Differential Equations, Vol. 40, No. 12, 2004, pp. 1747–1756. Translated from Differentsial'nye Uravneniya, Vol. 40, No. 12, 2004, pp. 1665–1674.

Flatness of Dynamically Linearizable Systems

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Received May 12, 2004

INTRODUCTION

The dynamic linearization problem [1] and the flatness problem [2] were posed for finding nonlinear control systems to which the linear theory can be generalized. The dynamic linearizability of an arbitrary flat system was proved and a dynamic feedback method was developed for solving control problems for flat systems in [3]. It turned out that numerous nonlinear control systems in various engineering fields are flat and control problems for them can be solved by this method (see references in [3]). Flatness conditions were obtained and a method for computing a flat output was devised in [4, 5], and they proved to be efficient in some cases. The problem on the flatness of an arbitrary dynamically linearizable system has so far been open.

The aim of the present paper is to bridge the gap. We show that a system is dynamically linearizable if and only if it can be covered by a trivial system (Theorem 5); moreover, a flat system can cover only a flat system (see Theorem 6). These two facts imply that each dynamically linearizable system is flat (see Theorem 1). A method for finding a plane observer for such systems is illustrated by Example 2. At the same time, Theorem 6 permits one to prove the nonflatness of a system by constructing a covering from this system to some nonflat system (see Example 1). Moreover, we analyze the regularity condition for a dynamic feedback [1] and represent three equivalent but more understandable conditions (see Theorem 3). These new conditions clarify the notion of dynamic feedback from various viewpoints; the regularity condition is still used in the verification (see Example 2). At the end of the present paper, we prove Theorems 3–6.

DYNAMIC FEEDBACK

We consider systems of the form

$$\dot{x} = f(t, x, u), \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m, \tag{1}$$

where $x = (x_1, \ldots, x_n)$ is the state of the system, $u = (u_1, \ldots, u_m)$ is the control (or the output), f is a smooth vector function, and $\dot{x} \equiv dx/dt$. Here and throughout the following, smoothness is understood as infinite differentiability.

A *dynamic feedback* (dynamic compensator) for system (1) is defined as a system of the form

$$\dot{\xi} = a(t, \xi, x, v), \qquad u = b(t, \xi, x, v), \qquad \xi \in \mathbb{R}^l, \qquad v \in \mathbb{R}^m, \tag{2}$$

with state ξ, input (x, v), and output u satisfying the solution correspondence condition: for each solution $(x(t), u(t))$ of system (1), there exist vector functions $\xi(t)$ and $v(t)$ that, together with the functions $x(t)$ and $u(t)$, identically satisfy Eq. (2). The set $(x(t), \xi(t), v(t))$ of functions thus obtained is a solution of the system

$$\dot{x} = f(t, x, b(t, \xi, x, v)), \qquad \dot{\xi} = a(t, \xi, x, v) \tag{3}$$

with state $(x, \xi) \in \mathbb{R}^{n+l}$ and control v. The second equation in system (2) defines the inverse mapping of the set of solutions of system (3) into the set of solutions of system (1). Therefore, the dynamic feedback (2) can be treated as a transformation of system (1) into system (3). Moreover, to each solution of system (3) there corresponds exactly one solution of system (1), and infinitely many solutions of system (3) can correspond to a solution of system (1).

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The regularity condition for a dynamic feedback was used in [1] instead of the correspondence of solutions. In what follows, we state this condition and prove that it is equivalent to the solution correspondence condition (see Theorem 3).

Let $s(t) = (x(t), u(t))$ be a solution of system (1). We say that system (1) is *dynamically linearizable* in a neighborhood of a solution $s(t)$ if there exist functions a and b that define a dynamic feedback (2) in the neighborhood of $s(t)$ such that system (3) obtained with the use of this feedback can be reduced by an invertible change of variables

$$z = Z(t, x, \xi), \qquad t = t, \qquad v = v \tag{4}$$

to a linear control system of the form $\dot{z} = Az + Bv$, where A and B are constant matrices. If the above-mentioned conditions are satisfied for $l = 0$, i.e., ξ is absent, then system (1) is said to be statically linearizable. Rigorous definitions of neighborhood of a solution and dynamic feedback in a neighborhood of a solution will be given below.

System (1) is said to be *flat* in a neighborhood of a solution $s(t)$ if in this neighborhood there exist functions

$$h_i\left(t, x, u, \dot{u}, \ldots, u^{(l_i)}\right), \qquad i = 1, \ldots, r, \tag{5}$$

such that the variables x and u can be expressed via t, the functions (5), and their derivatives along the trajectories of system (1) up to some finite order and if any finite set of functions (5), their derivatives along the trajectories of system (1), and the function t is functionally independent. The set of functions (5) is referred to as a *flat* (or *linearizing*) *output* (*observer*) of system (1).

It was shown in [3] that the flatness of a control system implies its dynamic linearizability. The converse statement is the main result of the present paper.

Theorem 1. *If system* (1) *is dynamically linearizable in a neighborhood of a solution* $s(t)$, *then it is flat in the neighborhood of* $s(t)$.

To state the remaining results and prove them, we use the infinite-dimensional geometric approach to control systems, which was developed earlier in [6, 7] for partial differential equations.

A GEOMETRIC MODEL OF CONTROL SYSTEMS

To system (1), we assign the infinite-dimensional space $\mathbb{R}^\infty$ with coordinates

$$t, x_1, \ldots, x_n, u_1^{(j)}, \ldots, u_m^{(j)}, \qquad j = 0, 1, \ldots, \tag{6}$$

where the coordinates $u_i^{(0)}$ correspond to the variables u_i and the coordinates $u_i^{(j)}$ correspond to the derivatives $d^j u_i/dt^j$, $j > 0$. The range of the variables (6) corresponding to system (1) is denoted by $\mathscr{E}^\infty$. Each solution $s(t) = (s_x(t), s_u(t))$ of system (1) and each point t_0 in whose neighborhood this solution is defined determine the point in $\mathscr{E}^\infty$ with coordinates

$$t_0, \quad x_0 = s_x(t_0), \quad u_0^{(l)} = \partial^l s_u(t_0)/\partial t^l, \quad l \geq 0,$$

which is referred to as the *infinite jet* of the solution $s(t)$ at the point t_0. Each point of $\mathscr{E}^\infty$ is the infinite jet of some solution (for the proof, see [5]). A *neighborhood* of a solution $s(t)$ is understood as a neighborhood of some infinite jet of that solution in $\mathscr{E}^\infty$. In particular, a basic neighborhood is a subset in $\mathscr{E}^\infty$ given by a system of inequalities of the form

$$|t - t_0| < \varepsilon, \qquad |x_i - x_{i,0}| < \varepsilon, \quad i = 1, \ldots, n,$$
$$\left|u_j^{(l)} - u_{j,0}^{(l)}\right| < \varepsilon, \quad j = 1, \ldots, m, \quad l = 0, \ldots, k,$$

where ε is a positive real number and k is a positive integer.

