Differential Equations, Vol. 40, No. 9, 2004, pp. 1330–1344. Translated from Differentsial'nye Uravneniya, Vol. 40, No. 9, 2004, pp. 1256–1270.

NUMERICAL METHODS

Integral Equations and Sound Propagation in a Shallow Sea

I. K. Lifanov and S. L. Stavtsev

Air-Force Engineering Academy, Moscow, Russia

Institute for Computational Mathematics, Russian Academy of Sciences, Moscow, Russia

Received March 15, 2004

The boundary value problem for the Helmholtz equation for the acoustic pressure in a shallow sea can be reduced to a system of integral equations (some of which can be hypersingular). We suggest a numerical method for solving this system. In this approach to the numerical solution of the sound propagation problem in a shallow sea, the surfaces of the sea, the seabed, and the layers can have an arbitrary geometric structure.

INTRODUCTION

Sound waves are the only waves of known physical nature that can propagate for hundreds of kilometers. This unique property of acoustic waves explains the practical interest of scientists and engineers of various specialities in the specific features of sound propagation in the ocean.

Underwater sound wave propagation is a very complicated process, which is hard to describe owing to a wide variety of phenomena and properties specific to various regions of the great oceans. First of all, this pertains to the shelf zones of the ocean, where the specific properties of sound wave propagation are related both to the interaction of the waves with the seabed and to the specific nature of hydrodynamic perturbations. At the same time, the shelf is most important for human activities and hence of most interest to researchers. The continental shelves are top priority in oil and gas exploration. By now, various research teams have obtained quite a few experimental and theoretical results [1–4], which permits one to speak of shallow ocean acoustics as an independent branch of ocean acoustics.

A considerable contribution to the development of the theory of sound propagation in a water layer is due to Brekhovskikh [5], who studied electromagnetic and sound waves in layered media. He introduced ray representations of electromagnetic and sound waves for the case in which the fields are excited by a monochromatic source.

The simplest model of a shallow-water waveguide possessing all main typical properties was suggested by Pekeris [6]. In this model, the water layer has a constant speed of sound $c(z) = c = \mathrm{const}$ and a constant density $\varrho(z) = \varrho = \mathrm{const}$; the seabed parameters c_1 and ϱ_1 are also constant. The Pekeris model is often used in qualitative considerations and quantitative estimates in the description of various shallow sea phenomena. A detailed analysis of the sound field behavior in the framework of this model (i.e., in a two-layer waveguide) was carried out in [5, 7, 8]. In this case, the problem can be reduced to a boundary value problem for the Helmholtz equation for the acoustic pressure. In the simplest cases, this boundary value problem can be solved in closed form. For example, this happens if the speed of sound in water is constant and (1) the ideal liquid (water) occupies the half-space bounded above by a plane (the free surface of water) and the upper half-space is filled with air, or (2) water occupies the upper half-space bounded below by a plane (seabed) that is an ideally reflecting surface. The solution has a more complicated form if the water layer is enclosed between two planes, one of which is a free surface and the other is an ideally reflecting seabed. Later [9–11], researchers dealt with problems in a water layer bounded by two planes in which the speed of sound depends on one vertical coordinate and with more complicated problems of this kind in which the seabed is not a horizontal plane but has a small constant slope. In the latter case, the solution was obtained with the use of ray representations and asymptotic ray representations. If the seabed is an arbitrary smooth surface, then the problem is difficult to solve with the use of such representations.

0012-2661/04/4009-1330 © 2004 MAIK "Nauka/Interperiodica"

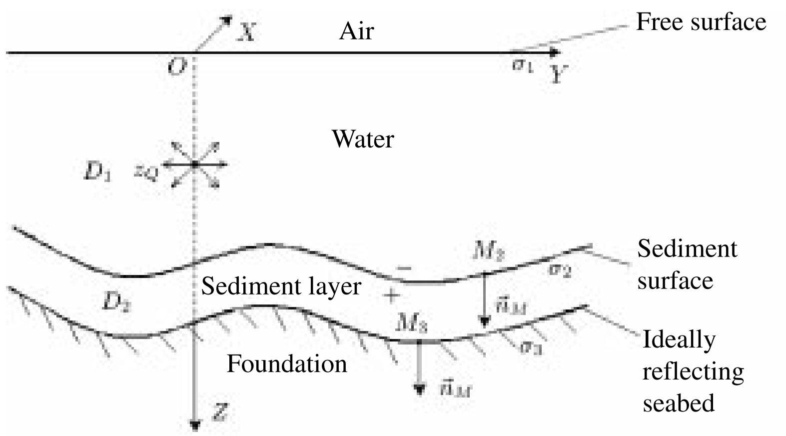

Fig. 1.

The integral equation method *a priori* imposes less restrictive constraints on the geometric properties of the water–air interface and the seabed. However, specialists in sound propagation in water have not used the integral equation method, possibly for the following reasons. If potential theory is used, then a simple layer potential should be placed on the water–air interface, since the pressure should be continuous here. Since the normal derivative of the acoustic pressure is continuous on the seabed, it follows that a double layer potential should be placed there. Now, by satisfying the boundary conditions, we obtain a system of two integral equations with two unknown functions. One of these integral equations is hypersingular. However, prior to 1980s, there were no sufficiently efficient numerical methods for solving hypersingular integral equations; therefore, scientists dealing with sound propagation in a water layer avoided such integral equations. Now there is a technique for numerically solving hypersingular integral equations in aerodynamics [12–14], and such boundary value problems can be solved numerically even if the seabed has edges and conical vertices, i.e., breaks and mountains.

In the present paper, we reduce various boundary value problems for the Helmholtz equation (to which some problems of sound propagation in a shallow sea can be reduced) to a system of integral equations and suggest a numerical solution method. In some cases, we obtain closed-form solutions of the integral equations; otherwise, a numerical solution is given.

1. GENERAL STATEMENT OF THE PROBLEM

Consider the problem of sound propagation in a water layer lying on a seabed consisting of a sediment layer (for example, silt) on a rigid ideally reflecting surface (see Fig. 1). In this case, to simplify the considerations, we first assume that the speeds of sound in water and in the layer on the seabed are constant but different. As the unknown function we take the acoustic pressure function $p(M)$. Then, by [5, 9–11], the problem of finding this function can be treated as the following boundary value problem.

Let a Cartesian coordinate system $OXYZ$ be given in R^3. Let the water layer occupy the domain D_1 with boundaries σ_1 and σ_2, and let the sediment layer fill the domain D_2 with boundaries σ_2 and σ_3, where σ_1 is the air–water interface, σ_2 is the water–sediment boundary, and σ_3 is the sediment boundary, which is a seabed ideally reflecting the sound. Let the surfaces σ_i, $i = 1, 2, 3$, be given by the equations

$$\psi_i = \psi_i(x, y), \qquad i = 1, 2, 3, \qquad (x, y) \in OXY,$$

where the functions $\psi_i(x, y)$, $i = 1, 2, 3$, are assumed to be continuous and piecewise smooth. Then we have

$$D_1 = \left\{ M(x, y, z) \in R^3,\ \psi_1(x, y) < z < \psi_2(x, y) \right\},$$
$$D_2 = \left\{ M(x, y, z) \in R^3,\ \psi_2(x, y) < z < \psi_3(x, y) \right\}.$$

