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On the fourth weight of generalized Reed–Muller codes

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Abstract

In this paper, we study the fourth weight of generalized Reed–Muller codes. Erickson in his Ph.D. thesis proved that the second weight of $R_q(a(q-1)+b, m)$ depends on the second weight $R_q(b, 2)$. Also, Leducq (Discret Math 338:1515–1535, 2015) proved that under the same condition, by the third weight of $R_q(b, 2)$ we can determine the third weight of $R_q(a(q-1)+b, m)$. In this paper we will show that the similar result does not hold for the fourth weight of generalized Reed–Muller codes. We will determine the fourth weight of generalized Reed–Muller codes of order $r = a(q-1)+b$ with $3 \leq b < \frac{q+4}{3}$.

Keywords Generalized Reed–Muller codes · Fourth weight · Affine subspace · Affine hyperplane

Mathematics Subject Classification 11T71 · 11G25

1 Introduction

Let F_q be the finite field with q elements and $m \geq 1$ an integer. We denote by B_m^q the F_q-algebra of the functions from F_q^m to F_q and by $F_q[X_1, \cdots, X_m]$ the F_q-algebra of polynomials in m variables with coefficients in F_q.

Let r be an integer such that $1 \leq r < m(q-1)$. The generalized Reed–Muller code of order r is the following subspace of the space $F_q^{q^m}$

$$R_q(r, m) = \left\{ (f(x))_{x \in F_q^m} | f \in F_q[X_1, \cdots, X_m] \text{ and } deg(f) \leq r \right\}$$

Throughout this article, we write $r = a(q-1)+b$, $0 \leq a \leq m-1$, $0 \leq b < q-1$, $d_r^m = (q-b)q^{m-a-1}$ and by W_i we denote the ith minimum weight of $R_q(r, m)$. The support of $f \in F_q[X_1, \cdots, X_m]$ is the set $\{x \in F_q^m : f(x) \neq 0\}$ and we denote by $|f|$ the cardinal of its support.

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There exist many questions concerning generalized Reed–Muller codes. Some of the most important questions are about the first minimum weights and weight distribution of generalized Reed–Muller codes. Not much is known for the mentioned questions for $q \geq 3$ and $r \geq 3$.

The minimum weight has been determined by Kasami et al. [9]. It has been proved that the minimal weight of the generalized Reed–Muller code $R_q(r, m)$ is $(q-b)q^{m-a-1}$ where $r = a(q-1) + b$ and $0 \leq b < q-1$. The codewords reaching this bound were described by Delsarte et al. [5] (see also [11]).

Erickson [7] proved that if we know the second weight of $R_q(b, 2)$ then, we can determine the second weight for all generalized Reed–Muller codes. He conjectured that the second weight of $R_q(b, 2)$ is $(q-b)q + b - 1$ and Bruen proved the conjecture using blocking set in [2]. This problem was studied by Geil using Gr$\ddot{o}$bner basis in [8] for $r < q$ and $r > (m-1)(q-1)$ and it was almost completely solved by Rolland [16]. Second weight codewords have been studied in [4, 17] and finally completely described in [12].

Leducq [13] got a full description of the third weight and the third weight codewords of generalized Reed–Muller codes of order $r = a(q-1) + b$ for $3 \leq b < \frac{q+4}{3}$.

The weight distribution of $R_q(2, m)$ was given by McEliece in [15] for any q and due to some mistakes in the computation, Li [14] provided a precise account for the weight distribution of second order generalized Reed–Muller codes. For $q = 2$, for all m and all r, the weight distribution is known in the range $[W_1, 2.5W_1]$ by a result of Kasami et al [10]. We refer the reader to [1, 3, 6] for further results.

In this paper, we want to determine the fourth weight of generalized Reed–Muller codes. The main result of this article is the determination of the fourth weight of $R_q(a(q-1)+b, m)$ the generalized Reed–Muller code of length q^m for prime number q and order $a(q-1)+b$ for $3 \leq b < \frac{q+4}{3}$.