On the set $\mathscr{E}^\infty$, one introduces the structure of an infinite-dimensional smooth manifold. This implies the definition of usual notions of smooth theory on $\mathscr{E}^\infty$: smooth functions, vector fields, differential forms, etc. More precisely, a smooth function on $\mathscr{E}^\infty$ is defined as a function smoothly depending on a finite (but arbitrary) set of variables (6). The algebra of smooth functions on $\mathscr{E}^\infty$ is denoted by $\mathscr{F}(\mathscr{E})$. Each differentiation of this algebra is a sum (in general, infinite) of the form

$$g_0\frac{\partial}{\partial t}+\sum_{l=1}^{n} g_l\frac{\partial}{\partial x_l}+\sum_{i=1}^{m}\sum_{j=0}^{\infty} g_i^{(j)}\frac{\partial}{\partial u_i^{(j)}},$$

where g_l, $l=0,\ldots,n$, and $g_i^{(j)}$, $i=1,\ldots,m$, $j=0,1,\ldots$, are some smooth functions on $\mathscr{E}^\infty$. Each differentiation of this kind is called a *vector field* on $\mathscr{E}^\infty$. The set of vector fields on $\mathscr{E}^\infty$ is a module over the algebra $\mathscr{F}(\mathscr{E})$ and is denoted by $\mathscr{D}(\mathscr{E})$.

A differential 1-form on $\mathscr{E}^\infty$ is defined as a 1-form depending on finitely many variables (6), that is, a finite sum

$$g_0dt+\sum_{l=1}^{n} g_l dx_l+\sum_{i=1}^{m}\sum_{j=0}^{k} g_i^{(j)}du_i^{(j)},\qquad g_0,g_l,g_i^{(j)}\in\mathscr{F}(\mathscr{E}).$$

By $\Lambda^1(\mathscr{E})$ we denote the $\mathscr{F}(\mathscr{E})$-module of differential 1-forms on $\mathscr{E}^\infty$. The algebra $\mathscr{F}(\mathscr{E})$ and the modulez $\mathscr{D}(\mathscr{E})$ and $\Lambda^1(\mathscr{E})$ are related by ordinary algebraic operations. In particular, the Lie derivative of a function g (respectively, a 1-form ω) along a vector field X is denoted by Xg (respectively, $X\omega$).

The vector field

$$D=\frac{\partial}{\partial t}+\sum_{l=1}^{n} f_l\left(t,x,u^{(0)}\right)\frac{\partial}{\partial x_l}+\sum_{i=1}^{m}\sum_{j=0}^{\infty} u_i^{(j+1)}\frac{\partial}{\partial u_i^{(j)}}$$

defined on $\mathscr{E}^\infty$ is called the *total derivative* with respect to t on $\mathscr{E}^\infty$. The Lie derivative along D coincides with the derivative along the trajectories of system (1), and the phase curves of this field coincide with the graphs of solutions of system (1) in $\mathscr{E}^\infty$ (see [7]). Therefore, as the geometric model of system (1), we take the pair $(\mathscr{E}^\infty, D)$, which is called the diffeotope (or the infinite continuation) of system (1) (for details, see [6, 7]).

THE GEOMETRIC INTERPRETATION OF FLATNESS AND DYNAMIC LINEARIZABILITY

Let $(\mathscr{S}^\infty, D_{\mathscr{S}})$ and $(\mathscr{E}^\infty, D_{\mathscr{E}})$ be two diffeotopes. A mapping

$$F:\mathscr{S}^\infty\longrightarrow\mathscr{E}^\infty \tag{7}$$

is said to be *smooth* if the corresponding induced mapping F^* takes each smooth function on $\mathscr{E}^\infty$ to a smooth function on $\mathscr{S}^\infty$, i.e., if $F^*(\mathscr{F}(\mathscr{E}))\subset\mathscr{F}(\mathscr{S})$, where, by definition, $F^*(g)=g\circ F$. The mapping (7) is a *diffeomorphism* if it is a smooth one-to-one mapping and the inverse mapping is also smooth.

A diffeomorphism (7) preserving the independent variable, i.e., such that $F^*(t)=t$, is called a $\mathscr{C}$-diffeomorphism (or a Lie–Bäcklund isomorphism) if

$$F_*\left(D_{\mathscr{S}}\right)=D_{\mathscr{E}}. \tag{8}$$

Systems are said to be *$\mathscr{C}$-diffeomorphic* if their diffeotopes are related by a $\mathscr{C}$-diffeomorphism. To obtain the definition of a $\mathscr{C}$-diffeomorphism in a neighborhood of a point $\theta\in\mathscr{S}^\infty$, one should replace the manifolds $\mathscr{S}^\infty$ and $\mathscr{E}^\infty$ in the above definitions by neighborhoods of the points $\theta\in\mathscr{S}^\infty$ and $F(\theta)\in\mathscr{E}^\infty$, respectively. Since the phase curves of the total derivative D coincide with the graphs of solutions of the corresponding system, it follows from condition (8) that each $\mathscr{C}$-diffeomorphism takes the solutions of one system to the solutions of the other system. Therefore, $\mathscr{C}$-diffeomorphic systems are equivalent, and a controllable system can be $\mathscr{C}$-diffeomorphic only to a controllable system. Throughout the following, by $\mathscr{E}^\infty$ we denote a diffeotope of system (1), and a system of the form

$$\dot y=v,\qquad y,v\in\mathbb{R}^r, \tag{9}$$

is said to be *trivial*. System (9) is flat, and $(y_1,\ldots,y_r)$ is its flat output. The following assertion was proved in [3].