We assume that there is a sound source at the point $M_Q\,(z_Q, 0, 0)$, $\psi_1(0,0) < z_Q < \psi_2(0,0)$ (see Fig. 1), which generates the sound pressure $p_Q(M)$ in the ambient space by the formula

$$p_Q(M) = \frac{Q}{4\pi}\frac{e^{ik_1 r_{MM_Q}}}{r_{MM_Q}}, \tag{1}$$

where the constant Q is the source intensity and r_{MM_Q} is the distance between points M and M_Q in space. The problem is to find the acoustic pressure function $p = p(M)$ defined in the domain $D = D_1 \cup D_2$ and satisfying the following conditions:

$$\Delta p(M) + k_i^2 p(M) = 0, \qquad M \in D_i, \qquad i = 1, 2, \tag{2}$$

where k_1 and k_2 are constants;

$$p(M) = 0, \qquad M \in \sigma_1, \tag{3}$$

$$[p(M)]_{\sigma_2} = 0, \qquad \left[\frac{1}{\varrho}\frac{\partial p(M)}{\partial n_M}\right]_{\sigma_2} = 0, \tag{4}$$

where $[\ ]_{\sigma_2}$ is the jump of the corresponding function across the surface σ_2, $\varrho = \varrho(M)$ is the layer density at a point M, $\varrho(M) = \varrho_i$, $M \in D_i$, $\varrho_i \neq 0$ is a constant, $i = 1, 2$, and $\vec{n}_M$ is the unit normal to the surface σ_2 at the point $M \in \sigma_2$;

$$\frac{\partial p(M)}{\partial n_M} = 0, \qquad M \in \sigma_3. \tag{5}$$

Let us clarify condition (4). At each point M of the surface σ_2, we choose a unit normal $\vec{n}_M$ in such a way that it varies continuously as the point moves on σ_2 (we assume that σ_2 is a smooth surface). For a function $f(M)$ defined in a neighborhood of the surface σ_2, by $f^{\pm}(M_0)$, $M_0 \in \sigma_2$, we denote the limit values of $f(M)$ as M approaches a point $M_0 \in \sigma_2$ from the side towards which the normal $\vec{n}_M$ is directed (the sign $+$) or from the opposite side (the sign $-$); see Fig. 1. Thus condition (4) can be represented in the form

$$p^+(M) = p^-(M), \qquad \left(\frac{1}{\varrho}\frac{\partial p(M)}{\partial n_M}\right)^+ = \left(\frac{1}{\varrho}\frac{\partial p(M)}{\partial n_M}\right)^-, \qquad M \in \sigma_2.$$

Note that the function $p_Q(M)$ is chosen so as to satisfy the radiation condition at infinity

$$\left(\frac{\vec{r}_M}{r_M}, \operatorname{grad} p_Q(M)\right) - ikp_Q(M) = O\left(\frac{1}{r_{MM_q}}\right) \tag{6}$$

as $r_{MM_Q} \to \infty$, where

$$r_M = |\vec{r}_M| = \left|x\vec{i} + y\vec{j} + z\vec{k}\right| = \sqrt{x^2 + y^2 + z^2}.$$

For a function $p(M)$ that is a solution of problem (2)–(5), we require the validity of condition (6) with r_{MM_Q} on the right-hand side replaced by the distance $\varrho(M, \sigma)$ from the point M to the boundary σ of the domain $D = D_1 \cup D_2$. But if $\varrho(M, \sigma) < C$ for each $M \in D$, then one should require that the solution contain no waves coming from infinity; this condition is satisfied provided that the solution is sought in the form of simple and double layer potentials with the function $\exp(ikr_{MM_0})$, $M_0 \in \sigma$, $M \in D$, where k_1 and k_2 are not eigenvalues for the domains D_1 and D_2, respectively.

2. SOME EXACT SOLUTIONS

To work through (i.e., test and justify) numerical schemes for complicated problems, one needs problems belonging to the class in question and admitting closed-form solutions.

First, consider the following problem. Let a water layer D occupy the half-space $z > 0$, let σ: $z = 0$ be the free surface, and let air fill the half-space $z < 0$. Suppose that at the point $M_Q\,(0, 0, z_Q)$, $z_Q > 0$, there is a sound source that generates the acoustic pressure $p_Q(M)$ in the ambient space by formula (1) with k_1 replaced by k. The problem is to find the pressure function $p(M)$ generated in the water layer by this source, i.e., find the function $p(M)$ satisfying the conditions

$$\Delta p(M) + k^2 p(M) = 0, \qquad M \in D = \{M(x, y, z),\ z > 0\}, \tag{7}$$

$$p(M) = 0, \qquad M \in \sigma, \tag{8}$$

and the radiation condition of the form (6) at infinity.

Obviously, the solution of problem (7), (8) has the form

$$p(M) = p_Q(M) + \tilde{p}_Q(M), \tag{9}$$

where

$$\tilde{p}_Q(M) = -\frac{Q}{4\pi}\frac{e^{ik_1 r_{M\tilde{M}_Q}}}{r_{M\tilde{M}_Q}}, \tag{10}$$

and $\tilde{M}_Q = M\,(0, 0, -z_Q)$; i.e., the function $\tilde{p}_Q(M)$ describes the acoustic pressure generated in the liquid by the source placed at the point $\tilde{M}_Q$ symmetric to the point M_Q around the plane OXY and having the opposite sign of Q. Indeed, the function $p_Q(M)$ and hence the function $\tilde{p}_Q(M)$ satisfy Eq. (7) at all points of D, and we have

$$p(M) = p(x, y, 0) = \frac{Q}{4\pi}\frac{\exp\left(ik\sqrt{x^2 + y^2 + (z_Q)^2}\right)}{\sqrt{x^2 + y^2 + (z_Q)^2}} + \frac{-Q}{4\pi}\frac{\exp\left(ik\sqrt{x^2 + y^2 + (z_Q)^2}\right)}{\sqrt{x^2 + y^2 + (z_Q)^2}} = 0$$

for points $M(x, y, 0)$ of the plane σ: $z = 0$.

Let us find the solution of problem (7), (8) in a different way. We model the water surface σ by a simple layer potential with density $g(M)$, $M \in \sigma$; i.e., the acoustic pressure $\tilde{p}_Q(M)$ due to the presence of the surface σ (the perturbed pressure) is sought in the form

$$\tilde{p}_Q(M) = \frac{1}{4\pi}\int\limits_{\sigma}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}g(M_0)\,d\sigma_{M_0}, \qquad M \notin \sigma,$$

and the desired acoustic pressure is sought in the form (9). The function $p(M) = p_Q(M) + \tilde{p}_Q(M)$ also satisfies Eq. (7) and the condition at infinity, and condition (8) is valid for the function $p(M)$ if the function $g(M)$ is found from the integral equation

$$\frac{1}{4\pi}\int\limits_{\sigma}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}g(M_0)\,d\sigma_{M_0} = -\frac{Q}{4\pi}\frac{e^{ikr_{MM_Q}}}{r_{MM_Q}}, \qquad M \in \sigma, \tag{11}$$

where $r_{MM_0} = \left((x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2\right)^{1/2} = \left|M\vec{M}_0\right| = |\vec{r}_{MM_0}|$.

Now consider another problem. The water occupies the domain D that is the half-space $z > 0$, and the ideally reflecting seabed is given by the surface (plane) $\sigma:\ z = 0$. At the point $M_Q\,(0, 0, z_Q)$, $z_Q > 0$, there is a sound source, which generates the acoustic pressure $p_Q(M)$ given by (1) in the ambient space.

The problem is to find the acoustic pressure $p(M)$ generated in the water layer by this source, i.e., find the function $p(M)$ satisfying Eq. (7) and the condition

$$\frac{\partial p(M)}{\partial n_M} = 0, \qquad M \in \sigma, \tag{12}$$

where the unit normal $\vec{n}_M$ to the surface σ at the point M is directed into the half-space

$$R_+^3 = \{M(x,y,z),\ z>0\}.$$

Obviously, the solution of problem (7), (12) has the form $p(M) = p_Q(M) + p_Q^*(M)$, where

$$p_Q^*(M) = \frac{Q}{4\pi}\frac{e^{ikr_{M\tilde{M}_Q}}}{r_{M\tilde{M}_Q}}$$

and $\tilde{M}_Q$ is the same point as in (10); i.e., the function $p_Q^*(M)$ describes the acoustic pressure generated in the fluid by the same source placed at the point $\tilde{M}_Q\,(0,0,-z_Q)$. Indeed, the functions $p_Q(M)$ and $p_Q^*(M)$ satisfy Eq. (7) in the domain D, and we have

