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SHORT COMMUNICATIONS

The Cauchy Problem for a Differential-Difference Nonlocal Wave Equation

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1. In the domain $\Omega = R^1 \times (0, +\infty)$, consider the equation

$$Lu(x,t) \equiv u_{tt}(x,t) - D_+^{2\gamma} u(\xi,t) = H(t-\tau)u(x,t-\tau), \tag{1}$$

where $0 < \tau \equiv \text{const}$, $1 < 2\gamma < 2$, $H(\xi)$ is the Heaviside function, and $D_+^{2\gamma}$ is the Riemann–Liouville fractional integro-differentiation operator [1, p. 85] acting on a function $u(x,t)$ with respect to the variable x:

$$D_\pm^{2\gamma} u(\xi,t) = \frac{\partial^2}{\partial x^2} D_\pm^{2\gamma-2} u(\xi,t) = \frac{\pm 1}{\Gamma(2-2\gamma)} \frac{\partial^2}{\partial x^2} \int\limits_{\mp\infty}^{x} |x-\xi|^{1-2\gamma} u(\xi,t)\, d\xi, \qquad x \in R^1.$$

Equation (1) arises in mathematical modeling of processes in media with fractal geometry [2].

We write $\Omega = \bigcup_{k=0}^{+\infty} \Omega_k$, where $\Omega_k = R^1 \times (k\tau, (k+1)\tau)$.

Problem K_γ. Find a solution $u(x,t)$ of Eq. (1) in the domain Ω such that

$$D_+^{2\gamma-2} u(\xi,t) \in C\left(\bar{\Omega}\right) \cap C^2(\Omega)$$

and u satisfies the conditions

$$u(x,t)|_{t=0} = \omega(x), \qquad u_t(x,t)|_{t=0} = \nu(x), \qquad x \in R^1, \tag{2}$$

$$\lim_{x\to\pm\infty} D_+^{2\gamma-2} u(\xi,t) = \lim_{x\to\pm\infty} D_+^{2\gamma-1} u(\xi,t) = 0, \qquad 0 \le t < +\infty, \tag{3}$$

where the functions $\omega(x)$ and $\nu(x)$ are twice and once continuously differentiable, respectively, and absolutely integrable on R^1; moreover,

$$\lim_{x\to\pm\infty} D_+^{2\gamma-2} \omega(\xi) = 0.$$

2. If there exists a solution of Problem K_γ, then it is unique, which follows from the fact that the solution of Problem K_γ with zero initial conditions (2) is trivial. To prove this, consider the total energy integral

$$E(t) = \int\limits_{-\infty}^{+\infty} \left[u_t^2(x,t) + u_x(x,t) D_+^{2\gamma-1} u(\xi,t)\right] dx, \tag{4}$$

which is positive definite and independent of t.

Indeed, by formula 2.5.3.10 in [3, p. 387],

$$|x-\xi|^{1-2\gamma} = \left(\Gamma(2\gamma-1)\cos\frac{\pi(2\gamma-1)}{2}\right)^{-1} \int\limits_0^{+\infty} s^{2\gamma-2} \cos s|x-\xi|\, ds,$$

and therefore, the assertion about positive definiteness is valid, since

$$\int_{-\infty}^{+\infty} u_x(x,t) D_+^{2\gamma-1} u(\xi,t)dx = \int_{-\infty}^{+\infty} u_x(x,t) D_+^{2\gamma-2} u_\xi(\xi,t)dx = \int_{-\infty}^{+\infty} u_x(x,t) D_-^{2\gamma-2} u_\xi(\xi,t)dx$$

$$= \frac{1}{2\Gamma(2-2\gamma)} \int_{-\infty}^{+\infty} u_x(x,t)dx \int_{-\infty}^{+\infty} |x-\xi|^{1-2\gamma} u_\xi(\xi,t)d\xi$$

$$= -\frac{\cos\pi\gamma}{\pi} \int_0^{+\infty} s^{2\gamma-2} \left[\left(\int_{-\infty}^{+\infty} u_x(x,t)\cos sx\,dx \right)^2 + \left(\int_{-\infty}^{+\infty} u_x(x,t)\sin sx\,dx \right)^2 \right] ds \ge 0.$$

(We have used the fact that $\cos\pi\gamma < 0$ for $1 < 2\gamma < 2$.)