Theorem 2. *System* (1) *is flat in a neighborhood of a point* $\theta \in \mathscr{E}^\infty$ *if and only if there exists a* $\mathscr{C}$*-diffeomorphism* F *of a neighborhood of this point into an open subset of the diffeotope of a trivial system* (9). *Furthermore, the functions* $F^*(y_1), \dots, F^*(y_r)$ *form a flat output of system* (1).

By $\mathscr{E}_1^\infty$ we denote the diffeotope of system (3), and by E_k we denote the $\mathscr{F}(\mathscr{E}_1)$-module spanned by the forms $dt, dx_i, d\ xi_s, db_j, dD_{\mathscr{E}_1}(b_j), \dots, dD^k_{\mathscr{E}_1}(b_j)$, $i = 1, \dots, n$, $s = 1, \dots, l$, $j = 1, \dots, m$. The dimension of a $\mathscr{F}(\mathscr{E}_1)$-submodule $E \subset \Lambda^1(\mathscr{E}_1)$ at a point $\theta \in \mathscr{E}_1^\infty$ is defined as the dimension of the space of covectors $\{\omega_\theta \mid \omega \in E\}$. By $\dim E$ we denote the integer-valued function on $\mathscr{E}_1^\infty$ whose value at a point θ is equal to the dimension of E at θ. The dynamic feedback (2) of system (1) is said to be *regular* in a neighborhood of a point $\theta \in \mathscr{E}_1^\infty$ if $\dim E_l - \dim E_{l-1} = m$ in this neighborhood.

Theorem 3. *Let the module* E_{l-1} *have a constant dimension in a neighborhood of a point of the diffeotope* $(\mathscr{E}_1^\infty, D_{\mathscr{E}_1})$ *of system* (3). *Then the following conditions are equivalent in the neighborhood of that point*:

(a) *the solution correspondence condition*;

(b) *the regularity condition for the dynamic feedback*;

(c) *each finite function set*

$$t, x_1, \dots, x_n, b_1, \dots, b_m, D_{\mathscr{E}_1}(b_1), \dots, D_{\mathscr{E}_1}(b_m), D^2_{\mathscr{E}_1}(b_1), \dots \tag{10}$$

is functionally independent on $\mathscr{E}_1^\infty$;

(d) *the set of variables* ξ *contains a subset* $\zeta = \left(\xi_{i_1}, \dots, \xi_{i_q}\right)$ *such that system* (3) *is equivalent to the system*

$$\dot{x} = f(t, x, u), \qquad \dot{\zeta} = g\left(t, \zeta, x, u, \dot{u}, \dots, u^{(l)}\right), \qquad \zeta \in \mathbb{R}^q; \tag{11}$$

moreover, the equivalence is given by the relations

$$x = x, \quad u = b(t, \xi, x, v), \qquad \zeta = \left(\xi_{i_1}, \dots, \xi_{i_q}\right). \tag{12}$$

One can show (e.g., see Lemma 4.4 in [5]) that the set of points of the diffeotope $\mathscr{E}_1^\infty$ in whose neighborhoods the module E_{l-1} has a constant dimension is open and dense everywhere in $\mathscr{E}_1^\infty$. Throughout the following, we consider only such points of the diffeotope $\mathscr{E}_1^\infty$.

Let $s(t) = (s_x(t), s_u(t))$ be some solution of system (1). We say that the dynamic feedback (2) is defined in a neighborhood of a solution $s(t)$ if system (11) equivalent to system (3) is defined, where $t, x, u, \dot{u}, \dots, u^{(l)}$ are coordinates of a point in a neighborhood of some infinite jet of the solution $s(t)$ and ζ are coordinates of a point in some open subset of $\mathbb{R}^q$.

A smooth mapping (7) satisfying the condition $F^*(t) = t$ is referred to as a *covering* if it satisfies condition (8), the tangent mapping $F_{*,\theta}$ is a vector space epimorphism at each point $\theta \in \mathscr{S}^\infty$, and the dimension of the kernel $F_{*,\theta}$ is constant for all $\theta \in \mathscr{S}^\infty$.

The *dimension* of the covering is defined as the dimension of the fiber of the mapping F, or, which is the same, the dimension of the kernel $F_{*,\theta}$. Any $\mathscr{C}$-diffeomorphism is a covering of zero dimension. If the mapping (7) is a covering and $(\mathscr{S}^\infty, D_{\mathscr{S}})$ and $(\mathscr{E}^\infty, D_{\mathscr{E}})$ are the diffeotopes of systems $\mathscr{S}$ and $\mathscr{E}$, respectively, then we say that the system $\mathscr{S}$ covers the system $\mathscr{E}$, or F is a covering of the system $\mathscr{E}$ by the system $\mathscr{S}$.

Theorem 4. *A dynamic feedback* (2) *for system* (1) *defines a finite-dimensional covering of system* (1) *by system* (3). *Each finite-dimensional covering of system* (1) *defines a dynamic feedback for system* (1).

System (1) is said to be *regular* at a point $\theta \in \mathscr{E}^\infty$ if the rank of the matrix $\partial f/\partial u$ at this point is equal to m.

