{H(N+1)}$ as follows (Takai and Kumar 2017):

– The state set $V_{H(N+1)}$ is defined as

$$V_{H(N+1)} = \{V_{H(N+1),0}, V_{H(N+1),1}, \ldots, V_{H(N+1),|V_{H(N+1)}|-1}\},$$

where, for any $k \in \{0, 1, \ldots, |V_{H(N+1)}| - 1\}$, $V_{H(N+1),k}$ is a max-SCC of $T_{H(N+1)}$. Without loss of generality, we assume that $r_{H(N+1),0} \in V_{H(N+1),0}$.

– The set $V_{H(N+1),mark}$ of marked states is defined as

$$V_{H(N+1),mark} = \{V_{H(N+1),k} \in V_{H(N+1)} \mid V_{H(N+1),k} \cap R_{H(N+1),mark} \neq \emptyset\}.$$

– The nondeterministic state transition function $\beta_{H(N+1)} : V_{H(N+1)} \times \Sigma_{H(N+1)} \to 2^{V_{H(N+1)}}$ is defined as

$$\begin{aligned}&\beta_{H(N+1)}(V_{H(N+1),k}, \sigma_{H(N+1)})\\&\quad = \{V_{H(N+1),k'} \in V_{H(N+1)} \mid k \neq k'\\&\qquad\quad \wedge[\exists r_{H(N+1)} \in V_{H(N+1),k}, \exists r'_{H(N+1)} \in V_{H(N+1),k'} :\\&\qquad\qquad \alpha_{H(N+1)}(r_{H(N+1)}, \sigma_{H(N+1)}) = r'_{H(N+1)}]\}\end{aligned}$$

for each $V_{H(N+1),k} \in V_{H(N+1)}$ and each $\sigma_{H(N+1)} \in \Sigma_{H(N+1)}$.

A labeling function $J_{HN} : V_{H(N+1)} \to 2^{I_{HN}}$ is defined as

$$\begin{aligned}&J_{HN}(V_{H(N+1),k})\\&\quad = \{i_{HN} \in I_{HN} \mid [\exists r_{H(N+1)} \in V_{H(N+1),k} : r_{H(N+1)}(i_{HN}) \in Q \times \{d\}]\\&\qquad\quad \wedge[\exists r_{H(N+1)}, r'_{H(N+1)} \in V_{H(N+1),k}, \exists \sigma_{H(N+1)} \in \Sigma_{H(N+1)} :\\&\qquad\qquad \alpha_{H(N+1)}(r_{H(N+1)}, \sigma_{H(N+1)}) = r'_{H(N+1)} \wedge \sigma_{H(N+1)}(i_{HN}) \neq \varepsilon]\}\end{aligned}$$

for each $V_{H(N+1),k} \in V_{H(N+1)}$.

Similar to Proposition 3, we have the following result.

Proposition 5 (Takai and Kumar 2017, Proposition 5) *Consider the pair* (L, K) *of regular languages. Then,* $H_{N+1}(m) \neq \emptyset$ *for any* $m \in \mathbb{N}$ *if and only if there exists a path* $V_{H(N+1),0} = V_{H(N+1),k_0} \xrightarrow{\sigma^{(k_0)}_{H(N+1)}} V_{H(N+1),k_1} \xrightarrow{\sigma^{(k_1)}_{H(N+1)}} \cdots \xrightarrow{\sigma^{(k_{h-1})}_{H(N+1)}} V_{H(N+1),k_h} \in V_{H(N+1),mark}$ *in the acyclic automaton* $\mathcal{T}_{H(N+1)}$ *such that*

$$\bigcup_{k\in\{k_0,k_1,\ldots,k_h\}} J_{HN}(V_{H(N+1),k}) = I_{HN}. \tag{14}$$

Then the following theorem is obtained, which can be used to verify N-inference diagnosability.

Theorem 2 (Takai and Kumar 2017, Theorem 8) *The pair (L, K) of regular languages is not N-inference diagnosable if and only if*

- *there exists a path* $V_{F(N+1),0} = V_{F(N+1),k_0} \xrightarrow{\sigma^{(k_0)}_{F(N+1)}} V_{F(N+1),k_1} \xrightarrow{\sigma^{(k_1)}_{F(N+1)}} \cdots \xrightarrow{\sigma^{(k_{h-1})}_{F(N+1)}} V_{F(N+1),k_h} \in V_{F(N+1),mark}$ *in the acyclic automaton $\mathcal{T}_{F(N+1)}$ that satisfies (13), and*
- *there exists a path* $V_{H(N+1),0} = V_{H(N+1),k_0} \xrightarrow{\sigma^{(k_0)}_{H(N+1)}} V_{H(N+1),k_1} \xrightarrow{\sigma^{(k_1)}_{H(N+1)}} \cdots \xrightarrow{\sigma^{(k_{h-1})}_{H(N+1)}} V_{H(N+1),k_h} \in V_{H(N+1),mark}$ *in the acyclic automaton $\mathcal{T}_{H(N+1)}$ that satisfies (14).*

Remark 3 Two different finite automata $T_{F(N+1)}$ and $T_{H(N+1)}$ are used above to test whether $F_{N+1}(m) \neq \emptyset$ and $H_{N+1}(m) \neq \emptyset$, respectively, for any $m \in \mathbb{N}$. It is possible to construct a single more general automaton to do the same, as follows, but with a higher complexity. To generate a nonfailure trace $s \in K$ in $T_{F(N+1)}$ and $T_{H(N+1)}$, we used the generator G_K of the nonfailure specification language $K \subseteq L$. Instead of G_K, the synchronous product $G \parallel \tilde{G}_K$ can be used by noting that, for any $s \in L$, $s \in K$ if and only if $\xi((q_0, q_{K,0}), s) \in Q \times Q_K$, i.e., the second element of $\xi((q_0, q_{K,0}), s)$ is not the dump state d. With this observation, we can define the following finite automaton:

$$T_{N+1} = (R_{N+1}, \Sigma_{N+1}, \alpha_{N+1}, r_{N+1,0}, R_{N+1}),$$

whose various elements are defined as follows:

- The state set R_{N+1} is defined as
$$R_{N+1} = \underbrace{(Q \times \tilde{Q}_K) \times (Q \times \tilde{Q}_K) \times \cdots \times (Q \times \tilde{Q}_K)}_{(2N+3) \text{ times}}.$$
- The initial state $r_{N+1,0} \in R_{N+1}$ is given as
$$r_{N+1,0} = ((q_0, q_{K,0}), (q_0, q_{K,0}), \ldots, (q_0, q_{K,0})).$$
- The event set Σ_{N+1} is defined as
$$\Sigma_{N+1} = \underbrace{(\Sigma \cup \{\varepsilon\}) \times (\Sigma \cup \{\varepsilon\}) \times \cdots \times (\Sigma \cup \{\varepsilon\})}_{(2N+3) \text{ times}} - \{(\varepsilon, \varepsilon, \ldots, \varepsilon)\}.$$
- For each
$$r_{N+1} = ((q, \tilde{q}_K), q_{10}, q_{11}, \ldots, q_{1N}, q_{20}, q_{21}, \ldots, q_{2N}) \in R_{N+1}$$
and each
$$\sigma_{N+1} = (\sigma, \sigma_{10}, \sigma_{11}, \ldots, \sigma_{1N}, \sigma_{20}, \sigma_{21}, \ldots, \sigma_{2N}) \in \Sigma_{N+1},$$
$\alpha_{N+1}(r_{N+1}, \sigma_{N+1})!$ if the following five conditions are satisfied:
  - $\sigma \neq \varepsilon \Rightarrow \xi((q, \tilde{q}_K), \sigma)!$,
  - $\forall i \in I, \forall i' \in \{0, 1, \ldots, k-1\} : \sigma_{ii'} \neq \varepsilon \Rightarrow \xi(q_{ii'}, \sigma_{ii'})!$,
  - $\forall i \in I : M_i(\sigma) = M_i(\sigma_{i0})$,
  - $\forall i' \in \{0, 1, \ldots, N-1\}$:
$$\begin{cases} M_2(\sigma_{1i'}) = M_2(\sigma_{1(i'+1)}), & \text{if } i' \text{ is an even number} \\ M_1(\sigma_{1i'}) = M_1(\sigma_{1(i'+1)}), & \text{if } i' \text{ is an odd number,} \end{cases}$$
  - $\forall i' \in \{0, 1, \ldots, N-1\}$:
$$\begin{cases} M_1(\sigma_{2i'}) = M_1(\sigma_{2(i'+1)}), & \text{if } i' \text{ is an even number} \\ M_2(\sigma_{2i'}) = M_2(\sigma_{2(i'+1)}), & \text{if } i' \text{ is an odd number.} \end{cases}$$

If $\alpha_{N+1}(r_{N+1}, \sigma_{N+1})!$, then

$$\alpha_{N+1}(r_{N+1}, \sigma_{N+1}) = ((q', \tilde{q}'_K), q'_{10}, q'_{11}, \ldots, q'_{1N}, q'_{20}, q'_{21}, \ldots, q'_{2N}),$$

where

$$(q', \tilde{q}'_K) = \begin{cases} \xi((q, \tilde{q}_K), \sigma), & \text{if } \sigma \neq \varepsilon \\ (q, \tilde{q}_K), & \text{otherwise} \end{cases}$$

and

$$q'_{ii'} = \begin{cases} \xi(q_{ii'}, \sigma_{ii'}), & \text{if } \sigma_{ii'} \neq \varepsilon \\ q_{ii'}, & \text{otherwise.} \end{cases}$$

Then both the conditions $F_{m+1}(m) \neq \emptyset$ and $H_{m+1}(m) \neq \emptyset$ for any $m \in \mathbb{N}$ can be described using the single automaton T_{N+1}. However, since $|Q_K| \leq |Q \times \tilde{Q}_K| = |Q| \times (|Q_K| + 1)$, we have $|R_{F(N+1)}| \leq |R_{N+1}|$ and $|R_{H(N+1)}| \leq |R_{N+1}|$, i.e., T_{N+1} will have a larger state space than $T_{F(N+1)}$ and $T_{H(N+1)}$.