The second assertion that $E(t) \equiv 0$, $0 \le t \le \tau$, is also valid, since, integrating by parts in the second and third terms of the expression

$$\frac{dE(t)}{dt} = \int_{-\infty}^{+\infty} \left[2u_t(x,t)u_{tt}(x,t) + u_{xt}(x,t) D_+^{2\gamma-1} u(\xi,t) + u_x(x,t) D_+^{2\gamma-1} u_t(\xi,t) \right] dx$$

and taking into account conditions (2) and (3), we obtain

$$\frac{dE(t)}{dt} = \int_{-\infty}^{+\infty} u_t(x,t)Lu(x,t)dx + \int_{-\infty}^{+\infty} u_t(-x,t)Lu(-x,t)dx = 0$$

owing to the fact that $Lu(x,t) = 0$ in Ω_0. Therefore, $E(t) \equiv \mathrm{const} = E(0) = 0$, $0 \le t \le \tau$.

Since the integral (4) is positive definite, it follows from the last relation that $u_t(x,t) \equiv 0$ and $u_t(x,t) \equiv 0$ in Ω_0, i.e., $u(x,t) \equiv \mathrm{const}$ in Ω_0. Since $u(x,0) = 0$ and $D_+^{2\gamma-2} u(\xi,t) \in C\left(\bar{\Omega}_0\right)$, we have $u(x,t) \equiv 0$ in $\bar{\Omega}_0$.

We use a similar argument for the domains Ω_k ($k = 1, 2, \ldots$), successively take into account the relations $u(x,t)|_{t=k\tau} = 0$ and $u_t(x,t)|_{t=k\tau} = 0$, $x \in R^1$, as well as $Lu(x,t) = 0$ in Ω_k [since $u(x,t) \equiv 0$ in $\bar{\Omega}_{k-1}$], and obtain $E(t) \equiv E(k\tau) = 0$, $k\tau \le t \le (k+1)\tau$, and $u(x,t) \equiv 0$ in $\bar{\Omega}_k$, i.e., $u(x,t) \equiv 0$ in $\bar{\Omega}$.

3. We seek a class of nontrivial solutions of Eq. (1) satisfying conditions (2) and (3) in the form

$$u(x,t) = \int_{-\infty}^{+\infty} e^{i\lambda x} \bar{u}(\lambda,t)d\lambda, \qquad (x,t) \in \bar{\Omega}, \tag{5}$$

where the Fourier amplitude $\bar{u}(\lambda,t)$ is determined from the problem

$$\begin{gathered} \frac{d^2}{dt^2}\bar{u}(\lambda,t) - (i\lambda)^{2\gamma}\bar{u}(\lambda,t) = H(t-\tau)\bar{u}(\lambda,t-\tau), \qquad t > 0, \\ \bar{u}(\lambda,t)|_{t=0} = \bar{f}(\lambda), \qquad \bar{u}_t(\lambda,t)|_{t=0} = \bar{g}(\lambda); \end{gathered} \tag{6}$$

here

$$\left\{ \begin{matrix} \bar{f}(\lambda) \\ \bar{g}(\lambda) \end{matrix} \right\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda\xi} \left\{ \begin{matrix} \omega(\xi) \\ \nu(\xi) \end{matrix} \right\} d\xi, \qquad i = \sqrt{-1}. \tag{7}$$

The solution of problem (6) can be represented in the form [4]

$$\bar{u}(\lambda,t) = \left\{ \bar{u}_k(\lambda,t),\ k\tau \le t \le (k+1)\tau\ (k = 0, 1, 2, \ldots) \right\}, \tag{8}$$

where

$$\bar{u}_k(\lambda,t) = \bar{f}(\lambda)A_{1k}(\lambda,t) + \bar{g}(\lambda)A_{2k}(\lambda,t), \tag{9}$$

$$A_{2k}(\lambda,t) = R(\lambda,t) + \sum_{m=1}^{k} \gamma_m \int_0^{t-m\tau} \eta\left((t-m\tau)^2-\eta^2\right)^{m-1} R(\lambda,\eta)d\eta,$$

$$A_{1k}(\lambda,t) = \frac{d}{dt}A_{2k}(\lambda,t), \qquad \gamma_m = \left(m!\Gamma(m)\times 2^{2m-1}\right)^{-1},$$

$$R(\lambda,t) = (i\lambda)^{-\gamma}\sinh\left((i\lambda)^{\gamma}t\right). \tag{10}$$

By substituting (8) into (5) and by taking into account (9) and (7), we obtain

$$u(x,t) = \left\{u_k(x,t),\ (x,t)\in\bar{\Omega}_k\ (k=0,1,2,\ldots)\right\},$$

where

$$u_k(x,t) = \int_{-\infty}^{+\infty} e^{i\lambda x}\bar{u}_k(\lambda,t)d\lambda = \int_{-\infty}^{+\infty}\omega(\xi)G_{kt}(x-\xi,t)d\xi + \int_{-\infty}^{+\infty}\nu(\xi)G_k(x-\xi,t)d\xi$$

and

$$G_k(x,t) = \bar{R}(x,t) + \sum_{m=1}^{k}\gamma_m \int_0^{t-m\tau}\eta\left((t-m\tau)^2-\eta^2\right)^{m-1}\bar{R}(x,\eta)d\eta;$$

moreover,

$$\bar{R}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda x}R(\lambda,t)d\lambda. \tag{11}$$

