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Reduction for block-transitive t-(k^2, k, λ) designs

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Abstract
In this paper, we study block-transitive automorphism groups of t-(k^2, k, λ) designs. We prove that a block-transitive automorphism group G of a t-(k^2, k, λ) design must be point-primitive, and G is either an affine group or an almost simple group. Moreover, the nontrivial t-(k^2, k, λ) designs admitting block-transitive automorphism groups of almost simple type with sporadic socle and alternating socle are classified.

Keywords t-design · Automorphism group · Primitivity · Block-transitivity

Mathematics Subject Classification 05B05 · 05B25 · 20B25

1 Introduction

Let $\mathcal{P}$ be a finite set with v elements, called points, and let $\mathcal{B}$ be a set of k-subsets of $\mathcal{P}$ called blocks. Then a t-(v, k, λ) design is a structure $\mathcal{D}$=$(\mathcal{P}, \mathcal{B})$, such that every set of t distinct points is contained in exactly λ blocks. A t-(v, k, λ) design is *nontrivial* if $t < k < v$ and *simple* if repeated blocks are not allowed. All the t-(v, k, λ) designs are supposed to be nontrivial and simple in this article. An *automorphism* of $\mathcal{D}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The full automorphism group of $\mathcal{D}$ is denoted by $\mathrm{Aut}(\mathcal{D})$ and any subgroup of $\mathrm{Aut}(\mathcal{D})$ is called an *automorphism group* of $\mathcal{D}$. If a subgroup G of $\mathrm{Aut}(\mathcal{D})$ acts transitively on the set of points (resp. blocks), then we say that $\mathcal{D}$ is *point-transitive* (resp. *block-transitive*). Owing to the result of Block [1], $\mathcal{D}$ is point-transitive if $\mathcal{D}$ is block-transitive. Similarly, if the automorphism group G acts primitively on points, then we say that $\mathcal{D}$ is *point-primitive*, otherwise, $\mathcal{D}$ is said to be *point-imprimitive*.

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In [9], Montinaro and Francot showed that a flag-transitive automorphism group G of a 2-(k^2, k, λ) design, with $\lambda \mid k$, is either an affine group or an almost simple group, and then classified the case of almost simple type [8, 9], and the case of affine type [7]. As a generalization of this result, in this paper we study block-transitive t-(k^2, k, λ) designs without the condition that $\lambda \mid k$. Since $(\mathcal{P}, B^G)$ is a block-transitive 1-design for any non-empty subset B of $\mathcal{P}$ if G is a transitive permutation group on $\mathcal{P}$, we always suppose that $t \geq 2$ in this paper. We now state the main results.

Theorem 1 *A block-transitive automorphism group G of a t-(k^2, k, λ) design must be point-primitive, and G is either an affine group or an almost simple group.*

Recall that for a primitive permutation group of almost simple type, the socle of the group is an alternating group, a classical group, an exceptional group of Lie type or a sporadic group. Based on Theorem 1, we analyze the sporadic case and alternating case in Section 4 and Section 5, respectively.

Theorem 2 *Let G be a block-transitive automorphism group of a t-(k^2, k, λ) design with sporadic socle. Then $\mathrm{Soc}(G) = HS$, the Higman-Sims simple group, and $t = 2$, $k = 10$. Moreover, one of the following holds:*

(1) $\lambda \in \{2^i \cdot 3 \cdot 5^2 \cdot 7 \mid i = 5, 6, 7, 8, 9\}$;
(2) $\lambda \in \{2^7 \cdot 3^2 \cdot 7\} \cup \{2^i \cdot 3^2 \cdot 5 \cdot 7 \mid i = 6, 7, 8, 9\}$;
(3) $\lambda \in \{2^i \cdot 3^2 \cdot 5^2 \cdot 7 \mid i = 0, 1, \ldots, 9\}$.

Remark 1 (a) $7 \nmid |G_B|$ for $B \in \mathcal{B}$. Let $\alpha \in \mathcal{P}$, then $G_\alpha = M_{22}$ if $G = HS$, and $G_\alpha = M_{22} : 2$ if $G = HS : 2$.
(b) In Theorem 2, Case (1) holds if and only if $3 \mid |G_B|$, Case (2) holds if and only if $5 \mid |G_B|$, Case (3) holds if and only if $3 \nmid |G_B|$ and $5 \nmid |G_B|$, and there is no design with $15 \mid |G_B|$, here B is a block of the 2-$(10^2, 10, \lambda)$ design.
(c) In Theorem 2, there exist block-transitive, but not flag-transitive 2-$(10^2, 10, \lambda)$ designs, which are given in Example 1. If $G = HS$ is flag-transitive, then $\lambda = 2^7 \cdot 3^2 \cdot 7$ or $2^6 \cdot 3^2 \cdot 5 \cdot 7$, and the designs are given in Lemma 4.4.

Theorem 3 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(k^2, k, λ) design, $G \leq \mathrm{Aut}(\mathcal{D})$ be block-transitive with alternating socle A_n. Then $G = A_n (n > 6)$, the point-stabilizer $G_\alpha = (S_\ell \times S_{n-\ell}) \cap A_n$, and one of the following hold:*

(1) *$\ell = 1$, $v = n = k^2$, and $\mathcal{D}$ is a t-$(k^2, k, \binom{k^2-t}{k-t})$ design, where $\mathcal{P} = \{1, 2, \ldots, k^2\}$, and $\mathcal{B} = \mathcal{P}^{\{k\}}$, the set of all k-subsets of $\mathcal{P}$;*
(2) *$\ell = 2$, $n = 9$, and $\mathcal{D}$ is a 2-$(6^2, 6, \lambda)$ design, where $\lambda \in \{2^2 \cdot 5, 3^2 \cdot 5, 2^3 \cdot 3 \cdot 5, 2^4 \cdot 3^2 \cdot 5\} \cup \{2^i \cdot 3^3 \cdot 5 \mid i = 2, 3, 4, 5\}$;*
(3) *$\ell = 3$, $n = 50$, and $\mathcal{D}$ is a 2-$(140^2, 140, \lambda)$ design with $43 \mid \lambda$. Especially, $19 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \mid \lambda$ if G is flag-transitive.*

Remark 2 In Theorem 3, the designs of Case (1) are complete, and all designs of Case (2) are given in Table 1. The existence of designs of Case (3) is an open question. Thus we ask:

Question: If a block-transitive 2-$(140^2, 140, \lambda)$ exists for some λ ?

2 Preliminaries

We first collect some useful results about t-(v, k, λ) designs. The first lemma is well known.

Lemma 2.1 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design, then $\mathcal{D}$ is a s-(v, k, λ_s) design for any s with $1 \le s \le t$, and*

$$\lambda_s = \frac{\lambda\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

As a generalization of the t-(v, k, λ) design, we give the definition of the t-(v, K, λ) design, where K is a set of positive integers such that $k \le v$ for every $k \in K$.

Definition 1 Let $\mathcal{P}$ be a finite set with v elements, called points, λ and t be positive integers where $2 \le t \le v$, K be a set of positive integers such that $t \le k \le v$ for every $k \in K$. Let $\mathcal{B}$ be a finite collection of subsets of $\mathcal{P}$, called blocks. Then the incidence structure $\mathcal{D}$=$(\mathcal{P}, \mathcal{B})$ is called a t-(v, K, λ) design if $|B| \in K$ for every $B \in \mathcal{B}$, and each subset of t distinct points is contained in exactly λ blocks.

Let $\mathcal{D}$ be a t-(v, K, λ) design. In particular, $\mathcal{D}$ is a t-(v, k, λ) design if $K = \{k\}$, and $\mathcal{D}$ is a pairwise balanced design if $t = 2$, that is a 2-(v, K, λ) design. Let $\alpha \in \mathcal{P}$, and $k \in K$. Let $r_\alpha^{(k)}$ be the number of blocks having size k through α, $b^{(k)}$ the number of blocks of size k, and r_α the number of blocks in $\mathcal{P}(\alpha)$, here $\mathcal{P}(\alpha)$ is the set of blocks through α. Then

$$\sum_{k\in K} r_\alpha^{(k)} = r_\alpha, \tag{1}$$

$$\sum_{k\in K} r_\alpha^{(k)}\binom{k-1}{t-1} = \binom{v-1}{t-1}\lambda, \tag{2}$$

and for each $k \in K$,

$$\sum_{\alpha\in\mathcal{P}} r_\alpha^{(k)} = b^{(k)}k. \tag{3}$$

Now let $\mathcal{D}$=$(\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design with $t \ge 2$. Let $\mathcal{P}'$ be a subset of $\mathcal{P}$ with $|\mathcal{P}'| = v_0$, and

$$\mathcal{B}' = \{B \cap \mathcal{P}' : |\mathcal{P}' \cap B| \ge t\}.$$

Then $(\mathcal{P}', \mathcal{B}')$ is a t-(v_0, K, λ) design, here the block-set $\mathcal{B}'$ is allowed to have the same elements, and $K = \{|B \cap \mathcal{P}'| : |\mathcal{P}' \cap B| \ge t\}$. We call the design $(\mathcal{P}', \mathcal{B}')$ the *induced design* by $\mathcal{P}'$, which helps us to analyze the parameters of $\mathcal{D}$.

The following lemma is a modification of the proposition first given and proved in [12, Lemma 2]. For completeness, we give detailed proof here.

Lemma 2.2 [12, Lemma 2] *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design. Let $G \le \mathrm{Aut}(\mathcal{D})$ be block-transitive, α be a point of $\mathcal{P}$, r be the number of blocks containing α, and let n be a nontrivial subdegree of G. Then the following statements hold:*

(i) *r divides $k \cdot |G_\alpha|$.*
(ii) *r divides $k\lambda_s\binom{n}{s-1}$, and then $(v-1)(v-2)\cdots(v-s+1)$ divides $k(k-1)\cdots(k-s+1)\binom{n}{s-1}$ for every s with $2 \le s \le t$.*

Proof Let $\mathcal{P}(\alpha)$ be the set of blocks of $\mathcal{D}$ containing the point α, and $B \in \mathcal{P}(\alpha)$. The point-transitivity and block-transitivity of G imply that

$$|G : G_{\alpha,B}| = |G : G_\alpha||G_\alpha : G_{\alpha,B}|,$$

and

$$|G : G_{\alpha,B}| = |G : G_B||G_B : G_{\alpha,B}|.$$

So

$$v|G_\alpha : G_{\alpha,B}| = b|G_B : G_{\alpha,B}|,$$

and then

$$k|G_\alpha : G_{\alpha,B}| = r|G_B : G_{\alpha,B}|,$$

which leads that r divides $k \cdot |G_\alpha|$.