$$\left.\frac{\partial p(M)}{\partial n_M}\right|_\sigma = \left.\frac{\partial p(M)}{\partial n_M}\right|_{z=0} = \left.\frac{\partial p_Q(M)}{\partial n_M}\right|_{z=0} + \left.\frac{\partial p_Q^*(M)}{\partial n_M}\right|_{z=0}$$
$$= \frac{Q}{4\pi}\left[\left.\left(e^{ikr_{MM_Q}}\frac{-(z_Q-z)\,ik}{\left(x^2+y^2+(z_Q-z)^2\right)^{1/2}}\frac{1}{r_{MM_Q}} + e^{ikr_{MM_Q}}\left(-r_{MM_Q}^{-2}\right)\frac{-(z_Q-z)}{r_{MM_Q}}\right)\right|_{z=0}\right.$$
$$\left. + \left.\left(\frac{ik\,(z_Q+z)}{r_{M\tilde{M}_Q}}e^{ikr_{M\tilde{M}_Q}}\frac{1}{r_{M\tilde{M}_Q}} + e^{ikr_{M\tilde{M}_Q}}\left(-r_{M\tilde{M}_Q}^{-2}\right)\frac{z_Q+z}{r_{M\tilde{M}_Q}}\right)\right|_{z=0}\right]$$
$$= \frac{Q}{4\pi}\frac{e^{ik(x^2+y^2+z_Q^2)^{1/2}}}{x^2+y^2+z_Q^2}z_Q\left(-ik - \left(x^2+y^2+z_Q^2\right)^{-1/2} + ik + \left(x^2+y^2+z_Q^2\right)^{-1/2}\right) \equiv 0, \qquad M\in\sigma.$$

Let us give a different solution of problem (7), (12). We model the seabed surface σ by a double layer potential with density $g(M)$, $M\in\sigma$; i.e., the acoustic pressure due to the presence of the seabed σ is sought in the form

$$p_Q^*(M) = \frac{1}{4\pi}\int_\sigma \frac{\partial}{\partial n_{M_0}}\left(\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\right) g\left(M_0\right) d\sigma_{M_0}, \qquad M\notin\sigma,$$

and the desired acoustic pressure is sought in the form (9). The function $p(M) = p_Q(M) + p_Q^*(M)$ also satisfies Eq. (7), and condition (12) is valid for the function $p(M)$ if the function $g(M)$ is a solution of the equation

$$\frac{1}{4\pi}\int_\sigma \frac{\partial}{\partial n_M}\frac{\partial}{\partial n_{M_0}}\left(\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\right) g\left(M_0\right) d\sigma_{M_0} = -\frac{Q}{4\pi}\frac{\partial}{\partial n_M}\left(\frac{e^{ikr_{MM_Q}}}{r_{MM_Q}}\right), \qquad M\in\sigma. \tag{13}$$

We have [15]

$$\frac{\partial}{\partial n_M}\frac{\partial}{\partial n_{M_0}}\left(\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\right) = e^{ikr_{MM_0}}\left[\frac{\left(1-ikr_{MM_0}\right)\left(\vec{n}_M,\vec{n}_{M_0}\right)}{r_{MM_0}^3}\right.$$
$$\left. + \left(\frac{k^2}{r_{MM_0}^3} - \frac{3\left(1-ikr_{MM_0}\right)}{r_{MM_0}^5}\right)\left(\vec{r}_{MM_0},\vec{n}_M\right)\left(\vec{r}_{MM_0},\vec{n}_{M_0}\right)\right], \tag{14}$$

$$\frac{\partial}{\partial n_M}\left(\frac{e^{ikr_{MM_Q}}}{r_{MM_Q}}\right) = e^{ikr_{MM_Q}}\frac{\left(1-ikr_{MM_Q}\right)\left(\vec{r}_{MM_Q},\vec{n}_M\right)}{r_{MM_Q}^3}. \tag{15}$$

For the special case in which the seabed is a plane, Eq. (13) acquires the form

$$\frac{1}{4\pi}\int_{\sigma} e^{ikr_{MM_0}}\left(\frac{1}{r^3_{MM_0}}-\frac{ik}{r^2_{MM_0}}\right)g\left(M_0\right)dx_0dy_0 = -\frac{Q}{4\pi}e^{ikr_{MM_Q}}\frac{z_Q\left(1-ikr_{MM_Q}\right)}{r^3_{MM_Q}}, \qquad M\in\sigma. \tag{16}$$

Now it is clear that the integrals occurring in (13) and (16) cannot be treated in the ordinary sense and should be understood as hypersingular integrals [13, 14]. Numerical methods for such equations also have specific features [13–15].

Remark 1. Since, in Eqs. (11) and (16), the surface σ is a plane, it follows that these equations can be reduced to one-dimensional integral equations. Indeed, on the plane $R^2(x,y)$, which is the integration domain in these equations, we introduce the polar coordinates $x=r\cos\varphi$ and $y=r\sin\varphi$, and in the space R^3, we introduce cylindrical coordinates. Then we have

$$r_{MM_Q}=\left(x^2+y^2+z_Q^2\right)^{1/2}=\left(r^2+z_Q\right)^{1/2},$$
$$r_{MM_0}=\left(\left(x_0-x\right)^2+\left(y_0-y\right)^2\right)^{1/2}=\left(r_0^2+r^2-2r_0r\cos\left(\varphi_0-\varphi\right)\right)^{1/2}.$$

Now one can write

$$\frac{e^{ikr_{MM_0}}}{r_{MM_0}}=K_1\left(r,r_0,\varphi_0-\varphi\right), \qquad e^{ikr_{MM_0}}\left(\frac{1}{r^3_{MM_0}}-\frac{ik}{r^2_{MM_0}}\right)=K_2\left(r,r_0,\varphi_0-\varphi\right), \tag{17}$$

where K_1 and K_2 are 2π-periodic functions of φ_0 and φ.

By (17), Eqs. (11) and (16) can be rewritten in the form

$$\frac{1}{4\pi}\int_0^{+\infty}\left(\int_0^{2\pi}K_1\left(r,r_0,\varphi_0-\varphi\right)g\left(r_0,\varphi_0\right)d\varphi_0\right)r_0dr_0 = -\frac{Q}{4\pi}\frac{\exp\left(ik\left(r^2+z_Q^2\right)^{1/2}\right)}{\left(r^2+z_Q^2\right)^{1/2}}, \qquad r\in(0,+\infty), \tag{18}$$

$$\frac{1}{4\pi}\int_0^{+\infty}\left(\int_0^{2\pi}K_2\left(r,r_0,\varphi_0-\varphi\right)g\left(r_0,\varphi_0\right)d\varphi_0\right)r_0dr_0 = -\frac{Q}{4\pi}\frac{\exp\left(ik\left(r^2+z_Q^2\right)^{1/2}\right)\left(1-ik\left(r^2+z_Q^2\right)^{1/2}\right)z_Q}{\left(r^2+z_Q^2\right)^{3/2}}, \qquad r\in(0,+\infty). \tag{19}$$

The right-hand sides of Eqs. (18) and (19) are independent of the parameter φ. Therefore, if these equations have a unique solution, then the solution can be sought in the form $g\left(r_0,\varphi_0\right)=g\left(r_0\right)$, which is justified by physical considerations. Indeed, since the source exciting the acoustic field is axisymmetric and the plane σ is axisymmetric, it follows that the simple or double layer potential modeling the plane also should have an axisymmetric density function. Therefore, we should solve the equations

$$\frac{1}{4\pi}\int_0^{+\infty}\left(\int_0^{2\pi}K_1\left(r,r_0,\varphi_0-\varphi\right)d\varphi_0\right)g\left(r_0\right)r_0dr_0 = -\frac{Q}{4\pi}\frac{\exp\left(ik\left(r^2+z_Q^2\right)^{1/2}\right)}{\left(r^2+z_Q^2\right)^{1/2}}, \qquad r\in(0,+\infty), \tag{20}$$

$$\frac{1}{4\pi}\int\limits_0^{+\infty}\left(\int\limits_0^{2\pi}K_2\left(r,r_0,\varphi_0-\varphi\right)d\varphi_0\right)g\left(r_0\right)r_0dr_0$$
$$=-\frac{Q}{4\pi}\frac{\exp\left(ik\left(r^2+z_Q^2\right)^{1/2}\right)\left(1-ik\left(r^2+z_Q^2\right)^{1/2}\right)z_Q}{\left(r^2+z_Q^2\right)^{3/2}},\qquad r\in(0,+\infty). \tag{21}$$

Since the integral over the period of a periodic function is invariant under shifts of the argument, we find that the integral in parentheses in (20) and (21) is independent of φ. Therefore, Eqs. (20) and (21) can be rewritten in the form

$$\frac{1}{4\pi}\int\limits_0^{+\infty}K_i^*\left(r,r_0\right)g\left(r_0\right)r_0dr_0=f_i(r),\qquad r\in(0,+\infty),\qquad i=1,2,$$

where the $f_i(r)$, $i=1,2$, are the right-hand sides of Eqs. (20) and (21), respectively, and the $K_i^*\left(r,r_0\right)$, $i=1,2$ are the expressions in parentheses on the left-hand sides in the respective equations.