Theorem 5. *A regular system* (1) *is dynamically linearizable in a neighborhood of a point* θ *of its diffeotope if and only if there exists a finite-dimensional covering of the neighborhood of* θ *by some open subset of the diffeotope of a trivial system.*

CONSTRUCTION OF A FLAT OBSERVER FOR DYNAMICALLY LINEARIZABLE SYSTEMS

Let $(\mathscr{E}^\infty, D_{\mathscr{E}})$ and $(\mathscr{E}_1^\infty, D_{\mathscr{E}_1})$ be the diffeotopes of system (1) and some flat system, respectively, let $\nu : \mathscr{E}_1^\infty \to \mathscr{E}^\infty$ be a covering, and let $y = (y_1, \ldots, y_r)$ be some flat output of the flat system. For $k \geq 0$, by $\Lambda_k^1(\mathscr{E}_1)$ we denote the module spanned over $\mathscr{F}(\mathscr{E}_1)$ by the 1-forms $dt, dy_1, \ldots, dy_r, dD_{\mathscr{E}_1}y_1, \ldots, dD_{\mathscr{E}_1}y_r, dD^2_{\mathscr{E}_1}y_1, \ldots, dD^k_{\mathscr{E}_1}y_r$. We also set

$$\mathscr{L}_k = \left\{\omega \in \Lambda^1(\mathscr{E}) \mid \nu^*(\omega) \in \Lambda_k^1(\mathscr{E}_1)\right\}.$$

The sets $\mathscr{L}_k$ are $\mathscr{F}(\mathscr{E})$-modules, since $\nu^*(f\omega) = \nu^*(f)\nu^*(\omega) \in \Lambda_k^1(\mathscr{E}_1)$ for $\omega \in \mathscr{L}_k$ and $f \in \mathscr{F}(\mathscr{E})$.

Let k_d be an integer such that the forms

$$\nu^*(dx_1), \ldots, \nu^*(dx_n), \nu^*(du_1), \ldots, \nu^*(du_m)$$

lie in $\Lambda_{k_d}^1(\mathscr{E}_1)$. A point $\theta \in \mathscr{E}^\infty$ in whose neighborhood the modules $\mathscr{L}_k$, $k = 0, \ldots, k_d$ have a constant dimension is said to be *ν-regular.*

Theorem 6. *If there exists a covering ν of some flat system in of the diffeotope of system* (1) *by the diffeotope of some flat system, then system* (1) *is flat in a neighborhood of any ν-regular point.*

Note that the covering in Theorem 6 can be infinite-dimensional. This permits one to prove nonflatness of systems. To this end, it suffices to find a covering of some nonflat system $\mathscr{S}$ by the system in question. It is convenient to take a system with one-dimensional control as $\mathscr{S}$, since for such systems flatness can be verified by simple methods [1, 3].

Example 1. The system of equations

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = x_3u_1^2 + x_2u_2, \qquad \dot{x}_3 = x_3u_2 \tag{13}$$

with state (x_1, x_2, x_3) and control (u_1, u_2) covers the nonflat system

$$\dot{z}_1 = u_1, \qquad \dot{z}_2 = u_1^2,$$

where $z_1 = x_1$ and $z_2 = x_2/x_3$. By Theorem 6, system (13) is not flat.

Theorem 5 and 6 readily imply the assertion of Theorem 1. In this case, a flat output of a dynamically linearizable system is constructed as follows. Let a dynamic feedback linearizing system (1) be constructed. It follows from the proof of Theorem 5 that such a feedback implies that system (3) is flat. Theorem 3 claims that systems (3) and (11) are equivalent. Let $y = (y_1, \ldots, y_r)$ be a flat output of system (11). If $q = 0$, then systems (1) and (11) coincide; therefore, y is a flat output of system (1) as well.

If $q > 0$, then for each $k \geq 0$ by $M^{(k)}$ we denote the submodule of elements of $\Lambda_k^1(\mathscr{E}_1)$ that do not contain $d\xi_{i_1}, \ldots, d\xi_{i_q}$ but contain only differentials of the functions (10). We construct a basis of the module $M^{(k)}$. Each module $M^{(k)}$ contains dt. The minimum value of k for which the module $M^{(k)}$ differs from the linear span of dt is denoted by k_0. It follows from the proof of Theorem 6 that the module $M^{(k_0)}$ has a basis that consists of the forms $dt, dh_1, \ldots, dh_{m_1}$. Moreover, the 1-forms $dt, dh_1, \ldots, dh_{m_1}$, $D(dh_1), \ldots, D(dh_{m_1})$ are linearly independent and lie in $M^{(k_0+1)}$. This set of forms is supplemented by the exact forms $dh_{m_1+1}, \ldots, dh_{m_2}$, where $m_1 \leq m_2 \leq m$, to form a basis in $M^{(k_0+1)}$. By successively constructing bases of the modules $M^{(k_0+2)}, \ldots, M^{(k_d)}$, we obtain the set of functions $h_1, \ldots, h_{m_d}$. Since the forms $dh_1, \ldots, dh_{m_d}$ contain only the coordinate differentials (10), it follows that $h_1, \ldots, h_{m_d}$ are the coordinate functions (10) and form a flat output of system (1) (see the proof of Theorem 6).

Example 2. The system of equations

$$\dot{x}_1 = u_2, \qquad \dot{x}_2 = x_2u_1, \qquad \dot{x}_3 = x_2 + u_1 \tag{14}$$

with state (x_1, x_2, x_3) and control (u_1, u_2) can be linearized by the dynamic feedback

$$\dot{\xi}_1 = v_1 + x_1, \qquad u_1 = v_1/x_2, \qquad u_2 = v_2. \tag{15}$$

The regularity condition is valid for this feedback, since $m = 2$, $l = 1$, the basis of the module E_0 consists of the forms $dt, dx_1, dx_2, dx_3, d\xi_1, dv_1, dv_2$, and the basis of the module E_1 consists of these forms and the forms $dv_1^{(1)}$ and $dv_2^{(1)}$. System (3) obtained with the use of the feedback (15) reads

$$\dot{x}_1 = v_2, \qquad \dot{x}_2 = v_1, \qquad \dot{x}_3 = x_2 + v_1/x_2, \qquad \dot{\xi}_1 = v_1 + x_1 \tag{16}$$

and has the flat output $h_1 = x_3 - \ln x_2$ and $h_2 = \xi_1 - x_2$.