4 Computation of delay bound

As in Section 3.3, we assume that $I = \{1, 2\}$ and $N \geq 1$. For the pair (L, K) of regular languages that is N-inference diagnosable (so there exists $m \in \mathbb{N}$ such that $F_{N+1}(m) = \emptyset$ or $H_{N+1}(m) = \emptyset$), let m^*_N be a minimum integer m:

$$m^*_N = \min\{m \in \mathbb{N} \mid F_{N+1}(m) = \emptyset \vee H_{N+1}(m) = \emptyset\}.$$

For any $m \geq m^*_N$, the decentralized diagnoser $\{D_i\}_{i \in I} : L \to C$ for which the local diagnosers are given by Eqs. 8–11 can detect any failure within m steps. Hence, m^*_N can be considered as the delay bound. Let $\mathbb{N}_{FN}$ and $\mathbb{N}_{HN}$ be the sets of delays under which (L, K) is "N-inference disjunctive diagnosable" and "N-inference conjunctive diagnosable", respectively, i.e., $\mathbb{N}_{FN} = \{m \in \mathbb{N} \mid F_{N+1}(m) = \emptyset\}$ and $\mathbb{N}_{HN} = \{m \in \mathbb{N} \mid H_{N+1}(m) = \emptyset\}$. Then we can define the minimum delays for the disjunctive and conjunctive cases, respectively, as

$$m^*_{FN} = \begin{cases} \min \mathbb{N}_{FN}, & \text{if } \mathbb{N}_{FN} \neq \emptyset \\ \text{undefined}, & \text{otherwise,} \end{cases}$$

$$m^*_{HN} = \begin{cases} \min \mathbb{N}_{HN}, & \text{if } \mathbb{N}_{HN} \neq \emptyset \\ \text{undefined}, & \text{otherwise,} \end{cases}$$

while the overall minimum delay as

$$m^*_N = \begin{cases} \min\{m^*_{FN}, m^*_{HN}\}, & \text{if } \mathbb{N}_{FN} \neq \emptyset \wedge \mathbb{N}_{HN} \neq \emptyset \\ m^*_{FN}, & \text{if } \mathbb{N}_{FN} \neq \emptyset \wedge \mathbb{N}_{HN} = \emptyset \\ m^*_{HN}, & \text{if } \mathbb{N}_{FN} = \emptyset \wedge \mathbb{N}_{HN} \neq \emptyset. \end{cases}$$

To compute m^*_N, we first develop methods for computing m^*_{FN} and m^*_{HN} when $\mathbb{N}_{FN} \neq \emptyset$ and $\mathbb{N}_{HN} \neq \emptyset$, respectively.

Note that m^*_{FN} is the minimum number of post-fault events the system must execute for a fault to be detected. We use the acyclic automaton $\mathcal{T}_{F(N+1)}$ for computing m^*_{FN} when $\mathbb{N}_{FN} \neq \emptyset$. Under this condition, we have $F_{N+1}(m^*_{FN}) = \emptyset$. Then from the inverse of Proposition 3, it holds for any path in the set $\mathcal{P}_{F(N+1)}$ of all paths from the initial state to marked states of the form $V_{F(N+1),0} = V_{F(N+1),k_0} \xrightarrow{\sigma^{(k_0)}_{F(N+1)}} V_{F(N+1),k_1} \xrightarrow{\sigma^{(k_1)}_{F(N+1)}} \cdots \xrightarrow{\sigma^{(k_{h-1})}_{F(N+1)}} V_{F(N+1),k_h} \in V_{F(N+1),mark}$, that $\bigcup_{k \in \{k_0, k_1, \ldots, k_h\}} J_{FN}(V_{F(N+1),k}) \neq I_{FN}$.

Then for a corresponding index $i_{FN} \in I_{FN} - \bigcup_{k\in\{k_0,k_1,\ldots,k_h\}} J_{FN}(V_{F(N+1),k})$, it holds that only the inter-max-SCC transitions in $T_{F(N+1)}$ (that are also the transitions of $\mathcal{T}_{F(N+1)}$) witness a post-fault event, whereas the intra-max-SCC transitions in $T_{F(N+1)}$ (max-SCCs of $T_{F(N+1)}$ are states in $\mathcal{T}_{F(N+1)}$) are simply ε-transitions (otherwise that index would already be included in $\bigcup_{k\in\{k_0,k_1,\ldots,k_h\}} J_{FN}(V_{F(N+1),k})$ by the definition of J_{FN}). Then for each such index, each transition along a path $p_{F(N+1)} \in \mathcal{P}_{F(N+1)}$ may contribute to a post-fault event count depending on whether or not that transition is a non-ε-transition for that index. Accordingly, for each path $p_{F(N+1)} : V_{F(N+1),0} = V_{F(N+1),k_0} \xrightarrow{\sigma^{(k_0)}_{F(N+1)}} V_{F(N+1),k_1} \xrightarrow{\sigma^{(k_1)}_{F(N+1)}} \cdots \xrightarrow{\sigma^{(k_{h-1})}_{F(N+1)}} V_{F(N+1),k_h} \in V_{F(N+1),mark}$ in $\mathcal{P}_{F(N+1)}$ and each index $i_{FN} \in I_{FN} - \bigcup_{k\in\{k_0,k_1,\ldots,k_h\}} J_{FN}(V_{F(N+1),k})$, we define an index-specific post-fault event count of each transition of the path as follows: For each $j \in \{1, 2, \ldots, h\}$,

$$
\begin{aligned}
&w_{FN,i_{FN}}(V_{F(N+1),k_{j-1}}, \sigma^{(k_{j-1})}_{F(N+1)}, V_{F(N+1),k_j}) \\
&:= \begin{cases} 1, \text{ if } [\sigma^{(k_{j-1})}_{F(N+1)}(i_{FN}) \neq \varepsilon] \\ \qquad \wedge [\exists r_{F(N+1)} \in V_{F(N+1),k_{j-1}}, \exists r'_{F(N+1)} \in V_{F(N+1),k_j} : \\ \qquad\quad \alpha_{F(N+1)}(r_{F(N+1)}, \sigma^{(k_{j-1})}_{F(N+1)}) = r'_{F(N+1)} \wedge r'_{F(N+1)}(i_{FN}) \in Q \times \{d\}] \\ 0, \text{ otherwise.} \end{cases}
\end{aligned}
$$

Using the index-specific post-fault event count of transitions, we define an index-specific post-fault event count for the path $p_{F(N+1)}$ by simply adding the index-specific counts along the transitions of $p_{F(N+1)}$:

$$w_{FN,i_{FN}}(p_{F(N+1)}) := \sum_{j=1}^{h} w_{FN,i_{FN}}(V_{F(N+1),k_{j-1}}, \sigma^{(k_{j-1})}_{F(N+1)}, V_{F(N+1),k_j}).$$

Next, a minimum among all indices in $I_{FN} - \bigcup_{k\in\{k_0,k_1,\ldots,k_h\}} J_{FN}(V_{F(N+1),k})$ is taken to determine the post-fault event count across all such indices for the path $p_{F(N+1)}$:

$$w_{FN}(p_{F(N+1)}) := \min_{i_{FN}\in I_{FN} - \cup_{k\in\{k_0,\ldots,k_h\}} J_{FN}(V_{F(N+1),k})} w_{FN,i_{FN}}(p_{F(N+1)}).$$

Finally a maximum of the counts along all paths in $\mathcal{P}_{F(N+1)}$ is taken to obtain the required delay bound of diagnosis:

$$w_{FN} := \begin{cases} \max_{p_{F(N+1)}\in\mathcal{P}_{F(N+1)}} w_{FN}(p_{F(N+1)}), & \text{if } \mathcal{P}_{F(N+1)} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{P}_{F(N+1)}$ is finite, w_{FN} is effectively computable.

The following theorem shows that the value m^*_{FN} can be computed as $m^*_{FN} = w_{FN}$.

Theorem 3 *For the pair (L, K) of regular languages that is N-inference diagnosable, if $\mathbb{N}_{FN} \neq \emptyset$ then $m^*_{FN} = w_{FN}$.*

Proof First, we show that $m^*_{FN} \leq w_{FN}$. For the sake of contradiction, we suppose that $w_{FN} < m^*_{FN}$. Since $m^*_{FN} = \min \mathbb{N}_{FN}$, we have $w_{FN} \notin \mathbb{N}_{FN}$, which implies $F_{N+1}(w_{FN}) \neq \emptyset$. We consider any $s_0 \in F_{N+1}(w_{FN})$. By Proposition 1, there exist $2(N+1)$ traces $s_{10}, s_{11}, \ldots, s_{1N}, s_{20}, s_{21}, \ldots, s_{2N} \in L$ such that the four conditions of the first part of Proposition 1 are satisfied for $m = w_{FN}$ and $k = N + 1$. Furthermore, by Proposition 2, there exists $s_{F(N+1)} := \sigma^{(0)}_{F(N+1)}\sigma^{(1)}_{F(N+1)} \cdots \sigma^{(l-1)}_{F(N+1)} \in L(T_{F(N+1)})$ $(l \geq 1)$ such that

$s_{F(N+1)}(i_N) = s_{i_N}$ for each $i_N \in I_N$. We consider the path $r_{F(N+1),0} = r^{(0)}_{F(N+1)} \xrightarrow{\sigma^{(0)}_{F(N+1)}} r^{(1)}_{F(N+1)} \xrightarrow{\sigma^{(1)}_{F(N+1)}} \cdots \xrightarrow{\sigma^{(l-1)}_{F(N+1)}} r^{(l)}_{F(N+1)}$ in $T_{F(N+1)}$ obtained by executing $s_{F(N+1)}$. For each $i_{FN} \in I_{FN}$, if $i_{FN} = 0$, then $s_{F(N+1)}(0) = s_0 \in F_{N+1}(w_{FN}) \subseteq L - K$. If $i_{FN} \neq 0$, then we have by the second condition of the first part of Proposition 1 that $s_{F(N+1)}(i_{FN}) \in F_0(w_{FN}) \subseteq L - K$. Thus, we have $r^{(l)}_{F(N+1)}(i_{FN}) \in Q \times \{d\}$ for each $i_{FN} \in I_{FN}$, i.e., $r^{(l)}_{F(N+1)} \in R_{F(N+1),mark}$.