Let $\Delta \neq \{\alpha\}$ be a nontrivial G_α-orbit with $|\Delta| = n$. For any integer s with $2 \leq s \leq t$, $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a s-(v, k, λ_s) design. This follows that $\mathcal{P}' = \Delta \cup \{\alpha\}$ induces a s-$(n+1, K, \lambda_s)$ design $\mathcal{D}'$ and

$$\sum_{k_0 \in K} r_\alpha^{(k_0)} \binom{k_0 - 1}{s-1} = \binom{n}{s-1} \lambda_s, \tag{4}$$

by Equation (2). For each $k_0 \in K$, we let $\mathcal{P}'(\alpha)^{(k_0)} = \{B_0 : |B_0 \cap \mathcal{P}'| = k_0, B_0 \in \mathcal{P}(\alpha)\}$, then $r_\alpha^{(k_0)} = |\mathcal{P}'(\alpha)^{(k_0)}|$ since the repeated blocks are allowed. Moreover, the fact $\Delta^{G_\alpha} = \Delta$ implies that $\mathcal{P}'(\alpha)^{(k_0)}$ is a union of orbits of G_α on $\mathcal{P}(\alpha)$. This together with the equality $k|G_\alpha : G_{\alpha,B}| = r|G_B : G_{\alpha,B}|$ for any $B \in \mathcal{P}(\alpha)$, gives that r divides $k|\mathcal{P}'(\alpha)^{(k_0)}|$, and then $r \mid kr_\alpha^{(k_0)}$. By Equation (4), we have r divides $k\lambda_s\binom{n}{s-1}$ for any s with $2 \leq s \leq t$. It follows that $(v-1)(v-2)\cdots(v-s+1)$ divides $k(k-1)\cdots(k-s+1)\binom{n}{s-1}$ by Lemma 2.1. □

According to Lemma 2.2, we get the following useful corollary:

Corollary 2.1 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ) design, α be a point of $\mathcal{P}$, and let n be a nontrivial subdegree of G. If $G \leq \mathrm{Aut}(\mathcal{D})$ is block-transitive, then*

$$\frac{v-1}{k-1} \mid kn.$$

In particular, if $\mathcal{D}$ is a t-(k^2, k, λ) design, then $k+1 \mid n$.

Proof By Lemma 2.1, we have $r = \lambda_1 = \frac{\lambda_2(v-1)}{k-1}$, thus $\frac{v-1}{k-1} \mid kn$, for r divides $k\lambda_2\binom{n}{2-1}$ by Lemma 2.2. □

The last proposition of this section gives the connection between groups and designs, and we will use it for searching the block-transitive t-designs.

Proposition 2.1 [2, Proposition 1.3] *Let G be a permutation group on $\mathcal{P}$, having orbits $O_1, \ldots, O_m$ on the set of t-subsets of $\mathcal{P}$, and B a k-subset of $\mathcal{P}$. Then $(\mathcal{P}, B^G)$ is a t-design if and only if the ratio of the number of members of O_i contained in B to the total number of members of O_i is independent of i. The group G acts block-transitively on the design, and is flag-transitive if and only if the set wise stabilizer of B in G acts transitively on B.*

For convenience, we call the k-subset B of $\mathcal{P}$ a *base block* of $(\mathcal{P}, B^G)$ if $(\mathcal{P}, B^G)$ is a t-design, here G is a permutation group of $\mathcal{P}$.

3 Reduction

We first prove that a block-transitive t-(k^2, k, λ) design must be point-primitive.

Lemma 3.1 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial t-(k^2, k, λ) design admitting a block-transitive automorphism group G. Then $t \leq 7$ and G is point-primitive.*

Proof Firstly, by [3, Theorem 1.1], for a nontrivial t-(v, k, λ) design $\mathcal{D}$, if $G \leq \mathrm{Aut}(\mathcal{D})$ is block-transitive, then we have $t \leq 7$. Secondly, by [2, Proposition 2.1(i)], we know that G is point-primitive if $t \geq 4$. So we assume that $t = 2$ or 3 in the following, and suppose for the contrary that G is point-imprimitive.

If $t = 3$, then by [6, Theorem 2.2], we have that

$$v \leq \frac{k(k-1)}{2} + 1.$$

It follows that $k^2 \leq \frac{k(k-1)}{2} + 1$ since $v = k^2$, which is impossible. Hence $t = 2$.

Assume that G preserves a partition $\mathcal{C}$ of the points into d imprimitivity classes of size c. Then $v = cd$, and the sizes of the intersections of each block with the imprimitivity classes determine a partition of k, say $\mathbf{x} = (x_1, x_2, ..., x_d)$ with $\sum_{i=1}^{d} x_i = k$. According to [2, Proposition 2.2],

$$\sum_{i=1}^{d} x_i(x_i - 1) = \frac{k(k-1)(c-1)}{v-1}. \tag{5}$$

It follows that $k+1$ divides $\gcd(c-1, v-1)$ since $v = k^2$. Recall that $v - 1 = cd - 1 = d(c-1) + d - 1$, which implies that $k+1$ divides $\gcd(c-1, d-1)$, and then $k + 1 \leq min\{c-1, d-1\}$. On the other hand, by the fact $v = cd = k^2$, we know that $min\{c-1, d-1\} \leq k-1$, hence $k+1 \leq k-1$, a contradiction. So the lemma is proved. □

The O'Nan-Scott theorem classifies primitive groups into five types: (i) Affine type; (ii) Almost simple type; (iii) Simple diagonal type; (iv) Product type; (v) Twisted wreath product type. More details refer to [5]. The following three propositions are devoted to prove Theorem 1 by combining the O'Nan-Scott theorem with the techniques developed in [9, 14].

Proposition 3.1 *Let G be block-transitive automorphism group of a t-(k^2, k, λ) design. Then G is not of simple diagonal type.*

Proof By Lemma 3.1, G is point-primitive. Assume that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$ is of simple diagonal type. Then there is a nonabelian simple group T such that $T^m \leq G \leq T^m : (\mathrm{Out}(T) \times S_m)$, where $m \geq 2$ and $v = |T|^{m-1}$. Let $\alpha \in \mathcal{P}$. From [9, Lemma 2.3], there is a nontrivial orbit Δ of G_α on $\mathcal{P}$ such that $|\Delta| \leq m|T|$. Thus $k + 1 \leq m|T|$ by Corollary 2.1. Since $v = k^2 = |T|^{m-1}$, we have

$$|T|^{\frac{m-1}{2}} < m|T|,$$

which implies that $|T|^{m-3} < m^2$. Therefore, $m = 2$ or $m = 3$.

By Lemma 2.2(i), $r \mid k|G_\alpha|$, we obtain that $(k+1) \mid |G_\alpha|$ since $r = \lambda_1 = (k+1)\lambda_2$ by Lemma 2.1. Recall that $|G_\alpha| = \frac{|G|}{v}$ and $G \leq T^m : (\mathrm{Out}(T) \times S_m)$, then $|G_\alpha| \mid |T||\mathrm{Out}(T)|m!$, hence $(k+1)$ divides $|T| \cdot |\mathrm{Out}(T)|m!$, which implies that $(k+1) \mid |\mathrm{Out}(T)|m!$ for $\gcd(k+1, |T|) = 1$. Then $k = |T|^{\frac{m-1}{2}} < k+1 < |\mathrm{Out}(T)|m!$, so we have $|T| < 4\,\mathrm{Out}(T)|^2$ when $m = 2$, and $|T| < 6|\mathrm{Out}(T)|$ when $m = 3$. At this point, the final part of the proof of [13, Proposition 3.4] can be applied to show that no cases occur. □

Proposition 3.2 *Let G be block-transitive automorphism group of a t-(k^2, k, λ) design. Then G is not of product type.*

Proof Assume that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design, and $G \leq \mathrm{Aut}(\mathcal{D})$ is point-primitive and of product type by Lemma 3.1. Then there is a group K with a primitive action (of almost simple or diagonal type) on a set Γ of size $v_0 \geq 5$, such that $\mathcal{P} = \Gamma^m$ and $G \leq K^m \rtimes S_m = K \wr S_m$, where $m \geq 2$. According to [9, Theorem 2.5], there is a nontrivial orbit Δ of G_α on $\mathcal{P}$ such that $|\Delta| \leq \frac{m(v_0-1)}{s-1}$, where s is the rank of K on Γ. It follows that $k+1 \leq \frac{m(v_0-1)}{s-1}$ from Corollary 2.1. Then

$$v_0^{\frac{m}{2}} = v^{\frac{1}{2}} = k < k+1 \leq \frac{m(v_0-1)}{s-1} < mv_0.$$

Thus $v_0^{m-2} < m^2$, which implies that $m = 2$, or $m = 3$ and $v_0 \leq 9$. At this point, the final part of the proof of [9, Theorem 2.5] can be applied to show that no cases occur. □

Proposition 3.3 *Let G be block-transitive automorphism group of a t-(k^2, k, λ) design. Then G is not of twisted wreath product type.*

Proof Assume that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$ is point-primitive and of twisted wreath product type by Lemma 3.1. Let $N = T_1 \times T_2 \times \cdots \times T_m$, where each $T_i \cong T$ $(i \in \{1, 2, \ldots, m\})$ is a nonabelian simple group and $m \geq 6$ [5]. Let $\alpha \in \mathcal{P}$, then $G = NG_\alpha$, N is is regular on $\mathcal{P}$ and $v = |T|^m$. Then there is an orbit Δ of G_α on $\mathcal{P} \setminus \{\alpha\}$ such that $|\Delta| \leq m|T|$ by [9, Proposition 2.4]. Then $k+1 \leq m|T|$ by Corollary 2.1. On the other hand, $k+1 > |T|^{\frac{m}{2}}$, since $k^2 = v = |T|^m$. Then $|T|^{\frac{m}{2}} < m|T|$, which is impossible for $m \geq 6$. □

Proof of Theorem 1 It follows immediately from Lemma 3.1 and Propositions 3.1–3.3. □

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(k^2, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$ be a block-transitive point-primitive group of almost simple type. Let $X \trianglelefteq G \leq Aut(X)$, where X is a non-abelian simple group. Then X is a sporadic group, an alternating group, a classical group, or an exceptional group of Lie type. In the following, we handle the first two cases separately.

4 Sporadic case

First, we give a lemma that is similar to Lemma 3.1 of [9].