From a geometric point of view a polynomial f defines a hypersurface in F_q^m and $Z(f)$ the set of points of this hypersurface (the set of zeros of f) is related to the support of the associated codeword by the following formula:

$$|f| = q^m - \#Z(f)$$

2 Preliminaries

2.1 Notation and preliminary remark

Let $f \in B_m^q$, $\lambda \in F_q$. We define $f_\lambda \in B_{m-1}^q$ by

$$\forall x = (x_2, ..., x_m) \in F_q^{m-1}, \; f_\lambda(x) = f(\lambda, x_2..., x_m).$$

Let $0 \leq r \leq (m-1)(q-1)$ and $f \in R_q(r, m)$. We denote by S the support of f. Consider H an affine hyperplane in F_q^m, by an affine transformation, we can assume $x_1 = 0$ is an equation of H. Then $S \cap H$ is the support of $f_0 \in R_q(r, m-1)$ or the support of $(1 - x_1^{q-1})f \in R_q(r + (q-1), m)$.

2.2 Useful lemmas

Here, we present some lemmas that we need to prove our main results. The Lemmas 1–8 are proved in [7]

Lemma 1 *Let $m \geq 1$, $q \geq 2$, $f \in B_m^q$ and $w \in F_q$. If for all $(x_2, \cdots, x_m)$ in F_q^{m-1}, $f(w, x_2, \cdots, x_m) = 0$, then for all $(x_1, x_2, \cdots, x_m) \in F_q^m$,*

$$f(x_1, \cdots, x_m) = (x_1 - w)g(x_1, \cdots, x_m)$$

with $deg_{x_1}(g) \leq deg_{x_1}(f) - 1$ and $deg(g) \leq deg(f) - 1$.

Lemma 2 *Let $m \geq 2$, $q \geq 3$, $0 \leq r \leq m(q-1)$. If $f \in R_q(r, m)$, $f \neq 0$ and there exists $y \in R_q(1, m)$ and $(\lambda_i)_{1 \leq i \leq n}$ n elements in F_q such that the hyperplanes of equation $y = \lambda_i$ do not meet the support of f, then*

$$|f| \geq (q-b)q^{m-a-1} + \begin{cases} n(b-n)q^{m-a-2} & \text{if} n < b, \\ (n-b)(q-1-n)q^{m-a-1} & \text{if} n \geq b. \end{cases}$$

where $r = a(q-1) + b$, $1 \leq b \leq q-1$.

Lemma 3 *If $f \in R_q(r, m)$ with $r \leq q-1$ and $|f| < (1 + \frac{1}{q})d_r^m$, then f is the product of r linear factors.*

Lemma 4 *Let $m \geq 2$, $q \geq 3$, $1 \leq b \leq q-1$. Assume $f \in R_q(b, m)$ is such that f depends only on x_1 and $g \in R_q(b-k, m)$, $1 \leq k \leq b$. Then either $f + g$ depends only on x_1 or $|f + g| \geq (q - b + k)q^{m-1}$.*

Lemma 5 *Let $m \geq 2$, $q \geq 3$, $1 \leq a \leq m-1$, $1 \leq b \leq q-2$. Assume $f \in R_q(a(q-1)+b, m)$ is such that $\forall x = (x_1, \cdots, x_m) \in F_q^m$,*

$$f(x) = (1 - x_1^{q-1})\widetilde{f}(x_2, \cdots, x_m)$$

and $g \in R_q(a(q-1) + b - k, m)$, $1 \leq k \leq q-1$, is such that $(1 - x_1^{q-1})$ does not divide g. Then either $|f + g| \geq (q - b + k)q^{m-a-1}$ or $k = 1$.

Lemma 6 *Let $m \geq 2$, $q \geq 3$, $1 \leq a \leq m-2$, $1 \leq b \leq q-2$ and $f \in R_q(a(q-1) + b, m)$. We set an order on the elements of F_q such that $|f_{\lambda_1}| \leq \cdots \leq |f_{\lambda_q}|$.*
If f has no linear factor and there exists $k \geq 1$ such that $(1 - x_2^{q-1})$ divides f_{λ_i} for $i \leq k$ but $(1 - x_2^{q-1})$ does not divide $f_{\lambda_{k+1}}$ then,

$$|f| \geq (q-b)q^{m-a-1} + k(q-k)q^{m-a-2}$$

Lemma 7 *Let $m \geq 2$, $q \geq 3$, $1 \leq a \leq m$ and $f \in R_q(a(q-1), m)$ such that $|f| = q^{m-a}$ and $g \in R_q(a(q-1)-k, m)$, $1 \leq k \leq q-1$, such that $g \neq 0$. Then, either $|f+g| = kq^{m-a}$ or $|f + g| \geq (k+1)q^{m-a}$.*

Lemma 8 *Let $m \geq 2$, $q \geq 3$, $1 \leq a \leq m-1$ and $f \in R_q(a(q-1), m)$. If for some $u, v \in F_q$, $|f_u| = |f_v| = q^{m-a-1}$, then there exists T an affine transformation fixing x_1 such that*

$$(f \circ T)_u = (f \circ T)_v$$

The following results can be found in [13].