For some $a_0, a_1, \ldots, a_{h-1} \in \{0, 1, \ldots, l\}$ ($h \geq 1$) such that $a_0 < a_1 < \cdots < a_{h-1}$, there exists a path $p_{F(N+1)} : V_{F(N+1),0} = V_{F(N+1),k_0} \xrightarrow{\sigma^{(a_0)}_{F(N+1)}} V_{F(N+1),k_1} \xrightarrow{\sigma^{(a_1)}_{F(N+1)}} \cdots \xrightarrow{\sigma^{(a_{h-1})}_{F(N+1)}} V_{F(N+1),k_h} \in V_{F(N+1),mark}$ that satisfies $\{r^{(0)}_{F(N+1)}, \ldots, r^{(a_0)}_{F(N+1)}\} \subseteq V_{F(N+1),k_0}, \{r^{(a_0+1)}_{F(N+1)}, \ldots, r^{(a_1)}_{F(N+1)}\} \subseteq V_{F(N+1),k_1}, \ldots, \{r^{(a_{h-1}+1)}_{F(N+1)}, \ldots, r^{(l)}_{F(N+1)}\} \subseteq V_{F(N+1),k_h}$ in the acyclic automaton $\mathcal{T}_{F(N+1)}$. Since $p_{F(N+1)} \in \mathcal{P}_{F(N+1)} \neq \emptyset$, the count $w_{FN}(p_{F(N+1)})$ of $p_{F(N+1)}$ satisfies $w_{FN}(p_{F(N+1)}) \leq w_{FN}$. Then there exists $i_{FN} \in I_{FN} - \bigcup_{k \in \{k_0,k_1,\ldots,k_h\}} J_{FN}(V_{F(N+1),k})$ such that $w_{FN,i_{FN}}(p_{F(N+1)}) = w_{FN}(p_{F(N+1)})$. By the definition of $w_{FN,i_{FN}}(p_{F(N+1)})$, there exists $a \in \{0, 1, \ldots, l-1\}$ such that $r^{(a)}_{F(N+1)}(i_{FN}) \notin Q \times \{d\}$, $r^{(a+1)}_{F(N+1)}(i_{FN}) \in Q \times \{d\}$, and

$$|\sigma^{(a)}_{F(N+1)}(i_{FN})\sigma^{(a+1)}_{F(N+1)}(i_{FN}) \ldots \sigma^{(l-1)}_{F(N+1)}(i_{FN})| = w_{FN}(p_{F(N+1)}).$$

Since $r^{(a)}_{F(N+1)}(i_{FN}) \notin Q \times \{d\}$, we have $s_{F(N+1)}(i_{FN}) \in K\Sigma^{\leq w_{FN}(p_{F(N+1)})} \subseteq K\Sigma^{\leq w_{FN}}$. This contradicts $s_{F(N+1)}(i_{FN}) \in F_0(w_{FN}) = L \cap (L-K)\Sigma^{\geq w_{FN}}$.

Next, we prove that $m^*_{FN} \geq w_{FN}$. For the sake of contradiction, we suppose that $m^*_{FN} < w_{FN}$. Since $0 \leq m^*_{FN}$, we have $0 < w_{FN}$, which implies $\mathcal{P}_{F(N+1)} \neq \emptyset$. In the acyclic automaton $\mathcal{T}_{F(N+1)}$, there exists a path $p_{F(N+1)} : V_{F(N+1),0} = V_{F(N+1),k_0} \xrightarrow{\sigma^{(k_0)}_{F(N+1)}} V_{F(N+1),k_1} \xrightarrow{\sigma^{(k_1)}_{F(N+1)}} \cdots \xrightarrow{\sigma^{(k_{h-1})}_{F(N+1)}} V_{F(N+1),k_h} \in V_{F(N+1),mark}$ ($h \geq 1$) whose count $w_{FN}(p_{F(N+1)})$ satisfies $w_{FN}(p_{F(N+1)}) = w_{FN}$. For any $i_{FN} \in \bigcup_{k \in \{k_0,k_1,\ldots,k_h\}} J_{FN}(V_{F(N+1),k})$, there exists $V_{F(N+1),k_{j_{i_{FN}}}}$ ($j_{i_{FN}} \in \{0, 1, \ldots, h\}$) such that

$$\begin{aligned}
&[\exists r_{F(N+1)} \in V_{F(N+1),k_{j_{i_{FN}}}} : r_{F(N+1)}(i_{FN}) \in Q \times \{d\}] \\
&\wedge[\exists r_{F(N+1)}, r'_{F(N+1)} \in V_{F(N+1),k_{j_{i_{FN}}}}, \exists \sigma_{F(N+1)} \in \Sigma_{F(N+1)} : \\
&\quad \alpha_{F(N+1)}(r_{F(N+1)}, \sigma_{F(N+1)}) = r'_{F(N+1)} \wedge \sigma_{F(N+1)}(i_{FN}) \neq \varepsilon].
\end{aligned} \tag{15}$$

Then there exists a path $r_{F(N+1),0} = r^{(0)}_{F(N+1)} \xrightarrow{\sigma^{(0)}_{F(N+1)}} r^{(1)}_{F(N+1)} \xrightarrow{\sigma^{(1)}_{F(N+1)}} \cdots \xrightarrow{\sigma^{(l-1)}_{F(N+1)}} r^{(l)}_{F(N+1)}$ ($l \geq 1$) in $T_{F(N+1)}$ such that, for some $a_0, a_1, \ldots, a_{h-1}$ with $a_0 < a_1 < \cdots < a_{h-1}$, $\{r^{(0)}_{F(N+1)}, \ldots, r^{(a_0)}_{F(N+1)}\} \subseteq V_{F(N+1),k_0}, \{r^{(a_0+1)}_{F(N+1)}, \ldots, r^{(a_1)}_{F(N+1)}\} \subseteq V_{F(N+1),k_1}, \ldots, \{r^{(a_{h-1}+1)}_{F(N+1)}, \ldots, r^{(l)}_{F(N+1)}\} \subseteq V_{F(N+1),k_h}$, $\sigma^{(a_p)}_{F(N+1)} = \sigma^{(k_p)}_{F(N+1)}$ ($p \in \{0, 1, \ldots, h-1\}$), and for $i_{FN} \in \bigcup_{k \in \{k_0,k_1,\ldots,k_h\}} J_{FN}(V_{F(N+1),k})$,

$$r^{(a_{j_{i_{FN}}-1}+1)}_{F(N+1)}(i_{FN}) \in Q \times \{d\}, \tag{16}$$

and

$$|\sigma^{(a_{j_{i_{FN}}-1}+1)}_{F(N+1)}(i_{FN})\sigma^{(a_{j_{i_{FN}}-1}+2)}_{F(N+1)}(i_{FN}) \ldots \sigma^{(l-1)}_{F(N+1)}(i_{FN})| \geq w_{FN} - 1. \tag{17}$$

In addition, for any $i_{FN} \in I_{FN} - \bigcup_{k\in\{k_0,k_1,\dots,k_h\}} J_{FN}(V_{F(N+1),k})$, since $w_{FN}(p_{F(N+1)}) = w_{FN}$, we have $w_{FN,i_{FN}}(p_{F(N+1)}) \geq w_{FN}$. Thus, there exists $j_{i_{FN}} \in \{0, 1, \dots, h\}$ such that Eqs. 16 and 17 hold.

Let $s_{F(N+1)} = \sigma^{(0)}_{F(N+1)}\sigma^{(1)}_{F(N+1)} \cdots \sigma^{(l-1)}_{F(N+1)} \in L(T_{F(N+1)})$. Since Eqs. 16 and 17 hold for each $i_{FN} \in I_{FN}$, it follows from $w_{FN} - 1 \geq m^*_{FN}$ that $s_{F(N+1)}(i_{FN}) \in L \cap (L - K)\Sigma^{\geq w_{FN}-1} \subseteq L \cap (L - K)\Sigma^{\geq m^*_{FN}} = F_0(m^*_{FN})$. In addition, for each $i_N \in I_N - I_{FN}$, we have $s_{F(N+1)}(i_N) \in K$. Then, by Propositions 1 and 2, we have $s_{F(N+1)}(0) \in F_{N+1}(m^*_{FN}) \neq \emptyset$, i.e., $m^*_{FN} \notin \mathbb{N}_{FN}$. This contradicts the definition of m^*_{FN}. □

Dually we use $\mathcal{T}_{H(N+1)}$ to compute m^*_{HN} when $\mathbb{N}_{HN} \neq \emptyset$. For each $p_{H(N+1)}$: $V_{H(N+1),0} = V_{H(N+1),k_0} \xrightarrow{\sigma^{(k_0)}_{H(N+1)}} V_{H(N+1),k_1} \xrightarrow{\sigma^{(k_1)}_{H(N+1)}} \cdots \xrightarrow{\sigma^{(k_{h-1})}_{H(N+1)}} V_{H(N+1),k_h} \in V_{H(N+1),mark}$ in the set $\mathcal{P}_{H(N+1)}$ of all paths from the initial state to marked states and each index $i_{HN} \in I_{HN} - \bigcup_{k\in\{k_0,k_1,\dots,k_h\}} J_{HN}(V_{H(N+1),k})$, we define an index-specific post-fault event count of each transition of the path as follows: For each $j \in \{1, 2, \dots, h\}$,

$$w_{HN,i_{HN}}(V_{H(N+1),k_{j-1}}, \sigma^{(k_{j-1})}_{H(N+1)}, V_{H(N+1),k_j}) := \begin{cases} 1, \text{ if } [\sigma^{(k_{j-1})}_{H(N+1)}(i_{HN}) \neq \varepsilon] \\ \quad \wedge[\exists r_{H(N+1)} \in V_{H(N+1),k_{j-1}}, \exists r'_{H(N+1)} \in V_{H(N+1),k_j} : \\ \quad\quad \alpha_{H(N+1)}(r_{H(N+1)}, \sigma^{(k_{j-1})}_{H(N+1)}) = r'_{H(N+1)} \wedge r'_{H(N+1)}(i_{HN}) \in Q \times \{d\}] \\ 0, \text{ otherwise.} \end{cases}$$

Then an index-specific post-fault event count for the path $p_{H(N+1)}$ is defined as

$$w_{HN,i_{HN}}(p_{H(N+1)}) := \sum_{j=1}^{h} w_{HN,i_{HN}}(V_{H(N+1),k_{j-1}}, \sigma^{(k_{j-1})}_{H(N+1)}, V_{H(N+1),k_j}).$$

Next, a minimum among all indices in $I_{HN} - \bigcup_{k\in\{k_0,k_1,\dots,k_h\}} J_{HN}(V_{H(N+1),k})$ is taken as the post-fault event count for the path $p_{H(N+1)}$:

$$w_{HN}(p_{H(N+1)}) := \min_{i_{HN}\in I_{HN}-\cup_{k\in\{k_0,\dots,k_h\}} J_{HN}(V_{H(N+1),k})} w_{HN,i_{HN}}(p_{H(N+1)}).$$

Finally a maximum of the counts along all paths in $\mathcal{P}_{H(N+1)}$ is taken as

$$w_{HN} := \begin{cases} \max_{p_{H(N+1)}\in\mathcal{P}_{H(N+1)}} w_{HN}(p_{H(N+1)}), & \text{if } \mathcal{P}_{H(N+1)} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{P}_{H(N+1)}$ is finite, w_{HN} is effectively computable.