Remark 2. As follows from the argument carried out in [16] for the Laplace equation, the integral equations (11) and (16) can be solved in closed form for the Helmholtz equation in a half-space (whose boundary surface is a plane). Indeed, we consider the Helmholtz equation

$$\Delta u(M)+k^2u(M)=0,\qquad M\in R_+^3, \tag{22}$$

in the half-space $R_+^3=\{M(x,y,z),\ z>0\}$ with the Dirichlet condition

$$u(M)=f(M),\qquad M\in R^2=\{M(x,y,0)\}. \tag{23}$$

We seek the solution of problem (22), (23) in the form of the simple layer potential

$$u(M)=\frac{1}{4\pi}\int\limits_{R^2}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\tilde{g}\left(M_0\right)dx_0dy_0,\qquad M_0\in R_+^3. \tag{24}$$

The function $u(M)$ is a solution of the Dirichlet problem whenever $\tilde{g}\left(M_0\right)$ is a solution of the integral equation

$$\frac{1}{4\pi}\int\limits_{R^2}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\tilde{g}\left(M_0\right)dx_0dy_0=f(M),\qquad M\in R^2. \tag{25}$$

If we seek the solution of the same problem in the form of the double layer potential

$$u(M)=\frac{1}{4\pi}\int\limits_{R^2}\frac{\partial}{\partial n_{M_0}}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}g\left(M_0\right)dx_0dy_0,\qquad M_0\in R_+^3, \tag{26}$$

then, by virtue of properties of a double layer potential, on the boundary (the normal $\vec{n}_{M_0}$ is directed into R_+^3), we obtain

$$\frac{1}{4\pi}\int\limits_{R^2}\frac{\partial}{\partial n_{M_0}}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}g\left(M_0\right)dx_0dy_0+\frac{g(M)}{2}=f(M),\qquad M\in R^2. \tag{27}$$

However, by virtue of (15), the integral on the left-hand side in (27) is zero, and we obtain $g(M)=2f(M)$, $M\in R^2$. Therefore, the function $u(M)$ in (26) acquires the form

$$u(M)=\frac{1}{4\pi}\int\limits_{R^2}\frac{\partial}{\partial n_{M_0}}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\left(2f\left(M_0\right)\right)dx_0dy_0,\qquad M_0\in R_+^3. \tag{28}$$

Since the Dirichlet problem for the Helmholtz equation in R_+^3 is uniquely solvable [16] under the radiation condition at infinity (which holds in our case), it follows that the functions $u(M)$ in (24) and (28) coincide in R_+^3. Consequently, their normal derivatives on R^2 also coincide; hence we have

$$\begin{gathered}\frac{1}{4\pi}\frac{\partial}{\partial n_M}\int\limits_{R^2}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\tilde{g}\left(M_0\right)dx_0dy_0-\frac{\tilde{g}(M)}{2}\\ =\frac{1}{4\pi}\int\limits_{R^2}\frac{\partial}{\partial n_M}\frac{\partial}{\partial n_{M_0}}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\left(2f\left(M_0\right)\right)dx_0dy_0,\qquad M\in R^2.\end{gathered}\tag{29}$$

Again by virtue of (15), the integral on the left-hand side in (29) is zero, and

$$\tilde{g}(M)=-\frac{1}{4\pi}\int\limits_{R^2}\frac{\partial}{\partial n_M}\frac{\partial}{\partial n_{M_0}}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\left(4f\left(M_0\right)\right)dx_0dy_0,\qquad M_0\in R^2,\tag{30}$$

i.e., we have found the solution of Eq. (25).

We have thereby proved the following assertion.

Theorem 1. *If the integral in* (30) *exists for the function $f(M)$ in the Dirichlet condition* (23), *then the integral equation* (25) *is solvable, and the solution is given by* (30).

Now suppose that the function $u(M)$ satisfies the Helmholtz equation (22) and the Neumann condition

$$\partial u(M)/\partial n_M=f(M),\qquad M\in R^2.\tag{31}$$

We seek the solution of problem (22), (31) in the form of the double layer potential (26). The function (26) is a solution of the Neumann problem if $g(M)$ satisfies the equation

$$\frac{1}{4\pi}\int\limits_{R^2}\frac{\partial}{\partial n_M}\frac{\partial}{\partial n_{M_0}}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}g\left(M_0\right)dx_0dy_0=f(M),\qquad M\in R^2.\tag{32}$$

If we seek the solution of the same problem in the form of the simple layer potential (24), then, by virtue of the properties of its normal derivative, we obtain

$$\frac{1}{4\pi}\frac{\partial}{\partial n_M}\int\limits_{R^2}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\tilde{g}\left(M_0\right)dx_0dy_0-\frac{\tilde{g}(M)}{2}=f(M),\qquad M\in R^2.\tag{33}$$

Since the integral on the left-hand side in (33) is zero, we have $\tilde{g}(M)=-2f(M)$, $M\in R^2$. Therefore, the function (24) acquires the form

$$u(M)=\frac{1}{4\pi}\int\limits_{R^2}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\left(-2f\left(M_0\right)\right)dx_0dy_0,\qquad M\in R_+^3.\tag{34}$$

Since the Neumann problem for the Helmholtz equation on R_+^3 is uniquely solvable, we see that the functions $u(M)$ given by (26) and (34) coincide in R_+^3; consequently, their limit values on R^2 also coincide:

$$\begin{gathered}\frac{1}{4\pi}\int\limits_{R^2}\frac{\partial}{\partial n_{M_0}}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}g\left(M_0\right)dx_0dy_0+\frac{g(M)}{2}\\ =\frac{1}{4\pi}\int\limits_{R^2}\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\left(-2f\left(M_0\right)\right)dx_0dy_0,\qquad M\in R^2.\end{gathered}\tag{35}$$

Again by virtue of (15), the integral on the left-hand side in (35) is zero; therefore,

$$g(M) = -\frac{1}{4\pi}\int\limits_{R^2} \frac{e^{ikr_{MM_0}}}{r_{MM_0}} \left(4f\left(M_0\right)\right) dx_0 dy_0, \qquad M \in R^2. \tag{36}$$

Thus we have found the solution of Eq. (32).

We have thereby justified the following assertion.

Theorem 2. *If the integral in* (36) *exists for the function* $f(M)$ *in the Neumann condition* (31), *then the hypersingular integral equation* (32) *is solvable, and the solution is given by* (36).

Remark 3. Note that the right-hand sides of the integral equations (11) and (10) are $p_Q(M)$ and $\partial p_Q(M)/\partial n_M$, respectively, where $p_Q(M)$ is given by (1). If the domain in question contains a dipole rather than a source, then all preceding considerations remain valid. The integral equations (11) and (13) preserve their form, but $p_Q(M)$ is computed on the basis of the right-hand side of (15).