Lemma 4.1 *Let $\mathcal{D}$ be a t-(k^2, k, λ) design admitting a block-transitive automorphism group G. If α is a point of $\mathcal{D}$, then $\frac{k+1}{\gcd(k+1, |\mathrm{Out}(X)|)}$ divides $|X_\alpha|$.*

Proof According to Theorem 1, G is point-primitive. Since $X \trianglelefteq G$, then X is point-transitive and $|X : X_\alpha| = |G : G_\alpha| = v$. It implies that $|G_\alpha| = \frac{|G|}{|X|}|X_\alpha|$. Moreover, we have $r \mid k|G_\alpha|$ and $r = (k+1)\lambda_2$, thus $k+1$ divides $|G_\alpha|$. Hence $k+1$ divides $\frac{|G|}{|X|}|X_\alpha|$, and then $k+1$ divides $|\mathrm{Out}(X)||X_\alpha|$. □

Next, we will use the following lemmas to prove Theorem 2. In the following, $p^i || x$ means that $p^i \mid x$, but $p^{i+1} \nmid x$, here p is a prime, and x, i are two positive intergers.

Lemma 4.2 *Let G be a block-transitive automorphism group of a t-(k^2, k, λ) design with sporadic socle. Then* $\mathrm{Soc}(G) = HS$, $t = 2$, $k = 10$, *and the number of blocks b is divided by* 7.

Proof Suppose that $X = \mathrm{Soc}(G)$ is sporadic. Then X is one of the 26 sporadic simple groups listed in [11]. Since G is point-primitive by Theorem 1, G_α is a maximal subgroup of G, and $v = k^2 = |G : G_\alpha| = |X : X_\alpha|$ for $X \lhd G$. Note that $|\mathrm{Out}(X)| = 1$ or 2 for the sporadic simple group X, thus $G = X$ or $G = X : 2$.

If $X = M_{11}$, then $G = M_{11}$ for $\mathrm{Out}(M_{11}) = 1$, and $k = 2, 3, 2^2, 2 \cdot 3$ or $2^2 \cdot 3$ since $|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $k^2 \mid |M_{11}|$. However, M_{11} does not have maximal subgroup of index $2^2, 3^2, 2^4$ or $2^2 \cdot 3^2$ by [11], thus $k = 12$ and $X_\alpha = F_{55}$, which is impossible as $13 \nmid |X_\alpha|$ by Lemma 4.1. Similarly, we can rule out the cases that $X \in \{M_{12}, J_2, McL\}$.

If $X = M_{22}$, then $G = M_{22}$ or $M_{22} : 2$. Then $k = 2^i 3^j$, where $i = 0, 1, 2, 3$, and $j = 0, 1$, for $k^2 \mid |M_{22}|$. However, G does not have maximal subgroup of index k^2 by [11]. It follows that $X \neq M_{22}$. Similarly, we can rule out the remaining the sporadic simple groups except for $X = HS$.

If $X = HS$, then $G = HS$ or $HS : 2$, and $k = 2^i 3^j 5^m$, where $i = 0, 1, 2, 3, 4$, $j = 0, 1$, $m = 0, 1$, for $|HS| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. Hence $k = 10$ and $X_\alpha = M_{22}$ by [11].

By Lemma 3.1, G is point-primitive and $t \leq 7$. If $t \geq 3$, then $\mathcal{D}$ is a 3-$(10^2, 10, \lambda_3)$ design by Lemma 2.1 and

$$b = \lambda_0 = \frac{v(v-1)(v-2)\lambda_3}{k(k-1)(k-2)} = \frac{5 \cdot 11 \cdot 7^2 \cdot \lambda_3}{2}.$$

Moreover, $b \mid |G|$ as G is block-transitive, which is impossible since $7^2 \nmid |G|$. So we may assume that $\mathcal{D}$ is a 2-$(10^2, 10, \lambda)$ design in the following. The primitive permutation representations of G on 100 points are given in [11]. We know that $7 \parallel |G|$ and G has a unique conjugate class of elements of order 7 if $G = HS$ or $HS : 2$, and each element of order 7 fixes only 2 points. This implies that G_B does not have an element of order 7 for any $B \in \mathcal{B}$, and then $7 \mid b$. □

Lemma 4.3 *Let $D = (\mathcal{P}, \mathcal{B})$ be a* 2-$(10^2, 10, \lambda)$ *design admitting a block-transitive automorphism group G with* $\mathrm{Soc}(G) = HS$. *Then one of the following holds:*

(1) $\lambda \in \{2^i \cdot 3 \cdot 5^2 \cdot 7 \mid i = 5, 6, 7, 8, 9\}$;
(2) $\lambda \in \{2^7 \cdot 3^2 \cdot 7\} \cup \{2^i \cdot 3^2 \cdot 5 \cdot 7 \mid i = 6, 7, 8, 9\}$;
(3) $\lambda \in \{2^i \cdot 3^2 \cdot 5^2 \cdot 7 \mid i = 0, 1, \ldots, 9\}$.

Proof Since $\mathrm{Soc}(G) = HS$ and $v = 100$, then $G = HS$ or $HS : 2$, and the point-stabiliser $G_\alpha = M_{22}$ or $M_{22} : 2$ respectively, here $\alpha \in \mathcal{P}$. If $G = HS : 2$, then $HS \unlhd G$ and $G_\alpha \cap HS = M_{22}$, this implies that HS is also primitive on P. By Atlas [11], the suborbit lengths of G and HS on $\mathcal{P}$ are the same, that is 1, 22, 77, then their orbit lengths on $\mathcal{P}^{\{2\}}$ are the same, here $\mathcal{P}^{\{2\}}$ is the set of 2-subsets of $\mathcal{P}$. On the other hand, each HS-orbit on $\mathcal{P}^{\{2\}}$ is contained in a $HS : 2$-orbits on $\mathcal{P}^{\{2\}}$. Thus HS-orbits and $HS : 2$-orbits on $\mathcal{P}^{\{2\}}$ are the same, it turns out that $(\mathcal{P}, B^{HS})$ is a 2-$(10^2, 10, \lambda)$ design if and only if $(\mathcal{P}, B^{HS:2})$ is a 2-$(10^2, 10, \lambda')$ design for a 10-subset B of $\mathcal{P}$ by Proposition 2.1, here $\lambda' = \lambda$ or 2λ. So we just consider the case of $G = HS$ in the following. Recall that $|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$.

Suppose that B is a base block of $\mathcal{D}$. Let $B^{\{2\}}$ be the set of 2-subsets of B, O_1 and O_2 be the two orbits of $G = HS$ on $\mathcal{P}^{\{2\}}$, then $|O_1| = 1100$ and $|O_2| = 3850$ for the nontrivial suborbits of G on $\mathcal{P}$ are 22 and 77. By Proposition 2.1,

$$\frac{|B^{\{2\}} \cap O_1|}{|O_1|} = \frac{|B^{\{2\}} \cap O_2|}{|O_2|},$$

then

$$|B^{\{2\}} \cap O_1| = 10, |B^{\{2\}} \cap O_1| = 35 \tag{6}$$

for $B^{\{2\}} = (B^{\{2\}} \cap O_1) \cup (B^{\{2\}} \cap O_2)$ and $O_1 \cap O_2 = \emptyset$.

Now we prove this lemma by considering the order of G_B. If $3 \mid |G_B|$, then there exists an element of order 3 in G_B for $3^2 \parallel |HS|$ and HS has no element of order 3^2. According to [11], G has a unique conjugate class of elements of order 3. Let $g \in G$ be a representative of the unique conjugate class, where

$$\begin{aligned} g =&(1, 56, 98)(2, 78, 67)(3, 32, 44)(4, 68, 29)(5, 48, 24)(7, 94, 85)(9, 82, 33)(10, 15, 66)\\ &(11, 99, 58)(12, 54, 70)(13, 55, 74)(14, 42, 97)(16, 51, 73)(17, 100, 60)(18, 57, 31)\\ &(19, 77, 36)(20, 95, 88)(21, 37, 26)(22, 92, 52)(25, 96, 91)(27, 64, 86)(30, 84, 38)\\ &(34, 93, 35)(39, 41, 90)(43, 89, 79)(46, 59, 53)(50, 87, 75)(61, 65, 63)\\ &(62, 71, 80)(72, 81, 76), \end{aligned}$$

and suppose that $g \in G_B$ by the block-transitivity of G. Thus B is a union of $\langle g \rangle$-orbits on $\mathcal{P}$. Using GAP [10], we obtained 4800 different base blocks from $\langle g \rangle$-orbits satisfying Equations (6), and among these designs, $\lambda \in \{2^i \cdot 3 \cdot 5^2 \cdot 7 \mid i = 5, 6, 7, 8\}$.

If $5 \mid |G_B|$, then there exists an element of order 5 in G_B for $5^3 \parallel |HS|$ and HS has no element of order 5^2 or 5^3. According to [11], G has 3 conjugate classes of elements of order 5. Let g_1, g_2, g_3 be representatives of the conjugate classes, where

$$\begin{aligned} g_1 =&(1, 2, 10, 50, 70)(3, 42, 9, 39, 18)(4, 34, 72, 53, 38)(5, 83, 22, 8, 19)(6, 99, 69, 48, 21)\\ &(7, 11, 62, 57, 16)(12, 75, 91, 15, 67)(13, 51, 43, 24, 85)(14, 64, 92, 79, 55)(17, 59, 93,\\ &71, 58)(20, 32, 30, 82, 94)(23, 37, 68, 47, 86)(25, 87, 78, 98, 40)(26, 97, 46, 74, 100)\\ &(27, 84, 81, 65, 52)(28, 29, 36, 49, 80)(31, 77, 61, 73, 76)(33, 60, 95, 63, 41)(35, 44,\\ &88, 89, 90)(45, 56, 54, 66, 96), \end{aligned}$$

$$\begin{aligned} g_2 =&(1, 32, 26, 61, 35)(2, 30, 97, 73, 44)(3, 13, 91, 38, 60)(4, 95, 42, 51, 15)(5, 47, 62,\\ &45, 59)(6, 92, 84, 29, 87)(7, 66, 58, 8, 37)(9, 43, 67, 34, 63)(10, 82, 46, 76, 88)\\ &(11, 96, 17, 19, 68)(12, 72, 41, 39, 24)(14, 52, 80, 40, 48)(16, 54, 71, 22, 23)\\ &(18, 85, 75, 53, 33)(20, 100, 77, 90, 70)(21, 64, 27, 28, 25)(31, 89, 50, 94, 74)\\ &(36, 78, 99, 79, 81)(49, 98, 69, 55, 65)(56, 93, 83, 86, 57), \end{aligned}$$

and

$$\begin{aligned} g_3 =&(2, 37, 34, 35, 74)(3, 95, 18, 55, 86)(4, 87, 49, 22, 17)(5, 41, 9, 33, 92)(6, 54, 43,\\ &60, 70)(7, 83, 65, 20, 88)(8, 36, 31, 61, 19)(11, 67, 45, 27, 13)(12, 48, 50, 68,\\ &71)(14, 52, 82, 69, 16)(15, 21, 78, 77, 63)(23, 46, 64, 26, 56)(24, 81, 75, 91, 38)\\ &(25, 90, 89, 42, 47)(28, 93, 66, 85, 99)(29, 40, 72, 51, 39)(30, 96, 73, 76, 100)\\ &(32, 59, 98, 58, 94)(62, 80, 79, 84, 97). \end{aligned}$$

By the block-transitivity of G, suppose that $g_i \in G_B$ for $i = 1, 2, 3$. Thus B is a union of $\langle g_i \rangle$-orbits on $\mathcal{P}$. Using GAP [10], we obtained 190 different base blocks, and among these designs, $\lambda \in \{2^7 \cdot 3^2 \cdot 7\} \cup \{2^i \cdot 3^2 \cdot 5 \cdot 7 | i = 6, 7, 8\}$.