Similar to Theorem 3, the following theorem holds, which shows that the value m^*_{HN} can be computed as $m^*_{HN} = w_{HN}$.

Theorem 4 *For the pair (L, K) of regular languages that is N-inference diagnosable, if $\mathbb{N}_{HN} \neq \emptyset$ then $m^*_{HN} = w_{HN}$.*

Remark 4 The result of Theorem 3 (respectively, Theorem 4) on computing the delay bound is reduced to that of Section IV of Qiu and Kumar (2006) (respectively, Theorem 2 of Yamamoto and Takai 2014) in the case of $N = 0$ and reduced to Theorem 3 (respectively, Theorem 4) of Yokota et al. (2017) in the case of $N = 1$.

Remark 5 In this remark, we discuss the complexity of computing the delay bound. To compute the delay bound m_N^* using Theorems 3 and 4, we first need to construct the finite automata $T_{F(N+1)}$ and $T_{H(N+1)}$, whose complexity is exponential with respect to the number n of local diagnosers and doubly exponential with respect to the levels N of inferencing, as shown in Table 1. Next, we construct the nondeterministic acyclic automata $\mathcal{T}_{F(N+1)}$ and $\mathcal{T}_{H(N+1)}$ by identifying all max-SCCs of $T_{F(N+1)}$ and $T_{H(N+1)}$. This complexity is $O((|R_{F(N+1)}| + |R_{H(N+1)}|) \times |\Sigma|^{1+\sum_{k=0}^{N} n(n-1)^k})$, where

$$|R_{F(N+1)}| = \begin{cases} O(|Q|^{1+\sum_{k=1}^{N/2} n(n-1)^{2k-1}} \cdot |Q_K|^{1+\sum_{k=0}^{N} n(n-1)^k}), & \text{if } N \text{ is an even number} \\ O(|Q|^{1+\sum_{k=1}^{(N+1)/2} n(n-1)^{2k-1}} \cdot |Q_K|^{1+\sum_{k=0}^{N} n(n-1)^k}), & \text{if } N \text{ is an odd number,} \end{cases}$$

$$|R_{H(N+1)}| = \begin{cases} O(|Q|^{\sum_{k=0}^{N/2} n(n-1)^{2k}} \cdot |Q_K|^{1+\sum_{k=0}^{N} n(n-1)^k}), & \text{if } N \text{ is an even number} \\ O(|Q|^{\sum_{k=0}^{(N-1)/2} n(n-1)^{2k}} \cdot |Q_K|^{1+\sum_{k=0}^{N} n(n-1)^k}), & \text{if } N \text{ is an odd number,} \end{cases}$$

and it is exponential with respect to the number n of local diagnosers and doubly exponential with respect to the levels N of inferencing. Then m_N^* can be obtained by exploring all paths of $\mathcal{T}_{F(N+1)}$ and $\mathcal{T}_{H(N+1)}$ that end with marked states. It turns out that the complexity of the delay bound computation is of the same order as that of verifying N-inference diagnosability—the former computes an exact delay bound while the latter checks only the existence of a finite delay bound.

Remark 6 Since the finite automaton $\mathcal{T}_{F(N+1)}$ is acyclic, the number of transitions of a path in $\mathcal{P}_{F(N+1)}$ is bounded by $|V_{F(N+1)}| - 1$, where $V_{F(N+1)}$ is the state set of $\mathcal{T}_{F(N+1)}$. By Theorem 3, m_{FN}^* is bounded by $|V_{F(N+1)}|-1$. Similarly, m_{HN}^* is bounded by $|V_{H(N+1)}|-1$, where $V_{H(N+1)}$ is the state set of $\mathcal{T}_{H(N+1)}$.

The following example illustrates the results on computing the delay bound.

Table 1 Complexity for constructing automata $T_{F(N+1)}$ and $T_{H(N+1)}$ (Takai and Kumar 2017)

	Complexity of constructing $T_{F(N+1)}$
Arbitrary even N	$O(\|Q\|^{1+\sum_{k=1}^{N/2} n(n-1)^{2k-1}} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{1+\sum_{k=0}^{N} n(n-1)^k})$
	$= O(\|Q\|^{O(n(n-1)^{N-1})} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{O(n(n-1)^N)})$
Arbitrary odd N	$O(\|Q\|^{1+\sum_{k=1}^{(N+1)/2} n(n-1)^{2k-1}} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{1+\sum_{k=0}^{N} n(n-1)^k})$
	$= O((\|Q\| \cdot \|Q_K\| \cdot \|\Sigma\|)^{O(n(n-1)^N)})$
$N=1$	$O(\|Q\|^{n^2-n+1} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{n^2+1})$
$N=2$	$O(\|Q\|^{n^2-n+1} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{n^2+n(n-1)^2+1})$
	Complexity of constructing $T_{H(N+1)}$
Arbitrary even N	$O(\|Q\|^{\sum_{k=0}^{N/2} n(n-1)^{2k}} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{1+\sum_{k=0}^{N} n(n-1)^k})$
	$= O((\|Q\| \cdot \|Q_K\| \cdot \|\Sigma\|)^{O(n(n-1)^N)})$
Arbitrary odd N	$O(\|Q\|^{\sum_{k=0}^{(N-1)/2} n(n-1)^{2k}} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{1+\sum_{k=0}^{N} n(n-1)^k})$
	$= O(\|Q\|^{O(n(n-1)^{N-1})} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{O(n(n-1)^N)})$
$N=1$	$O(\|Q\|^{n} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{n^2+1})$
$N=2$	$O(\|Q\|^{n+n(n-1)^2} \cdot (\|Q_K\| \cdot \|\Sigma\|)^{n^2+n(n-1)^2+1})$

Example 2 We consider the setting of Example 1, where (L, K) is 2-inference diagnosable. As shown in Example 1, we have $F_3(m) = H_3(m) = \emptyset$ for any $m \in \mathbb{N}$ with $m \geq 1$, which implies $\mathbb{N}_{F2} \neq \emptyset$ and $\mathbb{N}_{H2} \neq \emptyset$.

We compute the delay bound $m_2^* = \min\{m_{F2}^*, m_{H2}^*\}$ using Theorems 3 and 4. We construct the acyclic automaton $\mathcal{T}_{F3}$ to compute m_{F2}^*. A part of $\mathcal{T}_{F3}$, which includes a path from the initial state to a marked state in $V_{F3,mark}$, is shown in Fig. 3. (By the definition of $\mathcal{T}_{F3}$, each state of $\mathcal{T}_{F3}$ is a set of states of T_{F3}. Note that this example is a special case where each state of $\mathcal{T}_{F3}$ is a singleton.) Since $\mathcal{P}_{F3} \neq \emptyset$, we need to compute $w_{F2}(p_{F3})$ for all paths p_{F3} in $\mathcal{P}_{F3}$. For example, we consider the path $p_{F3} : V_{F3,0} = V_{F3,k_0} \xrightarrow{(e,e,e,e,e,e,e)} V_{F3,k_1} \xrightarrow{(f,b_2,b_2,b_2,a_2,a_2,a_2)} V_{F3,k_2} \xrightarrow{(\varepsilon,\varepsilon,f,\varepsilon,\varepsilon,f,\varepsilon)} V_{F3,k_3}$, shown in Fig. 3. Since $J_{F2}(V_{F3,k_0}) \cup J_{F2}(V_{F3,k_1}) \cup J_{F2}(V_{F3,k_2}) \cup J_{F2}(V_{F3,k_3}) = \emptyset$, the count of p_{F3} is given as

$$w_{F2}(p_{F3}) = \min_{i_{F2} \in I_{F2} = \{0,11,21\}} w_{F2,i_{F2}}(p_{F3}).$$

Since $w_{F2,i_{F2}}(p_{F3}) = 1$ for each $i_{F2} \in I_{F2}$, we have $w_{F2}(p_{F3}) = 1$. For other paths in $\mathcal{P}_{F3}$, their counts are computed in the same way. Then, by Theorem 3, we have $m_{F2}^* = w_{F2} = 1$. Similarly, by applying Theorem 4, we have $m_{H2}^* = w_{H2} = 1$. Consequently, the delay bound m_2^* is obtained as $m_2^* = \min\{m_{F2}^*, m_{H2}^*\} = 1$.

Remark 7 As shown in Example 1, this example is not 1-inference diagnosable, i.e., it is neither disjunctive-codiagnosable nor conjunctive-codiagnosable, and also it is nether conditionally disjunctive-codiagnosable nor conditionally conjunctive-codiagnosable, meaning that the delay bound for diagnosis in those schemes is not even defined (or may be considered to be infinity). In contrast the delay bound of 2-inference diagnosability is bounded, and in fact just 1.

We next show the following expected anti-monotonicity property that, for an N-inference diagnosable system, as the levels N of inferencing are increased, the delay bounds become smaller.

Theorem 5 *For any $N \in \mathbb{N}$ such that the pair (L, K) of regular languages is N-inference diagnosable, it holds that $m_{N+1}^* \leq m_N^*$.*

Proof Since (L, K) is N-inference diagnosable, $F_{N+1}(m_N^*) = \emptyset \vee H_{N+1}(m_N^*) = \emptyset$ holds, which implies $F_{N+2}(m_N^*) = \emptyset \vee H_{N+2}(m_N^*) = \emptyset$. Then, (L, K) is $(N + 1)$-inference

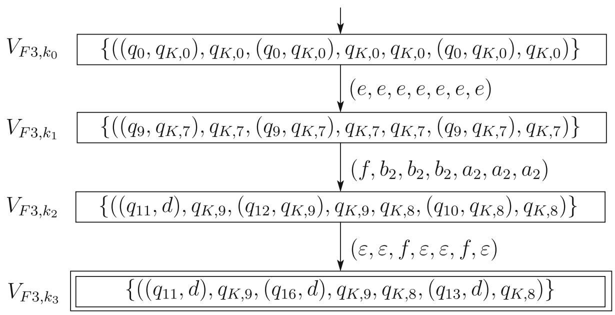

Fig. 3 A part of the automaton $\mathcal{T}_{F_3}$

diagnosable and $m_N^* \in \{m \in \mathbb{N} \mid F_{N+2}(m) = \emptyset \vee H_{N+2}(m) = \emptyset\}$, which implies $m_{N+1}^* \leq m_N^*$. □

As shown in the following example, the converse relation of Theorem 5 need not hold.