3. THE MATHEMATICAL MODEL OF THE PROBLEM FOR A WATER LAYER ON AN IDEALLY REFLECTING SEABED ON THE BASIS OF INTEGRAL EQUATIONS

Consider a water layer occupying the domain D shown in Fig. 2. The boundary of this domain consists of the surface σ_1 (the water–air interface) and the surface σ_2 (an ideally reflecting seabed). Suppose that, at the point $M_Q\,(0,0,z_Q) \in D$, there is a sound source generating the acoustic pressure $p(M)$ given by formula (1) with k_1 replaced by k. The problem is to find the acoustic pressure $p(M)$ in D satisfying Eq. (7), the conditions

$$p(M) = 0, \qquad M \in \sigma_1, \tag{37}$$

$$\frac{\partial p(M)}{\partial n_M} = 0, \qquad M \in \sigma_2, \tag{38}$$

and the radiation condition at infinity.

We seek $p(M)$ in the form

$$p(M) = p_Q(M) + p_1(M) + p_2(M), \tag{39}$$

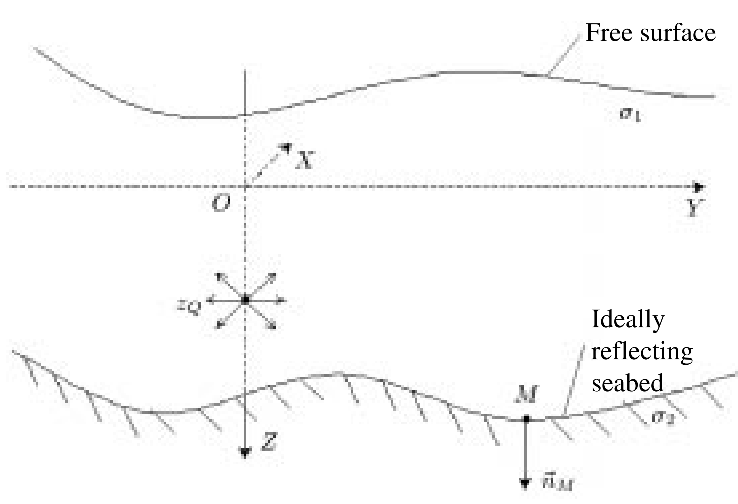

Fig. 2.

where

$$p_1(M) = \frac{1}{4\pi} \int\limits_{\sigma_1} \frac{e^{ikr_{MM_0}}}{r_{MM_0}} g_1(M_0)\, d\sigma_{1,M_0}, \qquad M \notin \sigma_1,$$
$$p_2(M) = \frac{1}{4\pi} \int\limits_{\sigma_2} \frac{\partial}{\partial n_{M_0}} \left(\frac{e^{ikr_{MM_0}}}{r_{MM_0}} \right) g_2(M_0)\, d\sigma_{2,M_0}, \qquad M \notin \sigma_2. \tag{40}$$

Then the function $p(M)$ given by (39) satisfies Eq. (7). It satisfies conditions (37) and (38) provided that the functions $g_1(M)$, $M \in \sigma_1$, and $g_2(M)$, $M \in \sigma_2$, are solutions of the system of integral equations

$$\frac{1}{4\pi} \int\limits_{\sigma_1} \frac{e^{ikr_{MM_0}}}{r_{MM_0}} g_1(M_0)\, d\sigma_{1,M_0} + \frac{1}{4\pi} \int\limits_{\sigma_2} \frac{\partial}{\partial n_{M_0}} \left(\frac{e^{ikr_{MM_0}}}{r_{MM_0}} \right) g_2(M_0)\, d\sigma_{2,M_0} = -\frac{Q}{4\pi} \frac{e^{ikr_{MM_Q}}}{r_{MM_Q}}, \qquad M \in \sigma_1, \tag{41}$$

$$\frac{1}{4\pi} \int\limits_{\sigma_1} \frac{\partial}{\partial n_M} \left(\frac{e^{ikr_{MM_0}}}{r_{MM_0}} \right) g_1(M_0)\, d\sigma_{1,M_0} + \frac{1}{4\pi} \int\limits_{\sigma_2} \frac{\partial}{\partial n_M} \frac{\partial}{\partial n_{M_0}} \left(\frac{e^{ikr_{MM_0}}}{r_{MM_0}} \right) g_2(M_0)\, d\sigma_{2,M_0} = -\frac{Q}{4\pi} \frac{\partial}{\partial n_M} \left(\frac{e^{ikr_{MM_Q}}}{r_{MM_Q}} \right), \qquad M \in \sigma_2. \tag{42}$$

Equation (41) is a weakly singular integral equation of the first kind for the function $g_1(M)$ on the surface σ_1, and Eq. (42) is a hypersingular integral equation of the first kind for the function $g_2(M)$ on the surface σ_2.

Since, in many problems on sound propagation in a water layer, the free water surface σ_1 can be treated as a plane, it follows that the solution of problem (7), (37), (38) can be reduced to the solution of a single hypersingular integral equation on the surface σ_2 rather than the solution of the system of integral equations (41) and (42). Let us outline the corresponding scheme.

We assume that σ_1 is the plane $z = 0$. At the point $M_Q(0, 0, z_Q) \in D$, $z_Q > 0$, there is a sound source, which generates the acoustic pressure $p_Q(M)$ as was mentioned above. Now consider a sound source skew-symmetric to the original source around the plane $z = 0$, that is, a sound source at the point $\tilde{M}_Q(0, 0, -z_Q)$, which generates the acoustic pressure $\tilde{p}_Q(M)$ by formula (10). Further, on the surface σ_2, we place a double layer potential with density $g_2(M)$, which generates the acoustic pressure $p_2(M)$ by formula (40). Consider the surface σ_2^* symmetric to the surface σ_2 around the plane $z = 0$. For example, if the equation of the surface σ_2 has the form $z = h(x, y)$, where $h(x, y)$ is a sufficiently smooth function in the space R^2 (on the plane OXY), and $h(x, y) \geq a > 0$ for each point $M(x, y) \in R^2$, then the equation of the surface σ_2^* has the form $z = -h(x, y)$, $(x, y) \in R^2$.

On the surface σ_2^*, we place a double layer potential skew-symmetric to the original one, i.e., having the density $g_2^*(M) = -g_2(x, y, h(x, y))$ at the point $M(x, y, -h(x, y))$, and generating the acoustic pressure $p_2^*(M)$ by the formula

$$p_2^*(M) = \frac{1}{4\pi} \int\limits_{\sigma_2^*} \frac{\partial}{\partial n_{M_0^*}} \left(\frac{e^{ikr_{MM_0^*}}}{r_{MM_0^*}} \right) g_2^*(M_0^*)\, d\sigma_{2,M_0^*}^*, \qquad g_2^*(x, y, -h(x, y)) = -g_2(x, y, h(x, y)).$$

Then the function $p(M)$ given by the formula

$$p(M) = p_Q(M) + \tilde{p}_Q(M) + p_2(M) + p_2^*(M), \qquad M \in D,$$

satisfies Eq. (7) and condition (37) on the boundary σ_1, that is, on the plane $z = 0$. This function $p(M)$ satisfies condition (38) on the surface σ_2 provided that $g_2(x, y, h(x, y))$ is found from the integral equation

$$\frac{1}{4\pi}\int\limits_{\sigma_2}\frac{\partial}{\partial n_M}\frac{\partial}{\partial n_{M_0}}\left(\frac{e^{ikr_{MM_0}}}{r_{MM_0}}\right)g_2\left(M_0\right)d\sigma_2+\frac{1}{4\pi}\int\limits_{\sigma_2^*}\frac{\partial}{\partial n_M}\frac{\partial}{\partial n_{M_0^*}}\left(\frac{e^{ikr_{MM_0^*}}}{r_{MM_0^*}}\right)g_2^*\left(M_0^*\right)d\sigma_2^*$$
$$=-\frac{Q}{4\pi}\frac{\partial}{\partial n_M}\left(\frac{e^{ikr_{MM_Q}}}{r_{MM_Q}}\right)+\frac{Q}{4\pi}\frac{\partial}{\partial n_M}\left(\frac{e^{ikr_{M\tilde{M}_Q}}}{r_{M\tilde{M}_Q}}\right),\qquad M\in\sigma_2. \tag{43}$$

Equation (43) can be rewritten as an equation on the domain of the function $z = h(x, y)$, that is, the surface $\sigma_1 = R^2$. To this end, we note that

$$d\sigma_2^*=d\sigma_2=\left(1+h'^2_{x_0}\left(x_0,y_0\right)+h'^2_{y_0}\left(x_0,y_0\right)\right)^{1/2}dx_0dy_0$$

and points on the surfaces σ_2 and σ_2^* have coordinates

$$M_0\left(x_0,y_0,h\left(x_0,y_0\right)\right)\quad\text{and}\quad M_0^*\left(x_0,y_0,-h\left(x_0,y_0\right)\right),$$

respectively.