It is obvious that $3 \nmid |G_B|$ if $5 \mid |G_B|$, and $5 \nmid |G_B|$ if $3 \mid |G_B|$. Now we consider the case that $3 \nmid |G_B|$ and $5 \nmid |G_B|$. Since $|G_B| = \frac{|G|}{b}$, $7 \mid b$ and $b = 110\lambda$, then $\lambda \in \{2^i \cdot 3^2 \cdot 5^2 \cdot 7 \mid i = 0, 1, \ldots, 9\}$, where b is the number of blocks. Therefore, the lemma holds. □

Proof of Theorem 2 It follows immediately from Lemmas 4.2–4.3. □

Let $\mathcal{D}$ be a 2-(v, k, λ) design $\mathcal{D}$ with $k \mid v$, and $G \leq \mathrm{Aut}(\mathcal{D})$. In 1984, Camina and Gagen prove that if G is block-transitive and $\lambda = 1$, then G is flag-transitive [4]. However, for the case that $\lambda > 1$, the block-transitivity and $k \mid v$ cannot imply flag-transitivity.

Example 1 Suppose that $G = HS$ acts primitively on $\mathcal{P} = \{1, 2, \ldots, 100\}$. Then there exist many block-transitive, but not flag-transitive 2-$(10^2, 10, \lambda)$ designs. The following are some examples.

(1) $D_1 = (\mathcal{P}, B_1^G)$, where $B_1 = \{1, 6, 8, 25, 28, 45, 56, 91, 96, 98\}$, $\lambda = 2^5 \cdot 3 \cdot 5^2 \cdot 7$.
(2) $D_2 = (\mathcal{P}, B_2^G)$, where $B_2 = \{1, 2, 10, 34, 35, 37, 44, 53, 57, 74\}$, $\lambda = 2^6 \cdot 3^2 \cdot 5 \cdot 7$.
(3) $D_3 = (\mathcal{P}, B_3^G)$, where $B_3 = \{1, 6, 8, 28, 43, 45, 56, 79, 89, 98\}$, $\lambda = 2^6 \cdot 3 \cdot 5^2 \cdot 7$.
(4) $D_4 = (\mathcal{P}, B_4^G)$, where $B_4 = \{2, 12, 34, 35, 37, 48, 50, 68, 71, 74\}$, $\lambda = 2^7 \cdot 3^2 \cdot 5 \cdot 7$.
(5) $D_5 = (\mathcal{P}, B_5^G)$, where $B_5 = \{2, 6, 8, 11, 28, 45, 58, 67, 78, 99\}$, $\lambda = 2^7 \cdot 3 \cdot 5^2 \cdot 7$.
(6) $D_6 = (\mathcal{P}, B_6^G)$, where $B_6 = \{2, 7, 20, 34, 35, 37, 65, 74, 83, 88\}$, $\lambda = 2^8 \cdot 3^2 \cdot 5 \cdot 7$.
(7) $D_7 = (\mathcal{P}, B_7^G)$, where $B_7 = \{1, 2, 6, 8, 28, 45, 56, 67, 78, 98\}$, $\lambda = 2^8 \cdot 3 \cdot 5^2 \cdot 7$.

Now we give a result on the flag-transitive 2-$(10^2, 10, \lambda)$ designs.

Lemma 4.4 *Let $D = (\mathcal{P}, \mathcal{B})$ be a 2-$(10^2, 10, \lambda)$ design admitting a flag-transitive automorphism group $G = HS$. Then $\lambda = 2^7 \cdot 3^2 \cdot 7$ or $2^6 \cdot 3^2 \cdot 5 \cdot 7$.*

Proof Let $G = HS$ acting primitively on $\mathcal{P} = \{1, 2, \ldots, 100\}$. If G is flag-transitive, then $10 \mid |G_B|$ for each $B \in \mathcal{B}$. Moreover, $|G_B| = \frac{|G|}{b} = \frac{|G|}{110\lambda}$, this implies that $\lambda \in \{2^7 \cdot 3^2 \cdot 7\} \cup \{2^i \cdot 3^2 \cdot 5 \cdot 7 | i = 6, 7, 8\}$ by Lemma 4.3, and $\mathcal{D}$ is one of the 190 designs which obtained in the proof of Lemma 4.3. Among these designs, if G is flag-transitive, then $\lambda = 2^7 \cdot 3^2 \cdot 7$ or $2^6 \cdot 3^2 \cdot 5 \cdot 7$, and up to isomorphism there are 2 designs which are listed in the following.

(i) $D_1 = (\mathcal{P}, B_1^G)$, where $B_1 = \{1, 7, 10, 20, 44, 53, 57, 65, 83, 88\}$, $\lambda = 2^7 \cdot 3^2 \cdot 7$.
(ii) $D_2 = (\mathcal{P}, B_2^G)$, where $B_2 = \{1, 5, 26, 32, 35, 45, 47, 59, 61, 62\}$, $\lambda = 2^6 \cdot 3^2 \cdot 5 \cdot 7$.

□

5 Alternative case

Now we deal with the case that $\mathrm{Soc}(G) = A_n$. In the following, we let $\Omega^{\{\ell\}}$ denote the set of all ℓ-subsets (i.e. subsets of size ℓ) of the set Ω, for $\ell = 1, 2, \ldots, n$.

Lemma 5.1 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(k^2, k, λ) design, $G \leq \mathrm{Aut}(\mathcal{D})$ be block-transitive with alternating socle A_n. Then $G = A_n (n > 6)$, the point-stabilizer $G_\alpha = (S_\ell \times S_{n-\ell}) \cap A_n$, and one of the following hold:*

(1) *$\ell = 1$, $v = n = k^2$, and $\mathcal{D}$ is a t-$(k^2, k, \binom{k^2-t}{k-t})$ design, where $\mathcal{P} = \{1, 2, \ldots, k^2\}$, and $\mathcal{B} = \mathcal{P}^{\{k\}}$, the set of all k-subsets of $\mathcal{P}$;*

Table 1 Twelve block-transitve 2-$(6^2, 6, \lambda)$ designs $\mathcal{D} = (\Omega_9^{\{2\}}, B^{A_9})$

Line	λ	Base block B	Flag-transitivity
1	$2^2 \cdot 5$	$\{\{1,2\},\{1,3\},\{2,3\},\{4,5\},\{4,6\},\{5,6\}\}$	Yes
2	$3^2 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{6,7\},\{8,9\}\}$	No
3	$2^3 \cdot 3 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{5,6\},\{5,7\},\{5,8\}\}$	Yes
4	$2^3 \cdot 3 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{5,6\},\{5,7\},\{6,7\}\}$	No
5	$2^3 \cdot 3 \cdot 5$	$\{\{1,2\},\{1,3\},\{2,4\},\{3,5\},\{4,6\},\{5,6\}\}$	Yes
6	$2^2 \cdot 3^3 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{2,6\},\{7,8\}\}$	No
7	$2^4 \cdot 3^2 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,6\},\{4,7\}\}$	No
8	$2^3 \cdot 3^3 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{4,5\},\{6,7\}\}$	No
9	$2^3 \cdot 3^3 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{5,6\},\{5,7\}\}$	No
10	$2^3 \cdot 3^3 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,5\},\{6,7\}\}$	No
11	$2^4 \cdot 3^3 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{5,6\},\{6,7\}\}$	No
12	$2^5 \cdot 3^3 \cdot 5$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,6\},\{5,7\}\}$	No

(2) $\ell = 2$, $n = 9$, *and* $\mathcal{D}$ *is a* 2-$(6^2, 6, \lambda)$ *design, where* $\lambda \in \{2^2 \cdot 5, 3^2 \cdot 5, 2^3 \cdot 3 \cdot 5, 2^4 \cdot 3^2 \cdot 5\} \cup \{2^i \cdot 3^3 \cdot 5 \mid i = 2, 3, 4, 5\}$*;*
(3) $\ell = 3$, $n = 50$, *and* $\mathcal{D}$ *is a* 2-$(140^2, 140, \lambda)$ *design.*

Proof Since Lemmas 3.3–3.5 of [9] also hold under our conditions, we have $G = A_n (n > 6)$, with $G_\alpha = (S_\ell \times S_{n-\ell}) \cap A_n$, and either $\ell \leq 2$, or $\ell = 3$ and $n = 50$. The detailed proofs are not provided here, and please refer to [9].

Firstly, we assume that $\ell = 1$. Then $\mathcal{P} = \Omega_n$, $v = n = k^2$, $G_\alpha = A_{n-1}$ for $\alpha \in \mathcal{P}$. Since A_{n-1} is $(n-3)$-transitive, then G_α is $(k-1)$-transitive for $k - 1 \leq n - 3$. It follows that G_α is transitive on $\mathcal{P}(\alpha)$, which implies that G is flag-transitive, $r = \binom{v-1}{k-1}$, and $b = \binom{v}{k}$. That is, $\mathcal{D} = (\Omega_n, \Omega_n^{\{k\}})$.

Secondly, assume that $\ell = 2$. Then $\mathcal{P} = \Omega_n^{\{2\}}$, $v = \frac{n(n-1)}{2}$, and the non-trivial orbits of G_α on $\mathcal{P}$ are $\Delta_i = \{\gamma : |\gamma \cap \alpha| = i\}$, $i = 0, 1$. It follows that subdegrees of G are $1, \binom{n-2}{2}$, and $2\binom{n-2}{1}$. By Corollary 2.1, $(k+1) \mid 2(n-1)$. It implies that $n = 9$, $k = 6$ for $k^2 = v = \frac{n(n-1)}{2}$. Thus $\mathcal{D}$ is a t-$(6^2, 6, \lambda)$ design. If $t \geq 3$, then

$$b = \frac{v(v-1)(v-2)\lambda_3}{k(k-1)(k-2)}.$$

Thus $17 \mid b$, and then $17 \mid |A_9|$ for G is block-transitive, which is impossible. Therefore, $t = 2$. Using GAP [10], we obtain all non-isomorphic designs by Proposition 2.1, which are listed in Table 1, and three of them are flag-transitive.