Example 3 We consider a plant modeled by the finite automaton G shown in Fig. 4a, which is obtained by slightly modifying the automaton of Fig. 2a. Let $\Delta_1 = \{a, a', c, c', d, e\}$, $\Delta_2 = \{b, b', c, c', d, e\}$, and

$$M_1(\sigma) = \begin{cases} a, & \text{if } \sigma \in \{a_1, a_2\} \\ a', & \text{if } \sigma \in \{a_1', a_2'\} \\ \sigma, & \text{if } \sigma \in \{c, c', d, e\} \\ \varepsilon, & \text{otherwise,} \end{cases}$$

$$M_2(\sigma) = \begin{cases} b, & \text{if } \sigma \in \{b_1, b_2\} \\ b', & \text{if } \sigma \in \{b_1', b_2'\} \\ \sigma, & \text{if } \sigma \in \{c, c', d, e\} \\ \varepsilon, & \text{otherwise.} \end{cases}$$

In addition, let $K \subseteq L$ be a closed regular language generated by the finite automaton G_K shown in Fig. 4b. We can verify that (L, K) is 1-inference diagnosable and that $\mathbb{N}_{F1} \neq \emptyset$ and $\mathbb{N}_{H1} \neq \emptyset$. Then it is also 2-inference diagnosable.

Similar to Example 2, we can obtain the delay bound $m_2^* = 1$ in the case of $N = 2$. We show that the delay bound m_1^* in the case of $N = 1$ is larger than m_2^*. To compute m_{F1}^* and m_{H1}^*, the acyclic automata $\mathcal{T}_{F2}$ and $\mathcal{T}_{H2}$ are constructed. A part of $\mathcal{T}_{F2}$ shown in Fig. 5

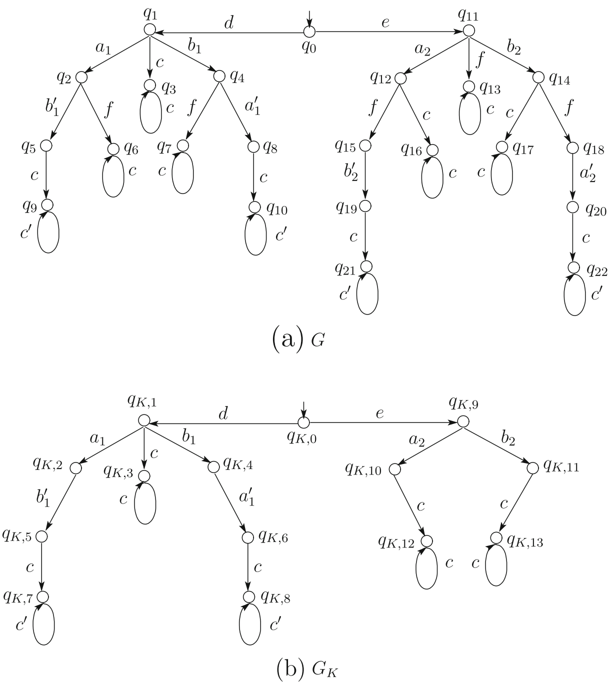

Fig. 4 Automata G and G_K of Example 3

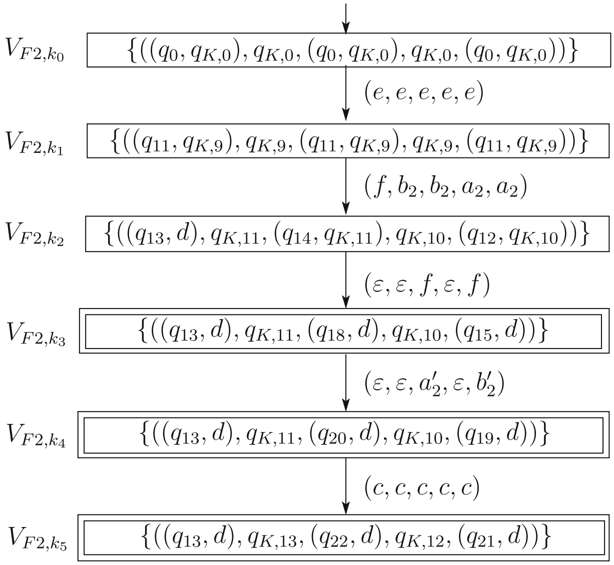

Fig. 5 A part of the automaton $\mathcal{T}_{F_2}$

includes a path $p_{F2} : V_{F2,0} = V_{F2,k_0} \xrightarrow{(e,e,e,e,e)} V_{F2,k_1} \xrightarrow{(f,b_2,b_2,a_2,a_2)} V_{F2,k_2} \xrightarrow{(\varepsilon,\varepsilon,f,\varepsilon,f)} V_{F2,k_3} \xrightarrow{(\varepsilon,\varepsilon,a_2',\varepsilon,b_2')} V_{F2,k_4} \xrightarrow{(c,c,c,c,c)} V_{F2,k_5}$ in $\mathcal{P}_{F2}$. For this path p_{F2}, since $J_{F1}(V_{F2,k_0}) \cup J_{F1}(V_{F2,k_1}) \cup J_{F1}(V_{F2,k_2}) \cup J_{F1}(V_{F2,k_3}) \cup J_{F1}(V_{F2,k_4}) \cup J_{F1}(V_{F2,k_5}) = \emptyset$, the count of p_{F2} is given as

$$w_{F1}(p_{F2}) = \min_{i_{F1} \in I_{F1} = \{0,11,21\}} w_{F1,i_{F1}}(p_{F2}).$$

Since $w_{F1,0}(p_{F2}) = 2$ and $w_{F1,11}(p_{F1}) = w_{F1,21}(p_{F2}) = 3$, we have $w_{F1}(p_{F2}) = 2$. For other paths in $\mathcal{P}_{F2}$, their counts are computed in the same way. Then, by Theorem 3, we have $m_{F1}^* = w_{F1} = 2$. Similarly, by applying Theorem 4, we have $m_{H1}^* = w_{H1} = 2$. Consequently, the delay bound m_1^* is obtained as $m_1^* = \min\{m_{F1}^*, m_{H1}^*\} = 2$, which is larger than the delay bound $m_2^*(= 1)$ in the case of $N = 2$.

For example, we consider a situation where the event c is executed after a failure trace $ef \in L - K$. The first diagnoser D_1 cannot distinguish $efc \in L - K$ from a nonfailure trace $eb_2c \in K$. In addition, the second diagnoser D_2 cannot distinguish $eb_2c \in K$ from a failure trace $eb_2fa_2'c \in L - K$. Thus, D_1 cannot detect the occurrence of the failure event f using a single-level of inferencing. Analogously, D_2 cannot detect the occurrence of f using a single-level of inferencing.

On the other hand, if the plant executes $eb_2fa_2'c \in L - K$, then D_1 can detect the occurrence of f unambiguously and can issue a failure decision with the ambiguity 0. Then, for $eb_2c \in K$, D_2 can issue a nonfailure decision with the ambiguity level 1 and, for $efc \in L - K$, D_1 can issue a failure decision with the ambiguity level 2. Similarly, D_2 can issue a failure decision with the ambiguity level 2 for $efc \in L - K$. Using 2 levels of inferencing, the occurrence of f is detected after c is executed, i.e., within one step delay.

5 Computation of ambiguity levels

Online diagnosis requires the computation of the ambiguity levels $n_i^f(M_i(s))$ and $n_i^h(M_i(s))$ of the local failure and nonfailure decisions, respectively, in accordance with Eqs. 8 and 9. (As before we continue to assume that $I = \{1, 2\}$). To compute the language $H_k(m)$ for each $m \in \mathcal{N}$ and each $k \in \{1, 2, \ldots, N+1\}$, we use the finite automaton T_{Hk_m} defined in Section 3.3, but replace each transition label $\sigma_{Hk} \in \Sigma_{Hk}$ with its first element $\sigma_{Hk}(0) \in \Sigma \cup \{\varepsilon\}$ to obtain a nondeterministic finite automaton

$$\hat{T}_{Hk_m} = (R_{Hk_m}, \Sigma, \hat{\alpha}_{Hk_m}, r_{Hk_m,0}, R_{Hk_m,mark})$$

over Σ, where the nondeterministic transition function $\hat{\alpha}_{Hk_m} : R_{Hk_m} \times (\Sigma \cup \{\varepsilon\}) \to 2^{R_{Hk_m}}$ is defined as follows: For each $r_{Hk_m} \in R_{Hk_m}$ and each $\sigma \in \Sigma \cup \{\varepsilon\}$,

$$\hat{\alpha}_{Hk_m}(r_{Hk_m}, \sigma) = \{r'_{Hk_m} \in R_{Hk_m} \mid \exists \sigma_{Hk} \in \Sigma_{Hk} : \sigma_{Hk}(0) = \sigma \wedge \alpha_{Hk_m}(r_{Hk_m}, \sigma_{Hk}) = r'_{Hk_m}\}.$$

In addition, for $k = 0$, we regard $G_K = (Q_K, \Sigma, \delta_K, q_{K,0}, Q_K)$ as a nondeterministic automaton

$$\hat{T}_{H0_m} = (R_{H0_m}, \Sigma, \hat{\alpha}_{H0_m}, r_{H0_m,0}, R_{H0_m,mark}),$$

i.e., the transition function $\hat{\alpha}_{H0_m} : R_{H0_m} \times (\Sigma \cup \{\varepsilon\}) \to 2^{R_{H0_m}}$ is defined as

$$\hat{\alpha}_{H0_m}(q_K, \sigma) = \begin{cases} \{\delta_K(q_K, \sigma)\}, & \text{if } \sigma \neq \varepsilon \wedge \delta_K(q_K, \sigma)! \\ \emptyset, & \text{otherwise} \end{cases}$$

for each $q_K \in R_{H0_m} = Q_K$ and each $\sigma \in \Sigma \cup \{\varepsilon\}$.