To compute the integral (11), we rewrite the expression (10) via the Fox H-function [5, p. 626]. To this end, we successively apply formulas 1.2 in [6, p. 118] and 8.4.51.7 in [5, p. 728] to (10). Then we obtain

$$R(\lambda,t) = 2^{-1}(i\lambda)^{-\gamma}\left[H_{12}^{11}\left(-(i\lambda)^{\gamma}t\Big|_{(0,1),(0,1)}^{(0,1)}\right) - H_{12}^{11}\left((i\lambda)^{\gamma}t\Big|_{(0,1),(0,1)}^{(0,1)}\right)\right].$$

In view of formula 2.25.2.4 in [5, p. 355], relation (11) becomes

$$\bar{R}(x,t) = \frac{1}{4\sqrt{\pi}}(2i)^{-\gamma}x^{\gamma-1}\left[M_+(x,t)+M_-(x,t)\right],$$

where

$$\begin{aligned}M_\pm(x,t) = {}& H_{32}^{12}\left(-\frac{t(\pm 2i)^{\gamma}}{x^{\gamma}}\Bigg|_{(0,1),(0,1)}^{((1+\gamma)/2,\gamma/2),(0,1),(\gamma/2,\gamma/2)}\right)\\ &- H_{32}^{12}\left(\frac{t(\pm 2i)^{\gamma}}{x^{\gamma}}\Bigg|_{(0,1),(0,1)}^{((1+\gamma)/2,\gamma/2),(0,1),(\gamma/2,\gamma/2)}\right)\\ &\pm iH_{32}^{12}\left(-\frac{t(\pm 2i)^{\gamma}}{x^{\gamma}}\Bigg|_{(0,1),(0,1)}^{(\gamma/2,\gamma/2),(0,1),((1+\gamma)/2,\gamma/2)}\right)\\ &\mp iH_{32}^{12}\left(\frac{t(\pm 2i)^{\gamma}}{x^{\gamma}}\Bigg|_{(0,1),(0,1)}^{(\gamma/2,\gamma/2),(0,1),((1+\gamma)/2,\gamma/2)}\right);\end{aligned}$$

moreover,

$$\frac{d}{dt}\bar{R}(x,t) = \frac{1}{4\sqrt{\pi}}x^{-1}\left[N_+(x,t) + N_-(x,t)\right],$$

$$N_\pm(x,t) = H_{32}^{12}\left(-\frac{t(\pm 2i)^\gamma}{x^\gamma}\Bigg|_{(0,1),(0,1)}^{(1/2,\gamma/2),(0,1),(0,\gamma/2)}\right) + H_{32}^{12}\left(\frac{t(\pm 2i)^\gamma}{x^\gamma}\Bigg|_{(0,1),(0,1)}^{(1/2,\gamma/2),(0,1),(0,\gamma/2)}\right)$$
$$\pm\, iH_{32}^{12}\left(-\frac{t(\pm 2i)^\gamma}{x^\gamma}\Bigg|_{(0,1),(0,1)}^{(0,\gamma/2),(0,1),(1/2,\gamma/2)}\right) \pm\, iH_{32}^{12}\left(\frac{t(\pm 2i)^\gamma}{x^\gamma}\Bigg|_{(0,1),(0,1)}^{(0,\gamma/2),(0,1),(1/2,\gamma/2)}\right).$$

It is known [5, p. 628] that the function $H_{pq}^{mn}\left(z\Big|_{[b_q,B_q]}^{[a_p,A_p]}\right)$ is analytic with respect to z in the sector $|\arg z| < a^*\pi/2$, where

$$a^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j.$$

This condition is satisfied in our case, since $a^* = 1$, $|\arg(i)^{\pm\gamma}| = \pi\gamma/2 < \pi/2$ for $1/2 < \gamma < 1$ and hence $|\arg\left(\pm t(\pm 2i)^\gamma x^{-\gamma}\right)| < \pi/2$.

By using the analyticity of the Fox H-function and its asymptotics [2], one can justify the solution of Problem K_γ.

If $\gamma = 1$ ($k = 0$), then our solution coincides with the solution of Problem K_1 for the string vibration equation.

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