Thirdly, we assume that $\ell = 3$ and $n = 50$. Then $\mathcal{P} = \Omega_{50}^{\{3\}}$, $v = 140^2$, and $k = 140$. If $t \geq 3$, then

$$b = \frac{v(v-1)(v-2)\lambda_3}{k(k-1)(k-2)}.$$

It implies that $239 \mid b$, and then $239 \mid |A_{50}|$, which is impossible. Therefore, $t = 2$, $b = 140r$, and $r = 141\lambda$. $\square$

Next, let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a 2-$(140^2, 140, \lambda)$ design, $\mathcal{P} = \Omega_{50}^{\{3\}}$, and $G = A_{50} \leq \mathrm{Aut}(\mathcal{D})$ be block-transitive. We will give some properties of this design, especially on the parameter λ.

Let $O_i = \{\{\alpha, \beta\} | |\alpha \cap \beta| = i, \alpha, \beta \in \mathcal{P}\}$, $i = 0, 1, 2$, be the G-orbits on $\mathcal{P}^{\{2\}}$. Then

$$|O_0| = \frac{\binom{50}{3} \cdot \binom{47}{3}}{2}, |O_1| = \frac{\binom{50}{3} \cdot \binom{47}{2} \cdot \binom{3}{1}}{2}, |O_2| = \frac{\binom{50}{3} \cdot \binom{47}{1} \cdot \binom{3}{2}}{2},$$

and $O_i(\alpha) = \{\beta | \{\alpha, \beta\} \in O_i\}$, $i = 0, 1, 2$, are G_α-orbits on $\mathcal{P} \setminus \{\alpha\}$ for and a point $\alpha \in \mathcal{P}$. It is obvious that

$$|O_0(\alpha)| = \binom{47}{3}, |O_1(\alpha)| = \binom{47}{2} \cdot \binom{3}{1}, |O_2(\alpha)| = \binom{47}{1} \cdot \binom{3}{2}.$$

Let B be a block and $B^{\{2\}}$ be the set of all 2-subsets of B. By Proposition 2.1, the ratio of the number of members of O_i contained in B to the total number of members of O_i is independent of i, $i = 0, 1, 2, 3$. Thus

$$\frac{|B^{\{2\}} \cap O_0|}{|O_0|} = \frac{|B^{\{2\}} \cap O_1|}{|O_1|} = \frac{|B^{\{2\}} \cap O_2|}{|O_2|},$$

which implies that

$$|B^{\{2\}} \cap O_0| : |B^{\{2\}} \cap O_1| : |B^{\{2\}} \cap O_2| = 115 : 23 : 1.$$

Thus $|B^{\{2\}} \cap O_0| = 70 \cdot 115 = 8050$, $|B^{\{2\}} \cap O_1| = 70 \cdot 23 = 1610$, $|B^{\{2\}} \cap O_2| = 70$ for $|B^{\{2\}}| = \binom{140}{2} = 70 \cdot 139$.

Furthermore, if G is flag-transitive, then $|B \cap O_i(\alpha)|r = \lambda |O_i(\alpha)|$ for $i = 1, 2, 3$, and so

$$\frac{|B \cap O_0(\alpha)|}{|O_0(\alpha)|} = \frac{|B \cap O_1(\alpha)|}{|O_1(\alpha)|} = \frac{|B \cap O_2(\alpha)|}{|O_2(\alpha)|} = \frac{\lambda}{r} = \frac{1}{141}.$$

Hence $|B \cap O_0(\alpha)| = 115$, $|B \cap O_1(\alpha)| = 23$, and $|B \cap O_2(\alpha)| = 1$.

From now on, we assume that the following hypothesis holds:

Hypothesis: Let p be a prime divisor of $|A_n|$ and $5 \leq p \leq 43$. Let $g \in A_n$ be a cycle of length p, $T = \langle g \rangle$, and $\bar{\Omega}_p = \mathrm{Fix}_{\Omega_{50}}(g)$.

Let $\alpha \in \mathcal{P}$, $\Gamma = \alpha^T$. It is clear that $\Gamma = \{\alpha\}$ if $|\alpha \cap \bar{\Omega}_p| = 3$. In the following proposition, we will consider the case that $|\alpha \cap \bar{\Omega}_p| < 3$. Let $\Gamma(i_0) = \{\beta | i_0 \in \beta, \beta \in \Gamma\}$, and $\Gamma^{\{2\}}(i_0) = \{\{\beta_1, \beta_2\} | i_0 \in \beta_1 \cap \beta_2, \{\beta_1, \beta_2\} \in \Gamma^{\{2\}}\}$ for $i_0 \in \Omega_{50} \setminus \bar{\Omega}_p$. Define two triples with respect to Γ:

$$\mu(\Gamma) = (|\Gamma^{\{2\}} \cap O_0|, |\Gamma^{\{2\}} \cap O_1|, |\Gamma^{\{2\}} \cap O_2|),$$

and

$$\nu_\beta(\Gamma) = (|\Gamma \cap O_0(\beta)|, |\Gamma \cap O_1(\beta)|, |\Gamma \cap O_2(\beta)|)$$

for $\beta \in \Gamma$. It is obvious that $|\Gamma^{\{2\}} \cap O_0| + |\Gamma^{\{2\}} \cap O_1| + |\Gamma^{\{2\}} \cap O_2| = |\Gamma^{\{2\}}| = \binom{p}{2}$, and $|\Gamma \cap O_0(\beta)| + |\Gamma \cap O_1(\beta)| + |\Gamma \cap O_2(\beta)| = p - 1$.

Proposition 5.1 *Let $\alpha \in \mathcal{P}$, $\Gamma = \alpha^T$. Then the following hold:*

(1) *If $|\alpha \cap \bar{\Omega}_p| = 2$, then $\mu(\Gamma) = (0, 0, \binom{p}{2})$, and $\nu_\beta(\Gamma) = (0, 0, p - 1)$.*
(2) *If $|\alpha \cap \bar{\Omega}_p| = 1$, then $\mu(\Gamma) = (0, \binom{p}{2} - p, p)$, and $\nu_\beta(\Gamma) = (0, p - 3, 2)$.*
(3) *If $|\alpha \cap \bar{\Omega}_p| = 0$, then $\mu(\Gamma) = (\binom{p}{2} - 3p, 3p, 0)$, or $(\binom{p}{2} - 2p, p, p)$, and $\nu_\beta(\Gamma) = (p - 7, 6, 0)$, or $(p - 5, 2, 2)$ respectively.*

Proof **(1)** Suppose that $|\alpha\cap\bar{\Omega}_p|=2$. Then $|\Gamma|=p$, and $|\beta_1\cap\beta_2|=|\alpha\cap\bar{\Omega}_p|=2$ for each $\{\beta_1,\beta_2\}\in\Gamma^{\{2\}}$, hence $|\Gamma^{\{2\}}\cap O_2|=\binom{p}{2}$ and $|\Gamma\cap O_2(\alpha)|=p-1$. Thus $\mu(\Gamma)=(0,0,\binom{p}{2})$, and $\nu_\alpha(\Gamma)=(0,0,p-1)$.

(2) Suppose that $|\alpha\cap\bar{\Omega}_p|=1$, and $\alpha=\{j_p,i_1,i_2\}$, here $\alpha\cap\bar{\Omega}_p=\{j_p\}$. Then $|\Gamma|=p$ and $\Gamma^{\{2\}}\cap O_0=\emptyset$, $\Gamma\cap O_0(\alpha)=\emptyset$ since $j_p\in\beta$, for any $\beta\in\Gamma$.

Let $i_1^{g_1}=i_0$ and $i_2^{g_2}=i_0$ for $i_0\in\Omega_{50}\backslash\bar{\Omega}_p$, here $g_1,g_2\in T$. Then $\Gamma(i_0)=\{\alpha^{g_1},\alpha^{g_2}\}$, and $\Gamma^{\{2\}}(i_0)=\{\{\alpha^{g_1},\alpha^{g_2}\}\}$. Since each point of Γ contains j_p, we have

$$|\Gamma^{\{2\}}\cap O_2|=|\bigcup_{i_0\in\Omega_{50}\backslash\bar{\Omega}_p}\Gamma^{\{2\}}(i_0)|=\sum_{i_0\in\Omega_{50}\backslash\bar{\Omega}_p}|\Gamma^{\{2\}}(i_0)|,$$

because $\Gamma^{\{2\}}(i_0)\cap\Gamma^{\{2\}}(j_0)=\emptyset$ for any two distinct $i_0,j_0\in\Omega_{50}\backslash\bar{\Omega}_p$. Otherwise, there exists one 2-subset $\{\beta_1,\beta_2\}\in\Gamma^{\{2\}}$ such that $\{j_p,i_0,j_0\}\in\beta_1\cap\beta_2$, which implies that $\beta_1=\beta_2$, a contradiction. Thus $|\Gamma^{\{2\}}\cap O_2|=p$, and

$$|\Gamma\cap O_2(\alpha)|=|\Gamma(i_1)\cup\Gamma(i_2)|-|\{\alpha\}|=|\Gamma(i_1)|+|\Gamma(i_2)|-|\Gamma(i_1)\cap\Gamma(i_2)|-|\{\alpha\}|=2.$$

Therefore $\mu(\Gamma)=(0,\binom{p}{2}-p,p)$, and $\nu_\alpha(\Gamma)=(0,p-3,2)$.