Further, we have

$$g_2^*\left(M_0^*\right)=g_2^*\left(x_0,y_0,-h\left(x_0,y_0\right)\right)=-g_2\left(x_0,y_0,h\left(x_0,y_0\right)\right)=-g_2\left(M_0\right),$$

$M_0 \in \sigma_2$, $M_0^* \in \sigma_2^*$. If $M_0 \in \sigma_2$, then

$$\vec{n}_{M_0}=\left(-h'_{x_0}\left(x_0,y_0\right)\vec{i}-h'_{y_0}\left(x_0,y_0\right)\vec{j}+\vec{k}\right)\left(1+h'^2_{x_0}\left(x_0,y_0\right)+h'^2_{y_0}\left(x_0,y_0\right)\right)^{-1/2},$$

and if $M_0^* \in \sigma_0^*$, then

$$\vec{n}_{M_0^*}=\left(-h'_{x_0}\left(x_0,y_0\right)\vec{i}-h'_{y_0}\left(x_0,y_0\right)\vec{j}-\vec{k}\right)\left(1+h'^2_{x_0}\left(x_0,y_0\right)+h'^2_{y_0}\left(x_0,y_0\right)\right)^{-1/2},$$

where $\vec{n}_{M_0^*}$ is the unit normal to σ_2^* at M_0^* symmetric to the vector $\vec{n}_{M_0}$ around the plane $z = 0$. Now by $K_{N,H}(M, M_0)$ we denote the right-hand side of (14), and by $f(M, M_Q)$ we denote the right-hand side of (15). Then Eq. (43) acquires the form

$$\frac{1}{4\pi}\int\limits_{R^2}\left(K_{N,H}\left(M,M_0\right)-K_{N,H}\left(M,M_0^*\right)\right)g_2\left(M_0\right)\left(1+h'^2_{x_0}\left(x_0,y_0\right)+h'^2_{y_0}\left(x_0,y_0\right)\right)^{1/2}dx_0dy_0$$
$$=\frac{Q}{4\pi}\left(-f\left(M,M_Q\right)+f\left(M,\tilde{M}_Q\right)\right),\qquad M\in\sigma_2, \tag{44}$$

where

$$M=M(x,y,h(x,y)),\qquad M_0=M_0\left(x_0,y_0,h\left(x_0,y_0\right)\right),\qquad M_0^*=M_0\left(x_0,y_0,-h\left(x_0,y_0\right)\right),$$
$$M_Q=M\left(0,0,z_Q\right),\qquad \tilde{M}_Q=M\left(0,0,-z_Q\right).$$

It follows from (14) that Eq. (44) is a hypersingular integral equation on the plane R^2.

Remark 4. These considerations show that one can consider the case near the sea coast.

The problem is again reduced to a hypersingular integral equation of the form (44), but the integral is taken over the half-plane R_L^2 bounded by the curve L that is the coastline (i.e., $L = \sigma_1 \cap \sigma_2$) rather than over the plane R^2, and the parametric point (x, y) also belongs to R_L^2.

4. THE MATHEMATICAL MODEL OF PROBLEM FOR A WATER LAYER ON A LAYERED SEABED ON THE BASIS OF INTEGRAL EQUATIONS

In this section, we consider problem (2)–(5) for a water layer on a seabed that consists of one layer lying on an ideally reflecting surface. We assume that, at the point $M_Q\,(0,0,z_Q) \in D_1$, there is a sound source generating the acoustic pressure $p_Q(M)$ by formula (1) in the domain in question.

The problem is actually to find $p_1(M)$, $M \in D_1$, and $p_2(M)$, $M \in D_2$, satisfying Eq. (2) in the domains D_1 and D_2, respectively. We seek these functions in the form

$$p_1(M) = p_Q(M) + p_{\sigma_1}(M) + p_{\sigma_2^-}(M), \qquad M \in D_1, \tag{45}$$

where $p_Q(M)$ is given by (1),

$$p_{\sigma_1}(M) = \frac{1}{4\pi}\int\limits_{\sigma_1} \frac{e^{ik_1 r_{MM_{01}}}}{r_{MM_{01}}} g_1\,(M_{01})\,d\sigma_1, \qquad M \in D_1, \qquad M \notin \sigma_1, \tag{46}$$

$$p_{\sigma_2^-}(M) = \frac{1}{4\pi}\int\limits_{\sigma_1} \frac{\partial}{\partial n_{M_{02}}} \frac{e^{ik_1 r_{MM_{02}}}}{r_{MM_{02}}} g_{2,-}\,(M_{02})\,d\sigma_2, \qquad M \in D_1, \qquad M \notin \sigma_2, \tag{47}$$

and

$$p_2(M) = p_{\sigma_2^+}(M) + p_{\sigma_3}(M), \qquad M \in D_2, \tag{48}$$

where

$$p_{\sigma_2^+}(M) = \frac{1}{4\pi}\int\limits_{\sigma_2} \frac{\partial}{\partial n_{M_{02}}} \frac{e^{ik_2 r_{MM_{02}}}}{r_{MM_{02}}} g_{2,+}\,(M_{02})\,d\sigma_2, \qquad M \in D_2, \qquad M \notin \sigma_2, \tag{49}$$

$$p_{\sigma_3}(M) = \frac{1}{4\pi}\int\limits_{\sigma_3} \frac{\partial}{\partial n_{M_{03}}} \frac{e^{ik_2 r_{MM_{03}}}}{r_{MM_{03}}} g_3\,(M_{03})\,d\sigma_3, \qquad M \in D_2, \qquad M \notin \sigma_3. \tag{50}$$

It follows from (45)–(47) that the function $p_1(M)$ satisfies Eq. (2) in the domain D_1, and formulas (48)–(50) imply that the function $p_2(M)$ satisfies Eq. (2) in the domain D_2. The functions $p_1(M)$ and $p_2(M)$ satisfy the boundary conditions (3)–(5) provided that the functions $g_1\,(M_{01})$, $g_{2,-}\,(M_{02})$, $g_{2,+}\,(M_{02})$, and $g_3\,(M_{03})$ satisfy the system of integral equations

$$p_{\sigma_1}^+\,(M_1) + p_{\sigma_2^-}\,(M_1) = -p_Q\,(M_1)\,, \qquad M_1 \in \sigma_1, \tag{51}$$

$$p_Q\,(M_2) + p_{\sigma_1}\,(M_2) + p_{\sigma_2^-}^-\,(M_2) = p_{\sigma_2^+}^+\,(M_2) + p_{\sigma_3}\,(M_2)\,, \qquad M_2 \in \sigma_2, \tag{52}$$

$$\frac{1}{\varrho_1^-}\left(\frac{\partial p_Q(M_2)}{\partial n_{M_2}} + \frac{\partial p_{\sigma_1}(M_2)}{\partial n_{M_2}} + \frac{\partial p_{\sigma_2^-}^-(M_2)}{\partial n_{M_2}}\right) = \frac{1}{\varrho_2^+}\left(\frac{\partial p_{\sigma_2^+}^+(M_2)}{\partial n_{M_2}} + \frac{\partial p_{\sigma_3}(M_2)}{\partial n_{M_2}}\right), \quad M_2 \in \sigma_2, \tag{53}$$

$$\frac{\partial p_{\sigma_2^+}\,(M_3)}{\partial n_{M_3}} + \frac{\partial p_{\sigma_3}^-\,(M_3)}{\partial n_{M_3}} = 0, \qquad M_3 \in \sigma_3. \tag{54}$$