Theorem 1 *Let $m \geq 2$, $q \geq 9$, $0 \leq a \leq m-2$ and $4 \leq b < \frac{q+4}{3}$. The third weight of $R_q(a(q-1) + b, m)$ is $W_3 = (q-2)(q-b+2)q^{m-a-2}$.*

Theorem 2 *Let $m \geq 3$, $q \geq 7$ and $0 \leq a \leq m-3$. The third weight of $R_q(a(q-1) + 3, m)$ is $W_3 = (q-1)^3 q^{m-a-3}$.*

Theorem 3 *For $q \geq 7$, $m \geq 2$, $0 \leq a \leq m-2$, $4 \leq b < \frac{q+4}{3}$, up to affine transformation, the third weight codewords of $R_q(a(q-1)+b, m)$ are of the form:*

$$f(x) = \prod_{i=1}^{a}(1 - x_i^{q-1})g(x_{a+1}, x_{a+2}) \quad \forall x = (x_1, \cdots, x_m) \in F_q^m$$

where $g \in R_q(b, 2)$ is such that $|g| = (q-2)(q-b+2)$.

Theorem 4 *For $q \geq 7$, $m \geq 3$, $0 \leq a \leq m-3$, up to affine transformation, the third weight codewords of $R_q(a(q-1)+3, m)$ are of the form:*

$$f(x) = \prod_{i=1}^{a}(1 - x_i^{q-1})x_{a+1}x_{a+2}x_{a+3} \quad \forall x = (x_1, \cdots, x_m) \in F_q^m.$$

3 A lower bound on the fourth weight

In this section, we determine a lower bound on the fourth weight of $R_q(a(q-1)+b, m)$ for the cases where $3 \leq b < \frac{q+4}{3}$. Throughout this paper, by hyperplane we mean affine hyperplane and q is a prime number. The validity of the results of this paper has been checked by a computer program.

Lemma 9 *Let $f \in R_q(b, m)$ be the product of b distinct linear factors such that $x_1 - \lambda_i$ for $i = 1, \cdots, k$ are k of these linear factors. If for some $j_0 \notin \{1, \cdots, k\}$, $|f_{\lambda_{j_0}}| = (q-b+k)q^{m-2}$ (the minimum weight of $R_q(b-k, m-1)$) then, for all $j \notin \{1, \cdots, k\}$, there is an integer t where $0 \leq t \leq b-k-1$ such that $|f_{\lambda_j}| = (q-b+k+t)q^{m-2}$.*

Proof We denote by H_{λ_i} the affine hyperplane with the equation $x_1 = \lambda_i$ for $i = 1, \cdots, q$. Assume that S denotes the support of f. By the assumption of the lemma, S does not meet the hyperplanes H_{λ_i} for $i = 1, \cdots, k$. Denote by $H^{(i)}$ $i = 1, \cdots, b-k$ the other affine hyperplanes corresponding to the other linear factors which do not meet S. Since for some $j_0 \notin \{1, \cdots, k\}$, $|f_{\lambda_{j_0}}| = (q-b+k)q^{m-2}$ then, $f_{\lambda_{j_0}}$ is a minimum weight codeword of $R_q(b-k, m-1)$. So $H_{\lambda_{j_0}} \cap H^{(i)} = P^{(i)}$ is an affine subspace of codimension 2 where $P^{(i)} \cap P^{(i')} = \emptyset$ for $i \neq i'$. We get that for each two hyperplanes $H^{(s)}$ and $H^{(s')}$, $H^{(s)} \cap H^{(s')}$ is either empty or an affine subspace of codimension 2 which is included in one of the hyperplanes H_{λ_i} for $i = 1, \cdots, q$. Denote by P^{ij} the affine subspace of codimension 2 $H_{\lambda_i} \cap H^{(j)}$ for $k+1 \leq i \leq q$ and $1 \leq j \leq b-k$ in which for $j \neq j'$, $P^{ij} \cap P^{ij'} = \emptyset$ or $P^{ij} = P^{ij'}$. So we get that $|f_{\lambda_i}| = q^{m-1} - (b-k-t)q^{m-2}$ in which $b-k-t$ is the number of distinct subspaces P^{ij} which is included in H_{λ_i}. □