(3) Suppose that $|\alpha\cap\bar{\Omega}_p|=0$. Then $|\Gamma|=p$, $|\Gamma(i_0)|=3$, and $|\Gamma^{\{2\}}(i_0)|=\binom{3}{2}=3$, for each element $i_0\in\Omega_{50}\backslash\bar{\Omega}_p$. Since $\Gamma^{\{2\}}(i_0)\cap\Gamma^{\{2\}}(i_1)\cap\Gamma^{\{2\}}(i_2)=\emptyset$ if $i_0,i_1,i_2\in\Omega_{50}\setminus\bar{\Omega}_p$ are different from each other, we have

$$\begin{aligned}|\Gamma^{\{2\}}\cap O_1|+|\Gamma^{\{2\}}\cap O_2|&=|\{\beta_1,\beta_2\}||\beta_1\cap\beta_2|>0,\{\beta_1,\beta_2\}\in\Gamma^{\{2\}}\}|\\&=|\bigcup_{i_0\in\Omega_{50}\backslash\bar{\Omega}_p}\Gamma^{\{2\}}(i_0)|\\&=\sum_{i_0\in\Omega_{50}\backslash\bar{\Omega}_p}|\Gamma^{\{2\}}(i_0)|-\sum_{i_0\neq j_0\in\Omega_{50}\backslash\bar{\Omega}_p}|\Gamma^{\{2\}}(i_0)\cap\Gamma^{\{2\}}(j_0)|\\&\le 3p,\end{aligned}\tag{7}$$

and

$$|\Gamma^{\{2\}}\cap O_2|=|\bigcup_{i_0\neq j_0\in\Omega_{50}\backslash\bar{\Omega}_p}(\Gamma^{\{2\}}(i_0)\cap\Gamma^{\{2\}}(j_0))|=\sum_{i_0\neq j_0\in\Omega_{50}\backslash\bar{\Omega}_p}|\Gamma^{\{2\}}(i_0)\cap\Gamma^{\{2\}}(j_0)|.$$

Since $\Gamma^{\{2\}}\cap O_1$ and $\Gamma^{\{2\}}\cap O_2$ are unions of T-orbits on $\Gamma^{\{2\}}$, then $p\mid|\Gamma^{\{2\}}\cap O_1|$ and $p\mid|\Gamma^{\{2\}}\cap O_2|$ for $\mathrm{Fix}_{\mathcal{P}}(T)\cap\Gamma=\emptyset$. Suppose that $|\Gamma^{\{2\}}\cap O_2|=mp$ $(0\le m\le 3)$, then

$$|\Gamma^{\{2\}}\cap O_1|+mp=|\bigcup_{i_0\in\Omega_{50}\backslash\bar{\Omega}_p}\Gamma^{\{2\}}(i_0)|=\sum_{i_0\in\Omega_{50}\backslash\bar{\Omega}_p}|\Gamma^{\{2\}}(i_0)|-mp=(3-m)p.$$

Therefore, $m\le 1$, and $\mu(\Gamma)=(\binom{p}{2}-3p,3p,0)$, or $(\binom{p}{2}-2p,p,p)$.

Next, we consider $\nu_\alpha(\Gamma)$. Since $\bigcap_{i_0\in\alpha}\Gamma(i_0)=\{\alpha\}$, we have

$$\begin{aligned}|\Gamma\cap O_1(\alpha)|+|\Gamma\cap O_2(\alpha)|&=|\{\beta|\beta\cap\alpha\neq\emptyset,\beta\in\Gamma\backslash\{\alpha\}\}|\\&=|\bigcup_{i_0\in\alpha}(\Gamma(i_0)\backslash\{\alpha\})|\\&=\sum_{i_0\in\alpha}|\Gamma(i_0)\backslash\{\alpha\}|-\sum_{i_0\neq j_0\in\alpha}|(\Gamma(i_0)\cap\Gamma(j_0))\backslash\{\alpha\}|\end{aligned}\tag{8}$$

For the case $|\Gamma^{\{2\}} \cap O_2| = 0$, we have $|\Gamma \cap O_2(\alpha)| = 0$ and $\Gamma(i_0) \cap \Gamma(j_0) = \{\alpha\}$ for any $i_0, j_0 \in \alpha, i_0 \neq j_0$. Then $|\Gamma \cap O_1(\alpha)| = \sum_{i_0 \in \alpha} |\Gamma(i_0) \backslash \{\alpha\}| = 3(3-1) = 6$ by Equation (8). Thus $\nu_\alpha(\Gamma) = (p-7, 6, 0)$.

For that case $|\Gamma^{\{2\}} \cap O_2| = p$, there is a point $\beta \in \Gamma$ such that $|\alpha \cap \beta| = 2$ for T is transitive on Γ. Thus the elements of α has an arrangement $i_1 i_2 i_3$, where $\alpha = \{i_1, i_2, i_3\}$, such that $i_1^{g_1} = i_2$, $i_2^{g_1} = i_3$, here $1 \neq g_1 \in T$. If $i_3^{g_1} = i_1$, then $\alpha^{g_1} = \alpha$, which is impossible. Thus $i_3^{g_1} \neq i_1$. It follows that

$$\{\alpha^{g_1}, \alpha^{g_1^{-1}}\} \subseteq \Gamma \cap O_2(\alpha).$$

Note that $\alpha^{g_1} \neq \alpha^{g_1^{-1}}$, otherwise $p \mid 2m$. If $|\Gamma \cap O_2(\alpha)| \geq 3$, then there exists another point $\beta \in \Gamma$ such that $|\beta \cap \alpha| = 2$. If $\beta \cap \alpha = \{i_1, i_2\}$, then there exists an element $g_2 \neq g_1 \in T$ such that $i_1^{g_2} = i_2$, $i_3^{g_2} = i_1$, this implies that $g_1 = g_2$, which is a contradiction. Similarly, $\beta \cap \alpha \neq \{i_2, i_3\}$. Thus $\beta \cap \alpha = \{i_1, i_3\}$. Let $\beta = \alpha^{g_2}$, $g_2 \in T$, here $g_2 \neq g_1 \in T$. Then $\{i_1, i_3\} = \{i_1, i_2\}^{g_2}$ or $\{i_1, i_3\} = \{i_2, i_3\}^{g_2}$. It turns out $\{i_1, i_2\} \in \alpha^{g_2^{-1}}$ or $\{i_2, i_3\} \in \alpha^{g_2^{-1}}$, thus $g_1 = g_2$ and $i_3^{g_1} = i_1$, which is impossible. Hence $|\Gamma \cap O_2(\alpha)| = 2$. By Equation (8), we have

$$\begin{aligned} |\Gamma \cap O_1(\alpha)| + 2 &= \sum_{i \in \alpha} |\Gamma(i) \backslash \{\alpha\}| - \sum_{i \neq j \in \alpha} |(\Gamma(i) \cap \Gamma(j)) \backslash \{\alpha\}| \\ &= \sum_{i \in \alpha} |\Gamma(i) \backslash \{\alpha\}| - |\{\alpha^{g^m}, \alpha^{g^{p-m}}\}|. \end{aligned}$$

Thus $|\Gamma \cap O_1(\alpha)| + 2 = 3(3-1) - 2 = 4$, and then $|\Gamma \cap O_1(\alpha)| = 2$. Therefore $\nu_\alpha(\Gamma) = (p-5, 2, 2)$.

Now the proposition follows from the fact that $|\Gamma \cap O_i(\beta)| = |\Gamma \cap O_i(\alpha)|$ for $i = 0, 1, 2$, $\beta \in \Gamma$. □

Remark 3 Note that, if $|\alpha \cap \bar{\Omega}_p| = 0$ and $p = 5$, then the elements of α must have an arrangement $i_1 i_2 i_3$, such that $i_1^{g_1} = i_2$, $i_2^{g_1} = i_3$, but $i_3^{g_1} \neq i_1$, where $\alpha = \{i_1, i_2, i_3\}$ and $1 \neq g_1 \in T$, since $|\alpha^{(2)}| = 6$, here $\alpha^{(2)}$ is the set of all ordered pairs of elements of α. Thus $|\Gamma^{\{2\}} \cap O_2| \neq 0$, it follows that $\mu(\Gamma) = (0, 5, 5)$ and $\nu_\beta(\Gamma) = (0, 2, 2)$.

Proposition 5.2 *Let $\Gamma_1 = \alpha_1^T$, $\Gamma_2 = \alpha_2^T$ be two T-orbits of length p on $\mathcal{P}$, and $\Gamma_{1,2}^{\{2\}} = \{\{\beta_1, \beta_2\} | \beta_1 \in \Gamma_1, \beta_2 \in \Gamma_2\}$. Then $p \mid |\Gamma_{1,2}^{\{2\}} \cap O_2|$, and the following hold:*

(1) *If $|\alpha_1 \cap \bar{\Omega}_p| = 1$, and $|\alpha_2 \cap \bar{\Omega}_p| = 0$, then $|\Gamma_{1,2}^{\{2\}} \cap O_0| \geq p(p-6)$.*
(2) *If $|\alpha_1 \cap \bar{\Omega}_p| = |\alpha_2 \cap \bar{\Omega}_p| = 0$, then $|\Gamma_{1,2}^{\{2\}} \cap O_0| \geq p(p-9)$.*

Proof Let $\Gamma_{1,2}^{\{2\}}(i_0) = \{\{\beta_1, \beta_2\} | i_0 \in \beta_1 \cap \beta_2, \beta_1 \in \Gamma_1, \beta_2 \in \Gamma_2\}$. Since $\Gamma_{1,2}^{\{2\}} \cap O_2$ is a union of T-orbits on $\Gamma_{1,2}^{\{2\}}$, and $\Gamma_{1,2}^{\{2\}} \cap \mathrm{Fix}_{O_2}(T) = \emptyset$ for both cases, we get $p \mid |\Gamma_{1,2}^{\{2\}} \cap O_2|$.

Moreover,

$$\begin{aligned}|\Gamma_{1,2}^{\{2\}}\cap O_1|+|\Gamma_{1,2}^{\{2\}}\cap O_2| =&|\{\{\beta_1,\beta_2\}||\beta_1\cap\beta_2|>0,\beta_i\in\Gamma_i\}|\\ =&|\bigcup_{i_0\in\Omega_{50}\setminus\bar{\Omega}_p}\Gamma_{1,2}^{\{2\}}(i_0)|\\ =&\sum_{i_0\in\Omega_{50}\setminus\bar{\Omega}_p}|\Gamma_{1,2}^{\{2\}}(i_0)|-\sum_{i_0\neq j_0\in\Omega_{50}\setminus\bar{\Omega}_p}|\Gamma_{1,2}^{\{2\}}(i_0)\cap\Gamma_{1,2}^{\{2\}}(j_0)|\\ \leq&\sum_{i_0\in\Omega_{50}\setminus\bar{\Omega}_p}|\Gamma_{1,2}^{\{2\}}(i_0)|\\ =&\sum_{i_0\in\Omega_{50}\setminus\bar{\Omega}_p}|\Gamma_1(i_0)|\cdot|\Gamma_2(i_0)|.\end{aligned}$$

Thus $|\Gamma_{1,2}^{\{2\}}\cap O_0|\geq p^2-p|\Gamma_1(i_0)|\cdot|\Gamma_2(i_0)|$, and the proposition holds. □

Remark 4 Since

$$|\Gamma_{1,2}^{\{2\}}\cap O_2|=|\bigcup_{i_0\neq j_0\in\Omega_{50}\setminus\bar{\Omega}_p}(\Gamma_{1,2}^{\{2\}}(i_0)\cap\Gamma_{1,2}^{\{2\}}(j_0))|=\sum_{i_0\neq j_0\in\Omega_{50}\setminus\bar{\Omega}_p}|\Gamma_{1,2}^{\{2\}}(i_0)\cap\Gamma_{1,2}^{\{2\}}(j_0)|,$$

we have $|\Gamma_{1,2}^{\{2\}}\cap O_2|=0$ if and only if $|\Gamma_{1,2}^{\{2\}}\cap O_1|=6p$ or $9p$, respectively.

Now we consider the parameter λ of the block-transitive 2-$(140^2, 140, \lambda)$ designs. It is clearly that $47\nmid\lambda$ by the fact $47\parallel|G|$, $b=2^2\cdot3\cdot5\cdot7\cdot47\cdot\lambda$ and the block-transitivity of G.