By taking account of the relationship between the direct values of simple and double layer potentials and their normal derivatives as well as the relationship between their limit values on the surfaces where the potentials sit, one can rewrite Eqs. (51)–(54) in the form

$$p_{\sigma_1}\,(M_1) + p_{\sigma_2^-}\,(M_1) = -p_Q\,(M_1)\,, \qquad M_1 \in \sigma_1, \tag{55}$$

$$p_Q(M_2) + p_{\sigma_1}(M_2) + p_{\sigma_2^-}(M_2) - \frac{g_{2,-}(M_2)}{2} = p_{\sigma_2^+}(M_2) + \frac{g_{2,+}(M_2)}{2} + p_{\sigma_3}(M_2)\,, \quad M_2 \in \sigma_2, \tag{56}$$

$$\frac{1}{\varrho_1^-}\left(\frac{\partial p_Q(M_2)}{\partial n_{M_2}} + \frac{\partial p_{\sigma_1}(M_2)}{\partial n_{M_2}} + \frac{\partial p_{\sigma_2^-}(M_2)}{\partial n_{M_2}}\right) = \frac{1}{\varrho_2^+}\left(\frac{\partial p_{\sigma_2^+}(M_2)}{\partial n_{M_2}} + \frac{\partial p_{\sigma_3}(M_2)}{\partial n_{M_2}}\right), \quad M_2 \in \sigma_2, \tag{57}$$

$$\frac{\partial p_{\sigma_2^+}\,(M_3)}{\partial n_{M_3}} + \frac{\partial p_{\sigma_3}^-\,(M_3)}{\partial n_{M_3}} = 0, \qquad M_3 \in \sigma_3. \tag{58}$$

Equation (55) is a weakly singular integral equation for the function $g_1(M_1)$, which is uniquely solvable for that function, Eq. (56) is a Fredholm integral equation of the second kind for the functions $g_{2,-}(M_2)$ and $g_{2,+}(M_2)$, Eq. (57) is a hypersingular equation for the functions $g_{2,-}(M_2)$ and $g_{2,+}(M_2)$, and Eq. (58) is a hypersingular integral equation of the first kind for the function $g_3(M_2)$. We have thereby obtained a system of four integral equations for four unknown functions.

Remark 5. It is of interest to note that if the surface σ_2 is a plane, then it follows from (15) that $p_{\sigma_2^-}(M_2) \equiv p_{\sigma_2^+}(M_2) \equiv 0$, $M_2 \in \sigma_2$; therefore, Eq. (56) is not an integral equation for the functions $g_{2,-}(M_2)$ and $g_{2,+}(M_2)$. From this equation, one of these functions can be expressed via the other, and the system of four equations for four functions can be reduced to a system of three integral equations for three functions.

Remark 6. If the free surface of water is also a plane, then the problem can be reduced to a system of two integral equations.

Indeed, in this case, we take the function $p_1(M)$, $M \in D_1$, in the form

$$p_1(M) = p_Q(M) + \tilde{p}_Q(M) + p_{\sigma_2^-}(M) + \tilde{p}_{\sigma_2^-}(M), \qquad M \in D_1,$$

where

$$\tilde{p}_Q(M) = -\frac{Q}{4\pi}\frac{e^{ik_1 r_{M\tilde{M}_Q}}}{r_{M\tilde{M}_Q}}, \qquad \tilde{M}_Q(0,0,-z_Q),$$

$$\tilde{p}_{\sigma_2^-}(M) = \frac{1}{4\pi}\int\limits_{\sigma_2^*}\frac{\partial}{\partial n_{M_0^*}}\left(\frac{e^{ik_1 r_{MM_{02}^*}}}{r_{MM_{02}^*}}\right) g_{2,-}(M_{02}^*)\,d\sigma_2,$$

σ_2^* is the surface symmetric to the surface σ_2 around the plane σ_1, $g_{2,-}^*(M_{02}^*) = -g_{2,-}(M_{02})$, and $M_{02}^* \in \sigma_2^*$ is the point symmetric to the point $M_{02} \in \sigma_2$ around the plane σ_1. Then the function $p_1(M)$ satisfies Eq. (2) in the domain D_1 and condition (3) on the surface σ_1. Therefore, it remains to satisfy conditions (4) and (5), and if the surface σ_2 is also a plane, then the original problem can be reduced to a system of two integral equations for two unknown functions.

5. EXAMPLES OF NUMERICAL COMPUTATIONS

For the numerical solution of the integral equations obtained in the preceding sections, we use a method analogous to the method of closed discrete vortex frames [13, 15].

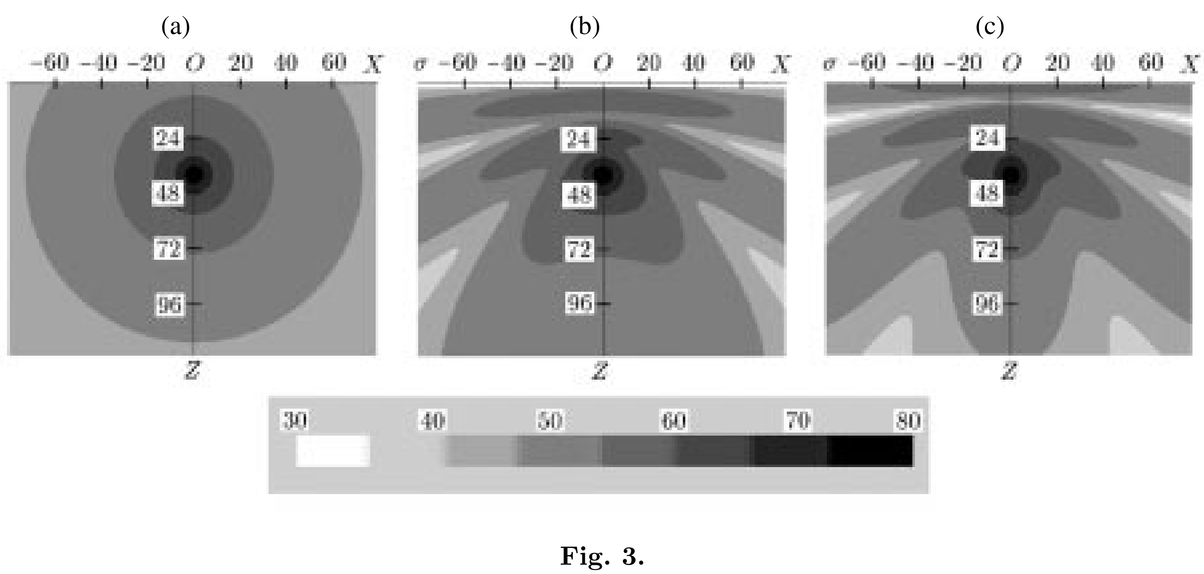

Fig. 3.

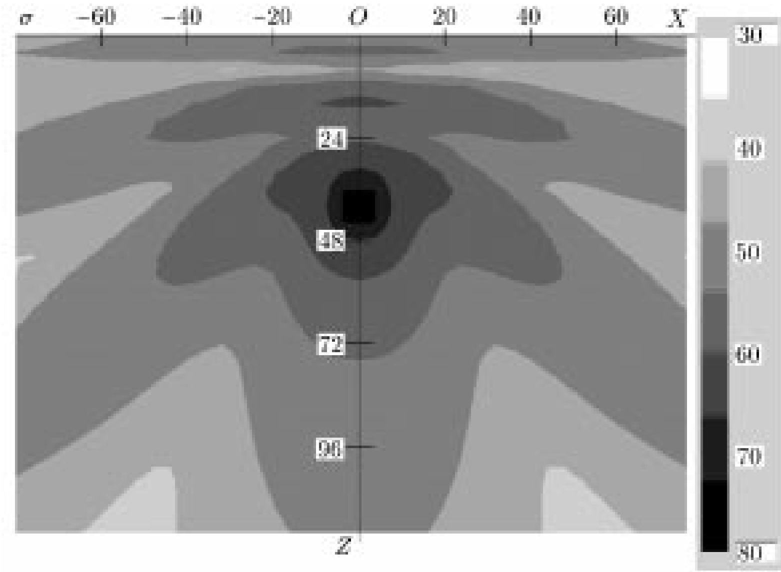

Fig. 4.