The diffeotope of system (16) has the coordinates $t, x_1, x_2, x_3, \xi_1, u_1, u_2, u_1^{(1)}, u_2^{(1)}, \ldots$ The forms dt, $dh_1 = dx_3 - dx_2/x_2$, and $dh_2 = d\xi_1 - dx_2$ form a basis of the module $\Lambda_0^1(\mathscr{E}_1)$. Therefore, the module $M^{(0)}$ is spanned by the forms dt and dh_1. Likewise, the set of forms dt, dh_1, $D(dh_1) = dx_2$, and $D(dh_2) = dx_1$ forms a basis of the module $M^{(1)}$, and the forms $D^2(dh_1) = d(x_2u_1)$ and $D^2(dh_2) = du_2$ complement that basis to a basis of $M^{(2)}$. Since $k_d = 2$, it follows that the functions $h_1 = x_3 - \ln x_2$ and $Dh_2 = x_1$ form a flat output of system (14).

PROOF OF THEOREM 3

First, we define auxiliary notions and state a lemma used in the proof of Theorems 3 and 6. Consider an arbitrary system with state $z = (z_1, \ldots, z_s)$ and control $v = (v_1, \ldots, v_m)$. Let $(\mathscr{E}_1^\infty, D)$ be the diffeotope of this system. By G_p we denote the $\mathscr{F}(\mathscr{E}_1)$-module spanned by the 1-forms $dt, dz_1, \ldots, dz_s, dv_1^{(0)}, \ldots, dv_m^{(0)}, dv_1^{(1)}, \ldots, dv_m^{(p)}$ if $p \ge 0$ and by the 1-forms $dt, dz_1, \ldots, dz_s$ if $p = -1$. Consider the quotient modules G_p/G_{p-1} for $p \ge 0$. By $[\Omega]_p$ we denote the coset in G_p/G_{p-1} of a form $\Omega \in G_p$. Note that $D(G_p) \subset G_{p+1}$. Therefore, for each $p \ge 0$ the Lie derivative along D induces a mapping

$$D : G_p/G_{p-1} \to G_{p+1}/G_p, \qquad D[\Omega]_p = [D\Omega]_{p+1}. \tag{17}$$

Lemma 1. *The mapping* (17) *is an isomorphism of modules, and its restriction to each point is an isomorphism of linear spaces.*

First, let us show that the mapping (17) is a homomorphism. If $\Omega \in G_p$, then $[\Omega]_{p+1} = 0$. Therefore,

$$\begin{aligned} D(f[\Omega]_p) &= D[f\Omega]_p = [D(f\Omega)]_{p+1} = [D(f)\Omega + fD\Omega]_{p+1} \\ &= D(f)[\Omega]_{p+1} + f[D\Omega]_{p+1} = f[D\Omega]_{p+1} = fD[\Omega]_p; \end{aligned}$$

i.e., the mapping (17) preserves the module structure.

The set of elements $\left[dv_1^{(p)}\right]_p, \ldots, \left[dv_m^{(p)}\right]_p$ is a basis of the quotient module G_p/G_{p-1}. The homomorphism (17) takes this basis to the basis $\left[dv_1^{(p+1)}\right]_{p+1}, \ldots, \left[dv_m^{(p+1)}\right]_{p+1}$ of the quotient module G_{p+1}/G_p. Therefore, the mapping (17) is an isomorphism.

The image and preimage of the isomorphism (17) are modules of constant dimension. Therefore, the restriction of D to any point is an isomorphism of linear spaces. This completes the proof of Lemma 1.

We apply Lemma 1 to system (3), where $z = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_l)$. We set $E_{-1} = G_{-1}$. Since $D(E_{k-1}) \subset E_k$, we see that the following restrictions of the isomorphism (17) are defined for nonnegative p and k:

$$D : \frac{E_{k-1} \cap G_p}{E_{k-1} \cap G_{p-1}} = \frac{E_{k-1} \cap G_p + G_{p-1}}{G_{p-1}} \longrightarrow \frac{E_k \cap G_{p+1} + G_p}{G_p} = \frac{E_k \cap G_{p+1}}{E_k \cap G_p}. \tag{18}$$

A restriction of a monomorphism is a monomorphism. On the other hand, the relations $E_k = D(E_{k-1}) + E_0$ and $E_0 \subset G_p$ for $p \ge 0$ prove that the mapping (18) is an epimorphism. Therefore, the mapping (18) is an isomorphism.

We set $d_{k,p} = \dim(E_k \cap G_p)$. By comparing the dimensions of the image and preimage of the isomorphism (18), we obtain $d_{k-1,p} - d_{k-1,p-1} = d_{k,p+1} - d_{k,p}$ for any $p, k \geq 0$. By summing the resulting relations for $p = 0, \ldots, k-1$, we obtain $d_{k-1,k-1} - d_{k-1,-1} = d_{k,k} - d_{k,0}$.

Since $G_{-1} \subset E_k \subset G_k$, we have $d_{k-1,-1} = \dim G_{-1}$ and $d_{k,k} = \dim E_k$. Therefore,

$$\dim E_k - \dim E_{k-1} = j_k, \qquad k \geq 0, \tag{19}$$

where $j_k = d_{k,0} - d_{k-1,-1} = \dim((E_k \cap G_0)/G_{-1})$. Therefore, the regularity condition can be represented in the form $j_l = m$. Note also that $E_{k-1} \subset E_k$. Therefore, $j_{k-1} \leq j_k \leq \dim(G_0/G_{-1}) = m$.

To prove the theorem, we successively prove the implications $(b) \Rightarrow (c)$, $(a) \Rightarrow (c) \Rightarrow (b)$, and $(c) \Rightarrow (d) \Rightarrow (a)$. By θ we denote a point of the diffeotope $\mathscr{E}_1^\infty$ occurring in the theorem.

$(b) \Rightarrow (c)$. Suppose the contrary: there exist j_0 and k such that in a neighborhood of the point θ the function $D^k b_{j_0}$ can be expressed via the remaining functions in the set $t, x_i, b_j, Db_j, \ldots, D^k b_j$, where $i = 1, \ldots, n$, $j = 1, \ldots, m$. This implies that the form $dD^k b_{j_0}$ is a $\mathscr{F}(\mathscr{E}_1)$-linear combination of differentials of the above-mentioned functions. By taking account of (19), we obtain $j_k < m$. If $k > l$, then this contradicts the regularity condition $j_l = m$, since $j_k \geq j_l$.