Lemma 5.2 *Let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a 2-$(140^2,140,\lambda)$ design admitting $G=A_{50}$ as a block-transitive automorphism group. Then $43\mid\lambda$.*

Proof Let $g\in G$ be a cycle of length 43. Suppose that there exists a block $B\in\mathcal{B}$ such that $g\in G_B$. Then B is a union of orbits of $T=\langle g\rangle$ on $\mathcal{P}$. Owing to $\mathrm{Fix}_{\mathcal{P}}(g)=\bar{\Omega}_{43}^{\{3\}}$ and $43\mid k$, we assume that $B=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Sigma$, where $\Gamma_i=\alpha_i^T$ $(i=1,2,3)$ are T-orbits of length 43, and $\Sigma\subseteq\mathrm{Fix}_{\mathcal{P}}(g)$, it is obvious that $|\alpha_i\cap\bar{\Omega}_{43}|=0$, 1 or 2.

Note that

$$70=|B^{\{2\}}\cap O_2|\geq|\Gamma_{1,2}^{\{2\}}\cap O_2|+|\Gamma_{1,3}^{\{2\}}\cap O_2|+|\Gamma_{2,3}^{\{2\}}\cap O_2|+\sum_{i=1}^{3}|\Gamma_i^{\{2\}}\cap O_2|.$$

By Proposition 5.1, $\mu(\Gamma_i)=(0,\binom{43}{2}-43,43)$ if $|\alpha_i\cap\bar{\Omega}_p|=1$, and $\mu(\Gamma_i)=(0,0,\binom{43}{2})$ if $|\alpha_i\cap\bar{\Omega}_p|=2$. Thus $|\alpha_i\cap\bar{\Omega}_{43}|\neq2$ for $i=1,2,3$, and at most, there is one point α_i, $i=1,2$ or 3, such that $|\alpha_i\cap\bar{\Omega}_{43}|=1$. Assume without loss of generality that $|\alpha_1\cap\bar{\Omega}_{43}|=1$. Then $\mu(\Gamma_1)=(0,\binom{43}{2}-43,43)$, thus $\mu(\Gamma_i)=(\binom{43}{2}-3\cdot43,3\cdot43,0)$ for $i=2,3$, and $|\Gamma_{i,j}^{\{2\}}\cap O_2|=0$ for $(i,j)=(1,2),(1,3)$ and $(2,3)$ by Proposition 5.2. This implies that $|\Gamma_{i,j}^{\{2\}}\cap O_1|=6\cdot43$ for $(i,j)=(1,2),(1,3)$ and $|\Gamma_{2,3}^{\{2\}}\cap O_1|=9\cdot43$. It turns out

$$\begin{aligned}|B^{\{2\}}\cap O_1|\geq&|\Gamma_{1,2}^{\{2\}}\cap O_1|+|\Gamma_{1,3}^{\{2\}}\cap O_1|+|\Gamma_{2,3}^{\{2\}}\cap O_1|+\sum_{i=1}^{3}|\Gamma_i^{\{2\}}\cap O_1|\\ =&2\cdot6\cdot43+9\cdot43+\left(\binom{43}{2}-43\right)+2\cdot3\cdot43\\ =&2021,\end{aligned}$$

which contradicts the fact that $|B^{\{2\}} \cap O_1| = 1610$. Hence, $|\alpha_i \cap \bar{\Omega}_{43}| = 0$ for $i = 1, 2$ and 3. Then

$$\begin{aligned}|B^{\{2\}} \cap O_0| \geq & |\Gamma_{1,2}^{\{2\}} \cap O_0| + |\Gamma_{1,3}^{\{2\}} \cap O_0| + |\Gamma_{2,3}^{\{2\}} \cap O_0| + \sum_{i=1}^{3} |\Gamma_i^{\{2\}} \cap O_0| + |\bigcup_{i=1}^{3} \Gamma_i| \cdot |\Sigma| \\ \geq & 3 \cdot 43(43-9) + 3\Big(\binom{43}{2} - 3 \cdot 43\Big) + 11 \cdot 129 \\ = & 8127.\end{aligned}$$

which contradicts the fact that $|B^{\{2\}} \cap O_0| = 8050$. Therefore, $\mathrm{Fix}_{\mathcal{B}}(g) = \emptyset$, and then $43 \mid b$. It follows that $43 \mid \lambda$ for $b = 140 \cdot 141\lambda$. □

Next, we consider the parameter λ of the flag-transitive 2-$(140^2, 140, \lambda)$ designs.

Lemma 5.3 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a 2-$(140^2, 140, \lambda)$ design admitting $G = A_{50}$ as a flag-transitive automorphism group. If p is a prime and $29 \leq p \leq 43$, then $p \mid \lambda$.*

Proof Let $g \in G$ is a cycle of length p. Suppose that there exists a block $B \in \mathcal{B}$ such that $g \in G_B$. Then B is a union of orbits of $T = \langle g \rangle$ on $\mathcal{P}$. Assume that $B = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m \cup \Sigma$, where $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are orbits of length p of T, and $\Sigma \subseteq \mathrm{Fix}_{\mathcal{P}}(g)$.

We first claim that $m \neq 0$. In fact, if $m = 0$, then $B \subseteq \mathrm{Fix}_{\mathcal{P}}(g)$. Let $\beta \in B$. Then $\beta \in \bar{\Omega}_p^{\{3\}}$, and there is a unique point $\beta' \in B$ such that $|\beta \cap \beta'| = 2$. Let

$$B = \{\beta_1, \beta_1'\} \cup \{\beta_2, \beta_2'\} \cup \cdots \cup \{\beta_{70}, \beta_{70}'\},$$

where $|\beta_i \cap \beta_i'| = 2$ for $i = 1, 2, \ldots, 70$. That is, B is parted into disjoint subsets.

Without loss of generality, we assume that $\beta_i = \{i_1, i_2, i_3\}$, $\beta_i' = \{i_1, i_2, i_4\}$ for $i = 1, 2, \ldots, 70$, and $S_{\beta_i, \beta_i'}^{\{2\}} = \{\{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\}, \{i_1, i_4\}, \{i_2, i_4\}\}$. Then $S_{\beta_i, \beta_i'}^{\{2\}} \cap S_{\beta_j, \beta_j'}^{\{2\}} = \emptyset$ for $i, j = 1, 2, \ldots, 70$ and $i \neq j$, because $|O_2(\beta) \cap B| = 1$ for each $\beta \in B$. Hence we have $\bigcup_{i=1}^{70} S_{\beta_i, \beta_i'}^{\{2\}} \subseteq \bar{\Omega}_p^{\{2\}}$, thus $5 \cdot 70 \leq \binom{50-p}{2}$, a contradiction for $p \geq 29$. It turns out $m \geq 1$.

Let $\Gamma_i = \alpha_i^T$ for $i = 1, 2, \ldots, m$, here $\alpha_i \in \mathcal{P}$. Then $|\alpha_i \cap \bar{\Omega}_p| = 0$, 1 or 2. By Proposition 5.1, $\nu_{\alpha_i}(\Gamma_i) = (0, 0, p-1)$ if $|\alpha_i \cap \bar{\Omega}_p| = 2$, and $\nu_{\alpha_i}(\Gamma_i) = (0, p-3, 2)$ if $|\alpha_i \cap \bar{\Omega}_p| = 1$. For both cases, $|\Gamma_i \cap O_2(\alpha_i)| \geq 2$, which is impossible for $|B \cap O_2(\alpha_i)| = 1$. Therefore, $|\alpha_i \cap \bar{\Omega}_p| = 0$ for $i = 1, 2, \ldots, m$.

Note that $p \nmid 140$ for $29 \leq p \leq 43$, it implies that $|\Sigma| > 0$. Then

$$\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m \subseteq O_0(\beta)$$

for $\beta \in \Sigma$, hence $mp \leq 115$ since $|B \cap O_0(\beta)| = 115$. Thus $m \leq 3$. Recall that $\alpha_i \cap \bar{\Omega}_p = \emptyset$, we have $|\Gamma_i(i_0)| = 3$, $i_0 \in \Omega_{50} \backslash \bar{\Omega}_p$, for $i = 1, 2, \ldots, m$. Then

$$\begin{aligned}|O_0(\alpha_1) \cap \Gamma_1| = & |\Gamma_1 \backslash \{\alpha_1\}| - |\bigcup_{i_0 \in \alpha_1} (\Gamma_1(i_0) \setminus \{\alpha_1\})| \\ \geq & |\Gamma_1 \backslash \{\alpha_1\}| - \sum_{i_0 \in \alpha_1} |\Gamma_1(i_0) \setminus \{\alpha_1\}| \\ = & p - 1 - 3 \cdot 2 = p - 7.\end{aligned}$$

Similarly, we have $|O_0(\alpha_1) \cap \Gamma_i| \geq p-9$ for $i = 2, \ldots, m$. Equalities hold if and only if $|O_2(\alpha_1) \cap \Gamma_i| = 0$ for each i. Since

$$\begin{aligned}|O_0(\alpha_1) \cap B| =& |O_0(\alpha_1) \cap \Gamma_1| + \sum_{i=2}^{m} |O_0(\alpha_1) \cap \Gamma_i| + |\Sigma| \\ \geq& p - 7 + (m-1)(p-9) + (140 - mp) = \quad 142 - 9m,\end{aligned}$$

and $|O_0(\alpha_1) \cap B| = 115$, then $m = 3$ and

$$\begin{aligned}|O_0(\alpha_1) \cap \Gamma_1| + |O_0(\alpha_1) \cap \Gamma_2| + |O_0(\alpha_1) \cap \Gamma_3| = \\ p - 7 + 2(p-9).\end{aligned}$$

Thus $|O_0(\alpha_1) \cap \Gamma_1| = p - 7$, $|O_0(\alpha_1) \cap \Gamma_i| = p - 9$ for $i = 2, 3$. By the above analysis, we have $|O_2(\alpha_1) \cap \Gamma_i| = 0$ for $i = 1, 2, 3$, and then $\sum_{i=1}^{3} |O_2(\alpha_1) \cap \Gamma_i| = 0$, which is impossible for $|O_2(\alpha_1) \cap B| = \sum_{i=1}^{3} |O_2(\alpha_1) \cap \Gamma_i| = 1$. Therefore, $\mathrm{Fix}_{\mathcal{B}}(g) = \emptyset$, and then $p \mid b$. Since $b = 140 \cdot 141\lambda$, it follows that $p \mid \lambda$. □

Lemma 5.4 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a 2-$(140^2, 140, \lambda)$ design admitting $G = A_{50}$ as a flag-transitive automorphism group. Then $19 \cdot 23^2 \mid \lambda$.*

Proof Let $p = 19$ or 23. Firstly, we prove $p \mid \lambda$ by proving $\mathrm{Fix}_{\mathcal{B}}(g) = \emptyset$, here $g = (i_1 i_2 \cdots i_p)(j_1 j_2 \cdots j_p)$ be a product of two disjoint p-cycles. Let $T = \langle g \rangle$, $\tilde{\Omega}_1 = \{i_1, i_2, \ldots i_p\}$, $\tilde{\Omega}_2 = \{j_1, j_2, \ldots j_p\}$, and $\tilde{\Omega}_3 = \mathrm{Fix}_{\Omega_{50}}(T)$.