The following proposition shows that the nondeterministic automaton $\hat{T}_{Hk_m}$ accepts the language $H_k(m)$.

Proposition 6 *For any $m \in \mathbb{N}$ and any $k \in \mathbb{N}$ such that $0 \leq k \leq N + 1$, $L_m(\hat{T}_{Hk_m}) = H_k(m)$.*

Proof First, we prove that $L_m(\hat{T}_{Hk_m}) \subseteq H_k(m)$. We consider any $s \in L_m(\hat{T}_{Hk_m})$. If $k = 0$, then $s \in L_m(G_K) = L(G_K) = K = H_0(m)$ by the definition of $\hat{T}_{H0_m}$. We consider the case of $0 < k \leq N + 1$. By the definition of $\hat{T}_{Hk_m}$, there exists $s_{Hk} \in L_m(T_{Hk_m})$ such that $s_{Hk}(0) = s$. Let $r_{Hk_m} = \alpha_{Hk_m}(r_{Hk_m,0}, s_{Hk})$ and $s_{ii'} = s_{Hk}(ii')$ for each $i \in I$ and each $i' \in \{0, 1, \ldots, k-1\}$. Since $r_{Hk_m}(0) = \delta_K(q_{K,0}, s) \in Q_K$, we have $s \in K = H_0(m)$. By Proposition 1, it suffices to show that the four conditions of the second part of Proposition 1 hold.

For each $i \in I$ and each $i' \in \{0, 1, \ldots, k-1\}$, if i' is an even number, then $ii' \in I_{HN}$. By the definition of the marked state set $R_{Hk_m,mark}$ of T_{Hk_m}, we have $r_{Hk_m}(ii') = \xi_m((q_0, q_{K,0}), s_{ii'}) \in Q \times \{d_m\}$, which implies $s_{ii'} \in L_m(G \parallel \tilde{G}_{K_m}) = F_0(m)$. If i' is an odd number, then $r_{Hk_m}(ii') = \delta_K(q_{K,0}, s_{ii'}) \in Q_K$, which implies $s_{ii'} \in K = H_0(m)$. It follows that the first condition of the second part of Proposition 1 holds. In addition, by Proposition 4, the last three of the four conditions of the second part of Proposition 1 are satisfied.

Next, we prove that $H_k(m) \subseteq L_m(\hat{T}_{Hk_m})$. We consider any $s \in H_k(m)$. If $k = 0$, then $s \in H_0(m) = K = L(G_K) = L_m(G_K) = L_m(\hat{T}_{H0_m})$. We consider the case of $0 < k \leq N + 1$. By Proposition 1, $s \in H_0(m)$ holds and there exist $s_{10}, s_{11}, \ldots, s_{1(k-1)}, s_{20}, s_{21}, \ldots, s_{2(k-1)} \in L$ that satisfy the four conditions of the second part of Proposition 1. Then, by Proposition 4, there exists $s_{Hk} \in L(T_{Hk_m})$ such that

$s_{Hk}(0) = s$ and $s_{Hk}(ii') = s_{ii'}$ for each $i \in I$ and each $i' \in \{0, 1, \ldots, k-1\}$. Let $r_{Hk_m} = \alpha_{Hk_m}(r_{Hk_m,0}, s_{Hk})$. For any $i \in I$ and any $i' \in \{0, 1, \ldots, k-1\}$, if $ii' \in I_{HN}$, then i' is an even number, so we have $s_{Hk}(ii') = s_{ii'} \in F_0(m) = L_m(G \parallel \tilde{G}_{K_m})$. It follows that $r_{Hk_m}(ii') = \xi_m((q_0, q_{K,0}), s_{Hk}(ii')) \in Q \times \{d_m\}$. By the definition of the marked state set $R_{Hk_m,mark}$, we have $r_{Hk_m} \in R_{Hk_m,mark}$ and $s_{Hk} \in L_m(T_{Hk_m})$. Since $s_{Hk}(0) = s$, we have $s \in L_m(\hat{T}_{Hk_m})$. □

Dually to $\hat{T}_{Hk_m}$, we obtain a nondeterministic finite automaton $\hat{T}_{Fk_m}$ from T_{Fk_m} as follows:

$$\hat{T}_{Fk_m} = (R_{Fk_m}, \Sigma, \hat{\alpha}_{Fk_m}, r_{Fk_m,0}, R_{Fk_m,mark}),$$

where the nondeterministic transition function $\hat{\alpha}_{Fk_m} : R_{Fk_m} \times (\Sigma \cup \{\varepsilon\}) \to 2^{R_{Fk_m}}$ is defined as follows: For each $r_{Fk_m} \in R_{Fk_m}$ and each $\sigma \in \Sigma \cup \{\varepsilon\}$,

$$\hat{\alpha}_{Fk_m}(r_{Fk_m}, \sigma) = \{r'_{Fk_m} \in R_{Fk_m} \mid \exists \sigma_{Fk} \in \Sigma_{Fk} : \sigma_{Fk}(0) = \sigma \wedge \alpha_{Fk_m}(r_{Fk_m}, \sigma_{Fk}) = r'_{Fk_m}\}.$$

In addition, for $k = 0$, we regard $G \parallel \tilde{G}_{K_m} = (Q \times \tilde{Q}_{K_m}, \Sigma, \xi_m, (q_0, q_{K,0}), Q \times \{d_m\})$ as a nondeterministic automaton

$$\hat{T}_{F0_m} = (R_{F0_m}, \Sigma, \hat{\alpha}_{F0_m}, r_{F0_m,0}, R_{F0_m,mark}),$$

i.e., the transition function $\hat{\alpha}_{F0_m} : R_{F0_m} \times (\Sigma \cup \{\varepsilon\}) \to 2^{R_{F0_m}}$ is defined as

$$\hat{\alpha}_{F0_m}((q, \tilde{q}_{K_m}), \sigma) = \begin{cases} \{\xi_m((q, \tilde{q}_{K_m}), \sigma)\}, & \text{if } \sigma \neq \varepsilon \wedge \xi_m((q, \tilde{q}_{K_m}), \sigma)! \\ \emptyset, & \text{otherwise} \end{cases}$$

for each $(q, \tilde{q}_{K_m}) \in R_{F0_m} = Q \times \tilde{Q}_{K_m}$ and each $\sigma \in \Sigma \cup \{\varepsilon\}$.

Similar to Proposition 6, the following proposition is obtained, which shows that the nondeterministic automaton $\hat{T}_{Fk_m}$ accepts the language $F_k(m)$.

Proposition 7 *For any $m \in \mathbb{N}$ and any $k \in \mathbb{N}$ such that $0 \leq k \leq N+1$, $L_m(\hat{T}_{Fk_m}) = F_k(m)$.*

For the online computation of the ambiguity level $n_i^f(M_i(s))$ of the local failure decision for each $M_i(s) \in M_i(L)$, we define a state estimate function $E_{Hk_m,i} : \Delta_i^* \to 2^{R_{Hk_m}}$ for each $i \in I$ and each $k \in \{0, 1, \ldots, N+1\}$ as follows:

- $E_{Hk_m,i}(\varepsilon) = UR_{\hat{T}_{Hk_m},i}(\{r_{Hk_m,0}\})$, and
- $\forall t \in \Delta_i^*, \forall \sigma_{M_i} \in \Delta_i$:

$$E_{Hk_m,i}(t\sigma_{M_i}) = UR_{\hat{T}_{Hk_m},i}\left(\bigcup_{r_{Hk_m} \in E_{Hk_m,i}(t)} \bigcup_{\sigma \in \Sigma \cap M_i^{-1}(\sigma_{M_i})} \hat{\alpha}_{Hk_m}(r_{Hk_m}, \sigma)\right).$$

Similarly, for the online computation of the ambiguity level $n_i^h(M_i(s))$ of the nonfailure decision for each $M_i(s) \in M_i(L)$, a state estimate function $E_{Fk_m,i} : \Delta_i^* \to 2^{R_{Fk_m}}$ is defined for each $i \in I$ and each $k \in \{0, 1, \ldots, N+1\}$ as follows:

- $E_{Fk_m,i}(\varepsilon) = UR_{\hat{T}_{Fk_m},i}(\{r_{Fk_m,0}\})$, and

– $\forall t \in \Delta_i^*, \forall \sigma_{M_i} \in \Delta_i$:

$$E_{Fk_m,i}(t\sigma_{M_i}) = UR_{\hat{T}_{Fk_m},i}\left(\bigcup_{r_{Fk_m}\in E_{Fk_m,i}(t)}\ \bigcup_{\sigma\in\Sigma\cap M_i^{-1}(\sigma_{M_i})} \hat{\alpha}_{Fk_m}(r_{Fk_m},\sigma)\right).$$

Then it holds that

$$E_{Hk_m,i}(t) = \{r_{Hk_m} \in R_{Hk_m} \mid \exists s \in M_i^{-1}(t) \cap L(\hat{T}_{Hk_m}) : r_{Hk_m} \in \hat{\alpha}_{Hk_m}(r_{Hk_m,0}, s)\},$$

$$E_{Fk_m,i}(t) = \{r_{Fk_m} \in R_{Fk_m} \mid \exists s \in M_i^{-1}(t) \cap L(\hat{T}_{Fk_m}) : r_{Fk_m} \in \hat{\alpha}_{Fk_m}(r_{Fk_m,0}, s)\},$$

$$t \notin M_i(H_k(m)) \Leftrightarrow E_{Hk_m,i}(t) \cap R_{Hk_m,mark} = \emptyset,$$

and

$$t \notin M_i(F_k(m)) \Leftrightarrow E_{Fk_m,i}(t) \cap R_{Fk_m,mark} = \emptyset$$

for each $t \in \Delta_i^*$. Accordingly, the following result on the online computation of the ambiguity levels $n_i^f(M_i(s))$ and $n_i^h(M_i(s))$ is obtained.

Theorem 6 *Consider any $m \in \mathbb{N}$ such that $F_{N+1}(m) = \emptyset$ or $H_{N+1}(m) = \emptyset$.*

1. *For each $s \in L$ and each $i \in I$, if $\{k \in \{0, 1, \ldots, N+1\} \mid E_{Hk_m,i}(M_i(s)) \cap R_{Hk_m,mark} = \emptyset\} \neq \emptyset$, then*

$$n_i^f(M_i(s)) = \min\{k \in \{0, 1, \ldots, N+1\} \mid E_{Hk_m,i}(M_i(s)) \cap R_{Hk_m,mark} = \emptyset\};$$

   otherwise $n_i^f(M_i(s)) = N+2$.