If $k \leq l$, then $dD^l b_{j_0} = D^{l-k}\left(dD^k b_{j_0}\right)$ is a linear combination of the remaining forms in the set $dt, dx_i, db_j, dDb_j, \ldots, dD^l b_j$, and we have arrived at a contradiction with the relation $j_l = m$.

$(a) \Rightarrow (c)$. Since the functions (6) are functionally independent on $\mathscr{E}^\infty$, it follows that in a neighborhood of the point θ for any function Φ of the variables (6) there exists a point $\tilde{\theta}$ at which the function Φ is nonzero. Let $\tilde{\theta}$ be the infinite jet of the solution $(x(t), u(t))$ of system (1). Then

$$\Phi\left(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), \frac{\partial u_1(t)}{\partial t}, \ldots, \frac{\partial u_m(t)}{\partial t}, \frac{\partial^2 u_1(t)}{\partial t^2}, \ldots\right) \not\equiv 0. \tag{20}$$

It follows from the solution correspondence condition that there exists a solution $(x(t), \xi(t), v(t))$ of system (3) such that $u(t) = b(t, \xi(t), x(t), v(t))$. The restriction of the function

$$\Phi\left(t, x_1, \ldots, x_n, b_1, \ldots, b_m, D(b_1), \ldots, D(b_m), D^2(b_1), \ldots\right)$$

to this solution coincides with the function (20) and hence does not vanish identically. Since Φ is an arbitrary function, we find that the variables (10) are functionally independent.

$(c) \Rightarrow (b)$. It follows from condition (c) that the number of functions (10) whose differentials lie in E_l is less than the dimension of the module E_l: $\dim E_l \geq 1 + n + m(l+1)$. On the other hand, relations (19) with $k = 0, \ldots, l$ and $\dim E_{-1} = 1 + n + l$ imply that $\dim E_l = 1 + n + l + j_0 + j_1 + \cdots + j_l$. Hence we obtain $l + j_0 + j_1 + \cdots + j_l \geq m(l+1)$. And since $j_0 \leq j_1 \leq \cdots \leq j_l$, it follows that $l \geq (l+1)(m - j_l)$, which is possible only for $j_l = m$. By taking account of relation (19), we obtain the regularity condition (19).

$(c) \Rightarrow (d)$. We take indices $i_1, \ldots, i_q$ so as to ensure that the covectors $d\xi_{i_1}|_\theta, \ldots, d\xi_{i_q}\big|_\theta$ complement the linearly independent set $dt|_\theta, dx_1|_\theta, \ldots, dx_n|_\theta, db_1|_\theta, \ldots, db_m|_\theta, dDb_1|_\theta, \ldots, dD^{l-1}b_m\big|_\theta$ to a basis of the space $E_{l-1}|_\theta$. The corresponding forms are linearly independent in some neighborhood of the point θ; since the module E_{l-1} has a constant dimension in a neighborhood of this point, it follows that these forms comprise a basis of this module. We have earlier proved the implication $(c) \Rightarrow (b)$. Consequently, the set of forms $dD^l b_1, \ldots, dD^l b_m$ complements the basis of E_{l-1} to a basis of E_l. Since $j_l \leq j_k \leq m$ for $k > l$, it follows from the relation $j_l = m$ and from (19) that $\dim E_k - \dim E_{k-1} = m$ for all $k > l$. Therefore, we can successively repeat our considerations for $E_{l+1}, E_{l+2}, \ldots$ Thus the differentials of the functions $\xi_{i_1}, \ldots, \xi_{i_q}$ and (10) are linearly independent. Moreover, they form a basis of the module $\Lambda^1(\mathscr{E}_1)$ of all 1-forms on $\mathscr{E}_1^\infty$ in a neighborhood of the point θ. Indeed, the forms $d\xi_1, \ldots, d\xi_l$ lie in E_{l-1} and hence can be expressed via the above-mentioned forms. Moreover, by using the definition of j_l, from the relation $j_l = m = \dim(G_0/G_{-1})$, we obtain $G_0 \subset E_l$. Hence it follows that the forms $dv_1, \ldots, dv_m$ lie in E_l. Consequently, the forms $dv_i^{(j)}$ for all i and j can be expressed via $d\xi_{i_1}, \ldots, d\xi_{i_q}$ and the differentials of the functions (10).

Therefore, the forms $dt, d\xi_s, dx_p, dv_i^{(j)}$, which comprise a basis of $\Lambda^1(\mathscr{E}_1)$, can be expressed via the above-mentioned forms. Therefore, the latter also form a basis of $\Lambda^1(\mathscr{E}_1)$, and the functions $\xi_{i_1}, \ldots, \xi_{i_q}$ and (10) form a coordinate system in some neighborhood of the point $\theta \in \mathscr{E}_1^\infty$.

We set $g_j = D\xi_{i_j}$, $j = 1, \ldots, q$. Since the form $d\xi_{i_j}$ lies in E_{l-1}, it follows that its derivative dg_j lies in E_l and hence g_j is a function of the variables $t, \zeta, x, u, \dot{u}, \ldots, u^{(l)}$. By comparing the total derivative in system (11) with respect to t and on $\mathscr{E}_1^\infty$ in the coordinates $\xi_{i_1}, \ldots, \xi_{i_q}$ and (10), we find that they coincide. This implies that the corresponding diffeotopes are $\mathscr{C}$-diffeomorphic; consequently, systems (3) and (11) are equivalent.

$(d) \Rightarrow (a)$. Let $(x(t), u(t))$ be some solution of system (1), and let $\zeta(t)$ be the corresponding solution of the system

$$\dot{\zeta} = g\left(t, \zeta, x(t), u(t), \dot{u}(t), \ldots, u^{(l)}(t)\right).$$

Then $(x(t), u(t), \zeta(t))$ is a solution of system (11). It follows from condition (d) that it is obtained by the transformation (12) from some solution $(x(t), \xi(t), v(t))$ of system (3). This implies the solution correspondence condition and the assertion of Theorem 3.

PROOF OF THEOREM 4

It follows from Theorem 3 that system (3) is equivalent and hence $\mathscr{C}$-diffeomorphic to system (11). System (11) is a covering of system (1); moreover, the corresponding mapping F preserves the variables t, x, and u and "forgets" the variables ζ; i.e., F is a covering of dimension q. Since the composition of a $\mathscr{C}$-diffeomorphism with a covering is a covering, we have a covering of system (1) by system (3).