Suppose for the contrary that there exists a block $B \in \mathcal{B}$ such that $g \in G_B$. Then B is a union of orbits of T on $\mathcal{P}$. Let $B = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m \cup \Sigma$, where $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are orbits of length p of T, and $\Sigma \subseteq \mathrm{Fix}_{\mathcal{P}}(T)$. Let $\Gamma_i = \alpha_i^T$ for $i = 1, 2, \ldots, m$, here $\alpha_i \in \mathcal{P}$. Since $p \nmid k$, $|\Sigma| \neq 0$. Let $\beta \in \Sigma$, and β_1 be the unique point such that $|\beta \cap \beta_1| = 2$. Then $\beta_1 \in \Sigma$, otherwise, $|O_2(\beta_1) \cap \beta_1^T| \geq p - 1$, a contradiction. Let

$$\Sigma = \{\beta_1, \beta_1'\} \cup \{\beta_2, \beta_2'\} \cup \cdots \cup \{\beta_w, \beta_w'\},$$

where $w = \frac{140 - pm}{2}$, and $|\beta_i \cap \beta_i'| = 2$ for $i = 1, 2, \ldots, w$. Similar to the proof of Lemma 5.3, we have

$$5 \cdot \frac{140 - pm}{2} = |\bigcup_{i=1}^{w} S_{\beta_i, \beta_i'}^{\{2\}}| \leq |\tilde{\Omega}_3^{\{2\}}| = \binom{50 - 2p}{2},$$

thus $m \geq 6$.

Since $|O_2(\alpha_i) \cap \Gamma_i| = p - 1$ if $|\alpha_i \cap \tilde{\Omega}_3| = 2$, it turns out $|\alpha_i \cap \tilde{\Omega}_3| = 0$ or 1 for $i = 1, 2, \ldots, m$. Assume without loss of generality that $|\alpha_1 \cap \tilde{\Omega}_3| = 1$. Note that $|O_2(\alpha_1) \cap \Gamma_1| = 2$ if $|\alpha_1 \cap \tilde{\Omega}_1| = 2$ or $|\alpha_1 \cap \tilde{\Omega}_2| = 2$. Thus $|\alpha_1 \cap \tilde{\Omega}_1| = 1$ and $|\alpha_1 \cap \tilde{\Omega}_2| = 1$. Assume that $\alpha_1 = \{i_1, j_1, k_1\}$, where $i_1 \in \tilde{\Omega}_1$, $j_1 \in \tilde{\Omega}_2$, and $k_1 \in \tilde{\Omega}_3$. Then $|O_1(\alpha_1) \cap \Gamma_1| = p - 1$ and $|O_2(\alpha_1) \cap \Gamma_1| = 0$. Suppose that $|O_2(\alpha_1) \cap \Gamma_2| = 1$, then $|O_2(\alpha_1) \cap \Gamma_i| = 0$ for $i = 3, 4, \ldots, m$. It follows that $|O_1(\alpha_1) \cap \Gamma_i| = |\Gamma_i(i_1)| + |\Gamma_i(i_2)|$, $i = 3, 4, \ldots, m$. Thus $|O_1(\alpha_1) \cap \Gamma_i| = 2$ if $|\alpha_i \cap \tilde{\Omega}_3| = 1$, and $|O_1(\alpha_1) \cap \Gamma_i| = 3$ if $|\alpha_i \cap \tilde{\Omega}_3| = 0$ for $i = 3, 4, \ldots, m$. It follows that

$$23 = |O_1(\alpha) \cap B| \geq p - 1 + 2(m-2) \geq 26,$$

a contradiction. Therefore $|\alpha_1 \cap \tilde{\Omega}_3| = 0$. Similarly, we have $|\alpha_i \cap \tilde{\Omega}_3| = 0$, $i = 2, \ldots, m$. Let $\beta \in \Sigma$, then $\bigcup_{i=1}^{m} \Gamma_i \subseteq O_0(\beta) \cap B$. This implies that $pm \leq 115$, and so $m = 6$ and $p = 19$.

Since $115 = |O_0(\beta) \cap B| = |\bigcup_{i=1}^{m} \Gamma_i| + |O_0(\beta) \cap \Sigma|$, then there exists a unique point $\gamma \in \Sigma$ such that $|\beta \cap \gamma| = 0$. Thus $|\delta \cap \beta| \geq 1$ and $|\delta \cap \gamma| \geq 1$ for each point $\delta \in \Sigma \setminus \{\beta, \gamma\}$, because of $|O_1(\alpha) \cap B| + |O_2(\alpha) \cap B| = 24$ for each $\alpha \in B$. On the other hand, owing to the above discussion and $66 - 5 \cdot \frac{140-19\cdot 6}{2} = 1$, there is only one 2-subset $\{i_0, j_0\} \in \tilde{\Omega}_3^{\{2\}}$, such that $\{i_0, j_0\} \nsubseteq \alpha$ for any point $\alpha \in \Sigma$. Let $\{a_1, a_2\} \neq \{i_0, j_0\} \in \tilde{\Omega}_3^{\{2\}}$, then there exists a point $\delta \in \Sigma$ such that $\{a_1, a_2\} \subseteq \delta$. However, $|\delta \cap \beta| \geq 1$ and $|\delta \cap \gamma| \geq 1$, which is impossible. Therefore, $\mathrm{Fix}_{\mathcal{B}}(g) = \emptyset$, and then $19 \cdot 23 \mid \lambda$.

Now let $g_0 \in G$ be a cycle of length p, $T_0 = \langle g_0 \rangle$. Recall that $\bar{\Omega}_p = \mathrm{Fix}_{\Omega_p}(T)$. Suppose that $g_0 \in G_B$ for a block $B \in \mathcal{B}$. Then B is a union of T_0-orbits on B. Let $B = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m \cup \Sigma$, where $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are orbits of length p of T_0, and $\Sigma \subseteq \mathrm{Fix}_{\mathcal{P}}(T_0)$. Let $\Gamma_i = \alpha_i^{T_0}$ for $i = 1, 2, \ldots, m$, here $\alpha_i \in \mathcal{P}$.

If $m > 1$, similar to the proof of Lemma 5.3, we have that $|\alpha_i \cap \bar{\Omega}_p| = 0$, and $\mu(\Gamma_i) = (\binom{p}{2} - 3p, 3p, 0)$ and $\nu_{\alpha_i}(\Gamma_i) = (p-7, 6, 0)$ for $i = 1, 2, \ldots, m$, because of $|O_2(\alpha_i) \cap \Gamma_i| \leq 1$. Since $|O_2(\alpha_1) \cap B| = \sum_{i=1}^{m} |O_2(\alpha_1) \cap \Gamma_i| = 1$, then there exists a unique point $\beta \in \bigcup_{i=2}^{m} \Gamma_i$ such that $|\alpha_1 \cap \beta| = 2$. Without loss of generality, we assume that $|\alpha_2 \cap \alpha_1| = 2$, then

$$|O_1(\alpha_1) \cap \Gamma_2| + |O_2(\alpha_1) \cap \Gamma_2| = |\bigcup_{i_0 \in \alpha_1} \Gamma_2(i_0)| = \sum_{i_0 \in \alpha_1} |\Gamma_2(i_0)| - \sum_{i_0 \neq j_0 \in \alpha_1} |\Gamma_2(i_0) \cap \Gamma_2(j_0)|.$$

Since $|O_2(\alpha_1) \cap \Gamma_2| = 1$, we have

$$\sum_{i_0 \neq j_0 \in \alpha_1} |\Gamma_2(i_0) \cap \Gamma_2(j_0)| = 1.$$

Thus $|O_1(\alpha_1) \cap \Gamma_2| = 7$. Similarly, we have $|O_1(\alpha_1) \cap \Gamma_i| = 9$ for $i = 3, \ldots, m$. Thus

$$|O_1(\alpha_1) \cap B| = |O_1(\alpha_1) \cap \Gamma_1| + \sum_{i=2}^{m} |O_1(\alpha_1) \cap \Gamma_i| = 6 + 7 + 9(m-2) = 23,$$

which is impossible.

If $m = 1$, then

$$|O_0(\alpha_1) \cap B| \geq |\Sigma| = 140 - p,$$

a contradiction. Therefore $m = 0$ and $B \subseteq \mathrm{Fix}_{\mathcal{P}}(T)$.

Similar to the proof of Lemma 5.3, we know that there are 350 different 2-subsets of $\bar{\Omega}_{23}$ which are contained in the points of B. On the other hand, $|\bar{\Omega}_{23}^{\{2\}}| = 351$, it turns out, there is only one 2-subset $\{i_0, j_0\} \in \bar{\Omega}_{23}^{\{2\}}$, such that $\{i_0, j_0\} \nsubseteq \beta$ for any point $\beta \in B$. Choosing $\alpha = \{i_1, i_2, i_3\} \in B$ such that $\alpha \cap \{i_0, j_0\} = \emptyset$. Then for each $j \in \bar{\Omega}_{23} \backslash \alpha$, there is a point $\gamma \in B$ such that $\{i_e, j\} \subseteq \gamma$, here $e = 1, 2, 3$, and at most, γ contains two of these 2-subsets. Thus $\frac{3 \cdot 24}{2} \leq |(O_1(\alpha) \cup O_2(\alpha)) \cap B| = 24$, which leads a contradiction. Therefore, $\mathrm{Fix}_{\mathcal{B}}(g_0) = \emptyset$ if $p = 23$.

Note that $23^2 \parallel |G|$, the group $G = A_{50}$ has no element of order 23^2 and exactly has two conjugate classes of elements of order 23. It turns out, $23 \nmid |G_B|$ for each $B \in \mathcal{B}$. Therefore, $23^2 \mid b$, and then $23^2 \mid \lambda$. □

Remark 5 Let $g \in G$ be an element of order 19. Then $B \subseteq \mathrm{Fix}_{\mathcal{P}}(g)$ if $g \in G_B$ for some $B \in \mathcal{B}$, and g is a cycle of length 19 according to the proof of Lemma 5.4.

Proof of Theorem 3 It follows immediately from Lemmas 5.1–5.4. □

Declarations

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