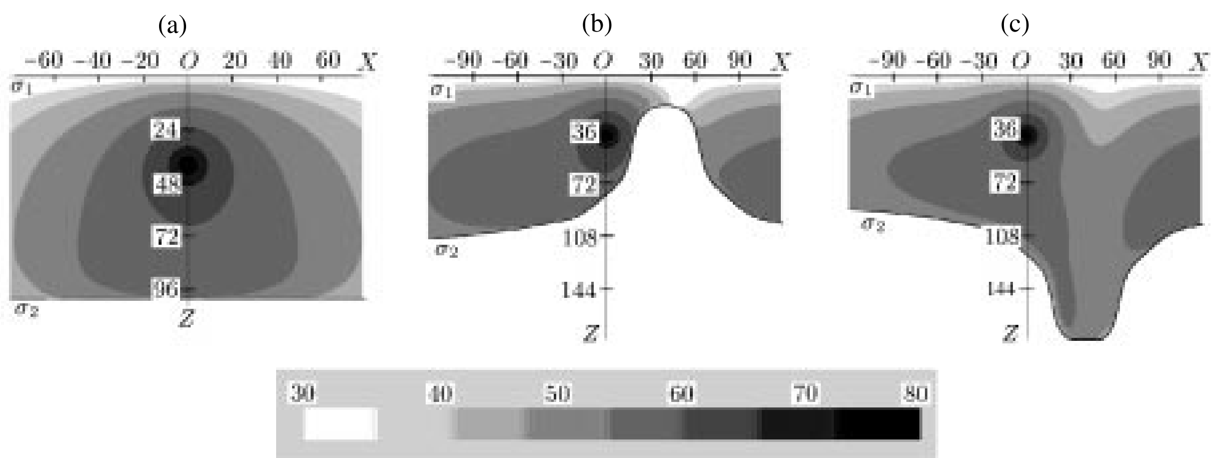

Fig. 5.

We compute the amplitude $p_{\max}(M) = ((\operatorname{Re}(p(M)))^2 + (\operatorname{Im}(p(M)))^2)^{1/2}$ of the excess pressure in decibels with respect to some reference pressure p_0 by the formula

$$N_p = 20 \lg \frac{p_{\max}(M)}{p_0} \text{ dB}.$$

In computations, we take $p_0 = 2 \times 10^{-5}$ Pa, which corresponds to the standard hearing.

Figure 3 represents the hydroacoustic field of a point sound source of intensity 1 W placed at a depth of 40 m. In this case, the hydroacoustic field is shown in the unbounded medium (Fig. 3a) and in the medium (Fig. 3b) bounded by the plane XOY on which condition (8) or condition (12) (Fig. 3c) is satisfied. Here we assume that the radiation frequency is 50 Hz and the speed of sound is 1450 m/sec.

In Fig. 3, one can see that if condition (8) is valid on the surface σ, then the sound loudness decreases and acoustic energy goes partly to the domain $z < 0$. But if condition (12) is valid on the surface σ, then, on the opposite, the loudness grows, and the sound is "reflected" in the surface σ.

We found the hydroacoustic field with the use of the numerical solution of the hypersingular integral equation (13) on the surface σ. The numerical results are shown in Fig. 4. Here σ is a disk of radius 1000 m on the plane OXY, and the number of partitions on this surface is 20×20. By comparing this figure with Fig. 3c, one can see that the pictures of the hydroacoustic field coincide except for the domain near the surface σ. This is explained by the fact that the integral equation (13) is solved numerically. The coincidence of the results obtained in the comparison of the hydroacoustic field of the solution in closed form and the numerical solution allows one to claim that the above-described method provides a numerical solution of the considered problem.

Consider problem (7), (37), (38) for the case in which the wave conductor has a constant depth. Let the depth of the waveguide be 100 m, and let the sound source be placed at a depth of 40 m. The hydroacoustic field computed in this waveguide is shown in Fig. 5a. The computation was performed for a frequency of 5 Hz with the use of the technique described in Section 3; we have taken account of the fact that the sea surface σ_1 is the plane OXY.

We have analyzed the influence of the seabed inhomogeneity on the hydroacoustic field under the assumption that the seabed is not even [for example, there is a "hill" (Fig. 5b)]. A similar result in the case of a "canyon" is represented in Fig. 5c.

ACKNOWLEDGMENTS

The work was financially supported by the Russian Foundation for Basic Research (project no. 02.01.00045) and the Program of Scientific Schools (project no. NSh-1784.001.01).

REFERENCES

1. Andreev, M.Yu., Katsnel'son, B.G., Kulapin, L.G., and Petnikov, V.G., *Akust. Zh.*, 1996, vol. 42, no. 4, pp. 459–464.
2. Bondar', L.F., Borisov, S.V., and Gritsenko, A.V., *Akust. Zh.*, 1994, vol. 40, no. 4, pp. 561–570.
3. Belov, A.I., Serebryany, A.N., and Zhuravlev, V.A., Observation of Internal Wave Effects on Acoustic and Oceanographic Field Study in the Middle Atlantic Bight, *Proc. Int. Conf. on Shallow Water Acoustics*, Beijing, 1997.
4. Caille, G.W., Dahl, P.H., Zijun Gan, *et al.*, Overview of the Joint China-U.S. "Yellow Sea '96" Experiment, *Proc. Int. Conf. on Shallow Water Acoustics*, Bejing, 1997.
5. Brekhovskikh, L.M., *Volny v sloistykh sredakh* (Waves in Layered Media), Moscow, 1957.
6. Pekeris, C.L., *Geol. Soc. Am. Mem.*, 1948, vol. 27, pp. 1–117.
7. *Ocean Acoustics*, De Santo, J., Ed., Berlin: Springer, 1979. Translated under the title *Akustika okeana*, Moscow, 1982.
8. Gazaryan, Yu.L., *Akust. Zh.*, 1958, vol. 4, no. 3, pp. 233–238.
9. Lependin, L.F., *Akustika* (Acoustics), Moscow: Vysshaya Shkola, 1978.
10. *Wave Propagation and Underwater Acoustics*, Keller, J.B. and Papadakis, J.S., Eds., Berlin: Springer, 1977. Translated under the title *Rasprostranenie voln i podvodnaya akustika*, Moscow, 1980.
11. Katsnel'son, B.G. and Petnikov, V.G., *Akustika melkogo morya* (Acoustics of Shallow Sea), Moscow, 1997.
12. Belotserkovskii, S.M. and Lifanov, I.K., *Chislennye metody v singulyarnykh integral'nykh uravneniyakh i ikh primenenie v aerodinamike, teorii uprugosti, elektrodinamike* (Numerical Methods in Singular Integral Equations and Their Application in Aerodynamics, Elasticity, and Electrodynamics), Moscow: Nauka, 1985.
13. Lifanov, I.K., *Metod singulyarnykh integral'nykh uravnenii i chislennyi eksperiment v matematicheskoi fizike, aerodinamike, teorii uprugosti i difraktsii voln* (The Method of Singular Integral Equations and Numerical Experiment in Mathematical Physics, Aerodynamics, Elasticity, and Wave Diffraction), Moscow: Yanus, 1995.
14. Vainikko, G.M., Lifanov, I.K., and Poltavskii, L.N., *Chislennye metody v gipersingulyarnykh integral'nykh uravneniyakh i ikh prilozheniya* (Numerical Methods in Hypersingular Integral Equations and Their Applications), Moscow, 2001.
15. Dovgii, S.A. and Lifanov, I.K., *Metody resheniya integral'nykh uravnenii* (Methods for Solving Integral Equations), Kiev, 2002.
16. Sobolev, S.L., *Uravneniya matematicheskoi fiziki* (Equations of Mathematical Physics), Moscow, 1992.