Conversely, let F be a finite-dimensional covering of system (1), and let $\zeta_1, \ldots, \zeta_q$ be coordinates in the fiber of F. Then in the variables t, ζ, x, and u the system that is a covering of system (1) has the form (11) for some $l \geq 0$. The corresponding dynamic feedback is given by the system

$$\begin{gathered}\dot{\eta}^{(1)} = \eta^{(2)}, \qquad \ldots, \qquad \dot{\eta}^{(l)} = v, \qquad \dot{\zeta} = g\left(t, \zeta, x, \eta^{(1)}, \ldots, \eta^{(l)}, v\right), \\ u = \eta^{(1)}, \quad \eta^{(1)}, \ldots, \eta^{(l)} \in \mathbb{R}^m,\end{gathered}$$

with $(ml + q)$-dimensional state $\xi = \left(\eta^{(1)}, \ldots, \eta^{(l)}, \zeta\right)$.

PROOF OF THEOREM 5

It was shown in [3] that each linear control system is flat. And since an invertible change of variables of the form (4) defines a $\mathscr{C}$-diffeomorphism of system (3) into a linear control system, it follows from the dynamic linearizability of system (1) that system (3) is $\mathscr{C}$-diffeomorphic to a flat system. But each $\mathscr{C}$-diffeomorphism is a mapping of a flat system into a flat system. Therefore, system (3) is flat. This, together with Theorem 2, implies that system (3) is $\mathscr{C}$-diffeomorphic to a trivial system, and it follows from Theorem 4 that there exists a finite-dimensional covering of system (1) by system (3). Since the composition of a $\mathscr{C}$-diffeomorphism with a covering is a covering, it follows that system (1) can be covered by the trivial system.

Conversely, suppose that there exists a finite-dimensional covering of system (1) by a trivial system. Just as above, we consider the coordinates $\zeta_1, \ldots, \zeta_q$ in the fiber of F. The trivial system in the variables t, ζ, x, and u has the form (11). Let $y = (y_1, \ldots, y_r)$ be its flat output, and suppose that the variables x and ζ can be expressed via t and y and the derivatives of y of order $< \mu$. Then it follows from the regularity of system (1) that the variables u can be expressed via t, y, and the derivatives of y of order $< \mu + 1$. We choose functions $\eta_1, \ldots, \eta_l$ of the variables $t, y, \dot{y}, \ldots, y^{(\mu)}$ so as to ensure that the Jacobian matrix $\partial(\eta, x, \zeta)/\partial\tilde{y}$ is nondegenerate at the point θ, where $\tilde{y} = \left(y, \dot{y}, \ldots, y^{(\mu)}\right)$. Then the first derivatives $\dot{\zeta}$ and $\dot{\eta}$ of the functions ζ and η along the trajectories of system (11) depend on $t, y, \dot{y}, \ldots, y^{(\mu)}$ and $y^{(\mu+1)}$. We set $v = y^{(\mu+1)}$. By passing from the variables t, $\tilde{y}$, and $y^{(\mu+1)}$ to the variables t, η, x, ζ, and v, from the expressions for $\dot{\zeta}$, $\dot{\eta}$, and u we obtain a dynamic feedback for system (1). This feedback is a linearization of system (1), since the replacement of the variables η, x, and ζ by the variables $\tilde{y}$ reduces the corresponding system (3) to the linear control system $y^{(\mu+1)} = v$.

PROOF OF THEOREM 6

Consider a covering ν from the diffeotope $\mathscr{E}_1^\infty$ of a flat system into a diffeotope $\mathscr{E}^\infty$ of system (1) as well as a ν-regular point $\theta' \in \mathscr{E}^\infty$. By the definition of a ν-regular point, there exists a neighborhood $\mathscr{U} \subset \mathscr{E}^\infty$ in which the modules $\mathscr{L}_0, \dots, \mathscr{L}_{k_d}$ have a constant dimension. Let θ be a point of the diffeotope $\mathscr{E}_1^\infty$ such that $\nu(\theta) = \theta'$. Since the modules $\Lambda_k^1(\mathscr{E}_1)$ define integrable distributions, we have $d\Lambda_k^1(\mathscr{E}_1) \subset \Lambda_k^1(\mathscr{E}_1) \wedge \Lambda^1(\mathscr{E}_1)$ for each $k \geq 0$. Hence it follows that

$$\nu^*(d\mathscr{L}_k) = d\nu^*(\mathscr{L}_k) \subset d\Lambda_k^1(\mathscr{E}_1) \subset \Lambda_k^1(\mathscr{E}_1) \wedge \Lambda^1(\mathscr{E}_1). \tag{21}$$

Let us show that

$$(\nu^*)^{-1}\left(\Lambda_k^1(\mathscr{E}_1) \wedge \Lambda^1(\mathscr{E}_1)\right) = \mathscr{L}_k \wedge \Lambda^1(\mathscr{E}). \tag{22}$$

Since ν^* is a morphism of vector bundles, it suffices to prove the corresponding formula for the restrictions of the morphism ν^* to the fibers of the bundle, i.e., for homomorphisms of linear spaces. In turn, this is a consequence of the following lemma.

Lemma 2. *If $A : L \to M$ is a monomorphism of linear spaces and $K \subset M$ is a subspace, then $A^{-1}(K \wedge M) = A^{-1}(K) \wedge L$.*

To prove the lemma, it suffices to choose embedded bases of the spaces K, $A(L)$, and M and compare the bases of spaces on the left- and right-hand sides of the equation.