2. *For each $s \in L$ and each $i \in I$, if $\{k \in \{0, 1, \ldots, N+1\} \mid E_{Fk_m,i}(M_i(s)) \cap R_{Fk_m,mark} = \emptyset\} \neq \emptyset$, then*

$$n_i^h(M_i(s)) = \min\{k \in \{0, 1, \ldots, N+1\} \mid E_{Fk_m,i}(M_i(s)) \cap R_{Fk_m,mark} = \emptyset\};$$

   otherwise $n_i^h(M_i(s)) = N+2$.

Remark 8 In this remark we discuss the complexity of computing the ambiguity levels. This requires us to construct the finite automata T_{Fk_m} and T_{Hk_m} ($k = 1, 2, \ldots, N+1$). The complexity of their construction is exponential with respect to the number n of local diagnosers and doubly exponential with respect to the levels k of inferencing, as shown in Table 2. Next, we construct the nondeterministic automata $\hat{T}_{Fk_m}$ and $\hat{T}_{Hk_m}$ by replacing each event of the form $(\sigma, \sigma_{10}, \sigma_{11}, \ldots, \sigma_{1(k-1)}, \sigma_{20}, \sigma_{21}, \ldots, \sigma_{2(k-1)})$ in T_{Fk_m} and T_{Hk_m} with its first element σ. Note that the sizes of $\hat{T}_{Fk_m}$ and $\hat{T}_{Hk_m}$ are the same as those of T_{Fk_m} and T_{Hk_m}, respectively. In addition, for $k = 0$, $\hat{T}_{F0_m}$ and $\hat{T}_{H0_m}$ are the same as $G \parallel \tilde{G}_{K_m}$ and G_K, respectively, and the complexity of constructing $G \parallel \tilde{G}_{K_m}$ is $O(|Q| \cdot (|Q_K| + m) \cdot \Sigma)$.

In online diagnosis, the ambiguity levels $n_i^f(M_i(s))$ and $n_i^h(M_i(s))$ of the local failure and nonfailure decisions are computed by updating the values of the state estimates $E_{Hk_m,i}(M_i(s))$ and $E_{Fk_m,i}(M_i(s))$, respectively, upon each local event observation. The complexities of updating $E_{Hk_m,i}(M_i(s))$ and $E_{Fk_m,i}(M_i(s))$ are $O(|R_{Hk_m}| \cdot |\Sigma|^{1+\sum_{l=1}^{k} n(n-1)^{l-1}})$ and $O(|R_{Fk_m}| \cdot |\Sigma|^{1+\sum_{l=1}^{k} n(n-1)^{l-1}})$, i.e., linear in the sizes of T_{Fk_m} and T_{Hk_m}, respectively.

Remark 9 Note that if $m_N^* = m_{FN}^*$ (respectively, $m_N^* = m_{HN}^*$), then $F_{N+1}(m_N^*) = \emptyset$ (respectively, $H_{N+1}(m_N^*) = \emptyset$), which implies $t \notin M_i(F_{N+1}(m_N^*))$ (respectively, $t \notin$

Table 2 Complexity for constructing automata T_{Fk_m} and T_{Hk_m}

	Complexity of constructing T_{Fk_m}
Arbitrary even k	$O((\|Q\| \cdot (\|Q_K\| + m))^{1+\sum_{l=1}^{k/2} n(n-1)^{2l-1}} \cdot \|Q_K\|^{\sum_{l=1}^{k/2} n(n-1)^{2l-2}} \cdot \|\Sigma\|^{1+\sum_{l=1}^{k} n(n-1)^{l-1}}) = O((\|Q\| \cdot (\|Q_K\| + m))^{O(n(n-1)^{k-1})} \cdot \|Q_K\|^{O(n(n-1)^{k-2})} \cdot \|\Sigma\|^{O(n(n-1)^{k-1})})$
Arbitrary odd k ($k \geq 3$)	$O((\|Q\| \cdot (\|Q_K\| + m))^{1+\sum_{l=1}^{(k-1)/2} n(n-1)^{2l-1}} \cdot \|Q_K\|^{\sum_{l=1}^{(k+1)/2} n(n-1)^{2l-2}} \cdot \|\Sigma\|^{1+\sum_{l=1}^{k} n(n-1)^{l-1}}) = O((\|Q\| \cdot (\|Q_K\| + m))^{O(n(n-1)^{k-2})} \cdot \|Q_K\|^{O(n(n-1)^{k-1})} \cdot \|\Sigma\|^{O(n(n-1)^{k-1})})$
$k = 1$	$O(\|Q\| \cdot (\|Q_K\| + m) \cdot \|Q_K\|^n \cdot \|\Sigma\|^{n+1})$
$k = 2$	$O((\|Q\| \cdot (\|Q_K\| + m))^{n^2-n+1} \cdot \|Q_K\|^n \cdot \|\Sigma\|^{n^2+1})$
	Complexity of constructing T_{Hk_m}
Arbitrary even k	$O((\|Q\| \cdot (\|Q_K\| + m))^{\sum_{l=1}^{k/2} n(n-1)^{2l-2}} \cdot \|Q_K\|^{1+\sum_{l=1}^{k/2} n(n-1)^{2l-1}} \cdot \|\Sigma\|^{1+\sum_{l=1}^{k} n(n-1)^{l-1}}) = O((\|Q\| \cdot (\|Q_K\| + m))^{O(n(n-1)^{k-2})} \cdot \|Q_K\|^{O(n(n-1)^{k-1})} \cdot \|\Sigma\|^{O(n(n-1)^{k-1})})$
Arbitrary odd k ($k \geq 3$)	$O((\|Q\| \cdot (\|Q_K\| + m))^{\sum_{l=1}^{(k+1)/2} n(n-1)^{2l-2}} \cdot \|Q_K\|^{1+\sum_{l=1}^{(k-1)/2} n(n-1)^{2l-1}} \cdot \|\Sigma\|^{1+\sum_{l=1}^{k} n(n-1)^{l-1}}) = O((\|Q\| \cdot (\|Q_K\| + m))^{O(n(n-1)^{k-1})} \cdot \|Q_K\|^{O(n(n-1)^{k-2})} \cdot \|\Sigma\|^{O(n(n-1)^{k-1})})$
$k = 1$	$O((\|Q\| \cdot (\|Q_K\| + m))^n \cdot \|Q_K\| \cdot \|\Sigma\|^{n+1})$
$k = 2$	$O((\|Q\| \cdot (\|Q_K\| + m))^n \cdot \|Q_K\|^{n^2-n+1} \cdot \|\Sigma\|^{n^2+1})$

$M_i(H_{N+1}(m_N^*)))$ for any $i \in I$ and any $t \in M_i(L)$. Thus, if $m_N^* = m_{HN}^*$, then $n_i^f(M_i(s))$ can be computed for each $s \in L$ as follows: If $\{k \in \{0, 1, \ldots, N\} \mid E_{Hk_m,i}(M_i(s)) \cap R_{Hk_m,mark} = \emptyset\} \neq \emptyset$, then

$$n_i^f(M_i(s)) = \min\{k \in \{0, 1, \ldots, N\} \mid E_{Hk_m,i}(M_i(s)) \cap R_{Hk_m,mark} = \emptyset\};$$

otherwise $n_i^f(M_i(s)) = N + 1$, i.e., we do not need to construct $\hat{T}_{H(N+1)_m}$ for $k = N + 1$. Similarly, if $m_N^* = m_{FN}^*$, then $n_i^h(M_i(s))$ can be computed for each $s \in L$ as follows: If $\{k \in \{0, 1, \ldots, N\} \mid E_{Fk_m,i}(M_i(s)) \cap R_{Fk_m,mark} = \emptyset\} \neq \emptyset$, then

$$n_i^h(M_i(s)) = \min\{k \in \{0, 1, \ldots, N\} \mid E_{Fk_m,i}(M_i(s)) \cap R_{Fk_m,mark} = \emptyset\};$$

otherwise $n_i^h(M_i(s)) = N + 1$.

Remark 10 It would appear that to construct a local diagnoser at the ith local site, we may not need to take the information of the ith local diagnoser into account in the finite automata $\hat{T}_{Fk_m}$ and $\hat{T}_{Hk_m}$ ($k = 1, 2, \ldots, N + 1$). But this does not really help in reducing the complexity. To see this, we consider any $s \in L$ such that $M_i(s) \in M_i(F_k(m))$ and $M_i(s) \in M_i(H_k(m))$, where $k \in \{0, 1, \ldots, N\}$. We can verify that

$$M_i(s) \notin M_i(H_{k+1}(m)) \Leftrightarrow M_i(s) \notin M_i(H_k(m) \cap M_j^{-1} M_j(F_k(m))), \tag{18}$$

$$M_i(s) \notin M_i(F_{k+1}(m)) \Leftrightarrow M_i(s) \notin M_i(F_k(m) \cap M_j^{-1} M_j(H_k(m))), \tag{19}$$

where $\{j\} = I - \{i\}$. By Eqs. 18 and 19, acceptors of the languages $H_k(m) \cap M_j^{-1} M_j(F_k(m))$ and $F_k(m) \cap M_j^{-1} M_j(H_k(m))$ can be used, instead of, $\hat{T}_{H(k+1)_m}$ and $\hat{T}_{F(k+1)_m}$, respectively, for computing the ambiguity levels using Eqs. 8 and 9. In the case of $k = 0$, for $H_0(m) \cap M_j^{-1} M_j(F_0(m))$ and $F_0(m) \cap M_j^{-1} M_j(H_0(m))$, there exist their acceptors whose state sets are $Q_K \times (Q \times \tilde{Q}_{K_m})$. Since $R_{H1_m} = Q_K \times (Q \times \tilde{Q}_{K_m}) \times (Q \times \tilde{Q}_{K_m})$ and $R_{F1_m} = (Q \times \tilde{Q}_{K_m}) \times Q_K \times Q_K$, such acceptors have smaller state sets than $\hat{T}_{H1_m}$ and $\hat{T}_{F1_m}$. However, for $k = 1$, if we construct an acceptor for $H_1(m) \cap M_j^{-1} M_j(F_1(m))$ by composing $\hat{T}_{H1_m}$ and $\hat{T}_{F1_m}$, then the state space $(Q_K \times (Q \times \tilde{Q}_{K_m}) \times (Q \times \tilde{Q}_{K_m})) \times ((Q \times \tilde{Q}_{K_m}) \times Q_K \times Q_K)$ of this acceptor is larger than the state space $Q_K \times (Q \times \tilde{Q}_{K_m}) \times Q_K \times (Q \times \tilde{Q}_{K_m}) \times Q_K$ of $\hat{T}_{H2_m}$, i.e., there is an increase in the size of the state space. For this reason, we chose the construction as reported above in this section.