Consider a point $\tilde{\theta}$ in a neighborhood of the point $\theta \in \mathscr{E}_1^\infty$ and set

$$M = \Lambda^1(\mathscr{E}_1)\big|_{\tilde{\theta}}, \qquad L = \Lambda^1(\mathscr{E})\big|_{\nu(\tilde{\theta})}, \qquad K = \Lambda_k^1(\mathscr{E}_1)\big|_{\tilde{\theta}}, \qquad A = \nu^*|_{\nu(\tilde{\theta})}.$$

We have $A^{-1}(K) = \mathscr{L}_k|_{\nu(\tilde{\theta})}$, and it follows from Lemma 2 that relation (22) is valid at an arbitrary point $\tilde{\theta}$. Finally, from (21) and (22), we obtain the Frobenius condition for the module $\mathscr{L}_k$:

$$d\mathscr{L}_k \subset (\nu^*)^{-1}\left(\Lambda_k^1(\mathscr{E}_1) \wedge \Lambda^1(\mathscr{E}_1)\right) = \mathscr{L}_k \wedge \Lambda^1(\mathscr{E}). \tag{23}$$

We take an arbitrary $k \in \{0, 1, \dots, k_d\}$. In the neighborhood $\mathscr{U}$, the dimension of the module $\mathscr{L}_k$ is constant; therefore, this module has a basis, which is denoted by $\{\omega_1, \dots, \omega_s\}$. Each form on $\mathscr{E}^\infty$ depends on finitely many coordinates (6). Let $\mathscr{E}_0$ be a manifold with the coordinates (6) on which the forms $\omega_1, \dots, \omega_s$ depend. Then the forms $\omega_1, \dots, \omega_s$ define a distribution on $\mathscr{E}_0$. It follows from (23) that this distribution is integrable. Therefore, for each $k \in \{0, 1, \dots, k_d\}$ the module $\mathscr{L}_k$ has a local basis of exact forms in the neighborhood $\mathscr{U}$.

Obviously, $dt \in \mathscr{L}_0$, since $\nu^*(t) = t$. Let k_0 be the minimum index for which the module $\mathscr{L}_{k_0}$ is not a linear span of dt, and let $\{dt, dh_1, \dots, dh_{m_1}\}$ be a basis of the module $\mathscr{L}_{k_0}$ in a neighborhood of the point θ'. Then the forms

$$\nu^*(dt),\ \nu^*(dh_1),\ \dots,\ \nu^*(dh_{m_1}) \tag{24}$$

lie in $\Lambda_{k_0}^1(\mathscr{E}_1)$ and are linearly independent at each point of the corresponding neighborhood in $\mathscr{E}_1^\infty$, since ν is a covering; consequently, ν^* is a monomorphism.

It follows from Lemma 1 that the form (24) and the forms

$$D(\nu^*(dh_1)) = \nu^*(Ddh_1),\ \dots,\ D(\nu^*(dh_{m_1})) = \nu^*(Ddh_{m_1}) \tag{25}$$

are linearly independent at each point. Indeed, a flat system can be treated as a system that has no state variables and whose flat output $(y_1, \dots, y_r)$ is a control. In addition, the module G_p [see (7)] is spanned by the forms $dt, dy_1^{(0)}, \dots, dy_r^{(0)}, dy_1^{(1)}, \dots, dy_r^{(p)}$. Since the forms (24) are linearly independent, it follows that the cosets $[\nu^*(dh_1)]_{k_0-1}, \dots, [\nu^*(dh_{m_1})]_{k_0-1}$ are also linearly independent. Then Lemma 1 implies that the cosets $[D\nu^*(dh_1)]_{k_0}, \dots, [D\nu^*(dh_{m_1})]_{k_0}$ are linearly independent, whence it follows that so are the forms (24) and (25) and hence the forms

$$dt,\ dh_1,\ \dots,\ dh_{m_1},\ d(Dh_1),\ \dots,\ d(Dh_{m_1}). \tag{26}$$

The forms (26) lie in $\mathscr{L}_{k_0+1}$. We complement this set of forms to a basis of the module $\mathscr{L}_{k_0+1}$; to this end, we add the exact forms $dh_{m_1+1}, \dots, dh_{m_2}$, where $m_2 = \dim \mathscr{L}_{k_0+1} - m_1 - 1$. This can be performed as follows. We have shown above that the module $\mathscr{L}_{k_0+1}$ has a basis of exact forms. Let that basis consist of the forms $dz_1, \ldots, dz_{m_2+m_1+1}$. We take a basis minor of the matrix $\dfrac{\partial t, h, Dh}{\partial z}$ at the point θ'. The functions z_i corresponding to nonbasis columns can be chosen as $h_{m_1+1}, \ldots, h_{m_2}$ in a neighborhood of θ'.

By successively repeating these considerations for $\mathscr{L}_{k_0+2}, \ldots, \mathscr{L}_{k_d}$, we obtain a basis of the module $\mathscr{L}_{k_d}$, which consists of the forms $dt, dh_1, \ldots, dh_{m_d}$ and some of their derivatives along the trajectories of system (1). Since the forms $dx_1, \ldots, dx_n, du_1, \ldots, du_m$ lie in $\mathscr{L}_{k_d}$, it follows that they are $\mathscr{F}(\mathscr{E})$-linear combinations of $dt, dh_1, \ldots, dh_{m_d}$ and the derivatives of $dh_1, \ldots, dh_{m_d}$. Therefore, $x_1, \ldots, x_n, u_1, \ldots, u_m$ are functions of $t, h_1, \ldots, h_{m_d}$ and the derivatives of $h_1, \ldots, h_{m_d}$ along the trajectories of system (1).

Just as above, one can show that each finite subset of the forms dt, $dD^j h_i$, $i = 1, \ldots, m_d$, $j \geq 0$, is $\mathscr{F}(\mathscr{E})$-linearly independent. Therefore, $(h_1, \ldots, h_{m_d})$ is a flat output, and system (1) is flat. This completes the proof of Theorem 6.

Finally, if $\{dt, dh_1, \ldots, dh_s\}$ is a basis of the $\mathscr{F}(\mathscr{E})$-module $\mathscr{L}_k$, then $\{dt, d\nu^*(h_1), \ldots, d\nu^*(h_s)\}$ is a basis of the $\mathscr{F}(\mathscr{E}_1)$-module $M^{(k)}$. This justifies the algorithm of constructing a flat observer for dynamically linearizable systems.

ACKNOWLEDGMENTS

The work was financially supported by the Education Ministry of the Russian Federation (project no. E02-1.0-211) and by the Russian Foundation for Basic Research (project no. 02-01-00704).

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