Example 4 We consider the setting of Example 1, where (L, K) is 2-inference diagnosable. As shown in Example 2, the delay bound is computed as $m_2^* = m_{F2}^* = m_{H2}^* = 1$.

We present an example of computing the ambiguity level $n_i^f(M_i(s))$ of the failure decision using Theorem 6. (The ambiguity level $n_i^h(M_i(s))$ of the nonfailure decision is computed in a similar way.) We assume that the plant G executes a failure trace $s := efc \in L - K$. For computing $n_i^f(M_i(s))$, we need to construct $\hat{T}_{Hk_m}$ for $k = 0, 1, 2$ and $m = m_2^* = 1$. By the definition of $\hat{T}_{H0_1}$, it is the same as G_K shown in Fig. 2b. A part of T_{H1_1} and its corresponding part $\hat{T}_{H1_1}$ are shown in Fig. 6a and b, respectively. In addition, a part of T_{H2_1} and its corresponding part $\hat{T}_{H2_1}$ are shown in Fig. 7a and b, respectively.

For $k = 0$, we have by Fig. 2b that

$$\begin{aligned} E_{H0_1,1}(M_1(s)) \cap R_{H0_1,mark} &= E_{H0_1,1}(ec) \cap Q_K \\ &= \{q_{K,11}\} \\ &\neq \emptyset \end{aligned}$$

and

$$\begin{aligned} E_{H0_1,2}(M_2(s)) \cap R_{H0_1,mark} &= E_{H0_1,2}(ec) \cap Q_K \\ &= \{q_{K,10}\} \\ &\neq \emptyset. \end{aligned}$$

For $k = 1$, we have by Fig. 6b that

$$\begin{aligned} ((q_{K,11}, (q_{11}, d_1), (q_{18}, d_1)) &\in E_{H1_1,1}(M_1(s)) \cap R_{H1_1,mark} \\ &= E_{H1_1,1}(ec) \cap R_{H1_1,mark} \\ &\neq \emptyset \end{aligned}$$

and

$$\begin{aligned} ((q_{K,10}, (q_{17}, d_1), (q_{11}, d_1)) &\in E_{H1_1,2}(M_2(s)) \cap R_{H1_1,mark} \\ &= E_{H1_1,2}(ec) \cap R_{H1_1,mark} \\ &\neq \emptyset. \end{aligned}$$

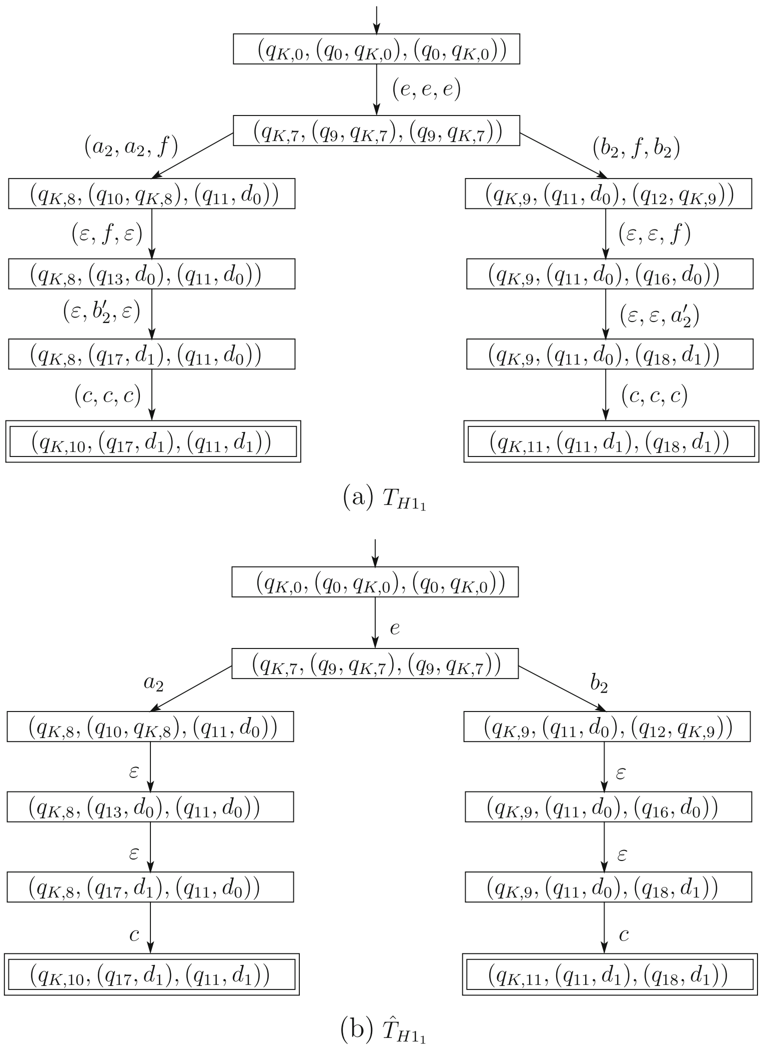

Fig. 6 A part of the automaton T_{H1_1} and its corresponding part of $\hat{T}_{H1_1}$

Furthermore, for $k = 2$, we have by Fig. 7b that

$$\begin{aligned} E_{H2_1,1}(M_1(s)) &= E_{H2_1,1}(ec) \\ &= \{(q_{K,11},(q_{15},q_{K,11}),q_{K,11},(q_{15},q_{K,11}),q_{K,11}), \\ &\quad (q_{K,11},(q_{11},d_1),q_{K,10},(q_{15},q_{K,11}),q_{K,11})\} \end{aligned}$$

and

$$\begin{aligned} E_{H2_1,2}(M_2(s)) &= E_{H2_1,2}(ec) \\ &= \{(q_{K,10},(q_{14},q_{K,10}),q_{K,10},(q_{14},q_{K,10}),q_{K,10}), \\ &\quad (q_{K,10},(q_{14},q_{K,10}),q_{K,10},(q_{11},d_1),q_{K,11})\}. \end{aligned}$$

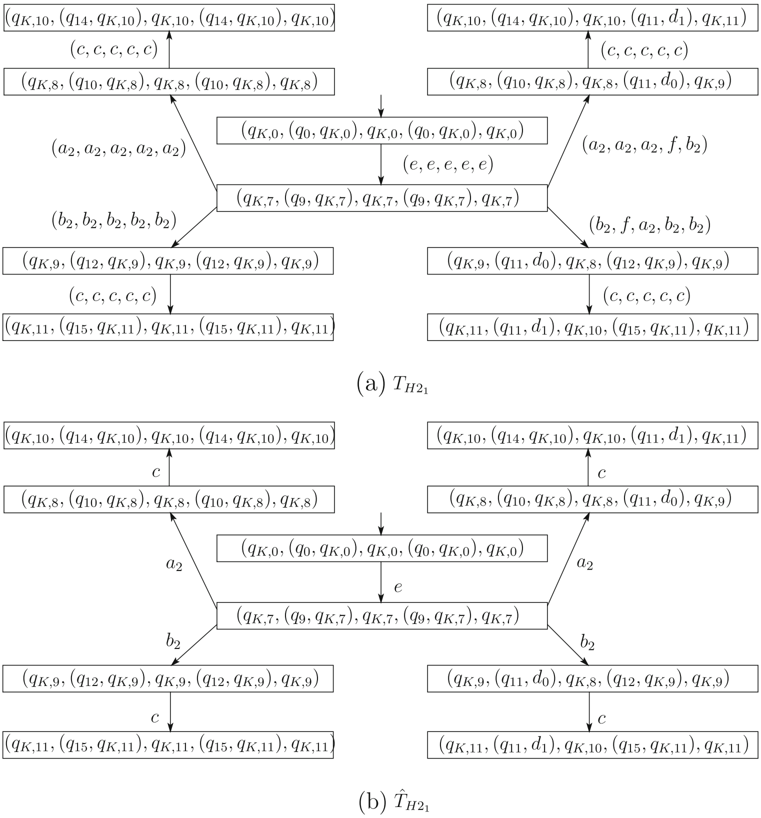

Fig. 7 A part of the automaton T_{H2_1} and its corresponding part of $\hat{T}_{H2_1}$

It follows that

$$E_{H2_1,1}(M_1(s)) \cap R_{H2_1,mark} = \emptyset$$

and

$$E_{H2_1,2}(M_2(s)) \cap R_{H2_1,mark} = \emptyset.$$

By Remark 9, the ambiguity level $n_i^f(M_i(s))$ is computed as

$$\begin{aligned} n_i^f(M_i(s)) &= \min\{k \in \{0,1,2\} \mid E_{Hk_1,i}(M_i(s)) \cap R_{Hk_1,mark} = \emptyset\} \\ &= 2 \end{aligned}$$

for each $i \in I$.

6 Conclusion

In Takai and Kumar (2017), we presented a framework for inference-based decentralized diagnosis, supporting multi-level inferencing for disjunctive as well as conjunctive decision making schemes, and introduced the notion of N-inference diagnosability to characterize the class of diagnosable systems. This paper presents results towards the implementation of the inference-based diagnosis scheme of Takai and Kumar (2017). To this end, a method for computing the delay bound of diagnosis was developed for N-inference diagnosable systems. This delay bound is smaller than the one achieved under any other framework subsumed by the N-inference diagnosability (such as disjunctive-codiagnosability and/or conjunctive-codiagnosability with or without conditional decisions). This delay bound was then used to identify a certain set of languages using which the local diagnosers can compute the ambiguity levels of their failure and nonfailure decisions online. The computation of the said set of languages was obtained, and a method for computing the ambiguity levels of local decisions was also reported in the paper. The complexity of offline computations (of the delay bound and the required set of languages) as well as of online diagnosis was discussed. The complexity increases with the levels of inferencing, but that also increases the class of diagnosable systems while decreases the delay of diagnosis. The complexity of the offline computations could be made more efficient through their structured representations (e.g., BDDs and symbolic computations), and can be a subject of future research.

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