Lemma 10 *Let $m \geq 3$, $q \geq 9$, $4 \leq b < \frac{q+4}{3}$ and $f \in R_q(b, m)$. If $|f| > (q-2)(q-b+2)q^{m-2}$, then $|f| \geq (q-1)^2(q-b+2)q^{m-3}$.*

Proof Let $f \in R_q(b, m)$ such that $|f| > (q-2)(q-b+2)q^{m-2}$. Assume $|f| < (q-1)^2(q-b+2)q^{m-3}$. Since

$$(q-1)^2(q-b+2)q^{m-3} \leq (1 + \frac{1}{q})d_b^m = (1 + \frac{1}{q})(q-b)q^{m-1} \tag{1}$$

for $b < \frac{q+4}{3}$, by Lemma 3 f is the product of b linear factors. For $y \in R_q(1, m)$, denote by n the number of distinct $\lambda \in F_q$ such that $y - \lambda$ divides f. Since $n \leq b$ by Lemma 2

$$(q-b)q^{m-1} + n(b-n)q^{m-2} < (q-1)^2(q-b+2)q^{m-3}$$

we get that $n \in \{1, 2, b-2, b-1, b\}$.
By applying an affine transformation we can assume that $x_1 = \lambda_1$, $\lambda_1 \in F_q$ is one of the linear factors.
If $n = b$, then for all $x = (x_1, \cdots, x_m) \in F_q^m$, we have

$$f(x) = \alpha \prod_{i=1}^{b} (x_1 - \lambda_i)$$

with $\lambda_i \in F_q$, $\lambda_i \neq \lambda_j$ for $i \neq j$. In this case f is a minimum weight codeword of $R_q(b, m)$ which is absurd.
If $n = b - 1$, then for all $x = (x_1, \cdots, x_m) \in F_q^m$, we have

$$f(x) = \prod_{i=1}^{b-1} (x_1 - \lambda_i) g(x)$$

with $\lambda_i \in F_q$, $\lambda_i \neq \lambda_j$ for $i \neq j$ and $g \in R_q(1, m)$. If $deg(g) = 0$, then f is a minimum weight codeword of $R_q(b-1, m)$. If $deg(g) = 1$, then f is a second minimum weight codeword of $R_q(b, m)$. Both cases give us a contradiction, since $(q-2)(q-b+2)q^{m-2} < |f| < (q-1)^2(q-b+2)q^{m-3}$.
If $n = b - 2$, then for all $x = (x_1, \cdots, x_m) \in F_q^m$, we have

$$f(x) = \prod_{i=1}^{b-2} (x_1 - \lambda_i) g(x)$$

with $\lambda_i \in F_q$, $\lambda_i \neq \lambda_j$ for $i \neq j$ and $g \in R_q(2, m)$. If $deg(g) = 0$, then f is a minimum weight codeword of $R_q(b-2, m)$. If $deg(g) = 1$, then f is a second minimum weight codeword of $R_q(b-1, m)$. Both cases give a contradiction. So $deg(g) = 2$. For all $i \geq b-1$, $f_{\lambda_i} \in R_q(2, m-1)$ and $|f_{\lambda_i}| = |g_{\lambda_i}| \geq (q-2)q^{m-2}$. Denote by $N := \#\{i \geq b-1; |f_{\lambda_i}| = (q-2)q^{m-2}\}$. For $\lambda \in F_q$, if $|f_\lambda| > (q-2)q^{m-2}$, then $|f_\lambda| \geq (q-1)^2 q^{m-3}$. Since $|f| < (q-1)^2(q-b+2)q^{m-3}$, we get that $N \geq 1$. So by Lemma 9 we conclude that for all $i \geq b-1$, $|f_{\lambda_i}| = (q-2)q^{m-2}$ or $|f_{\lambda_i}| = (q-1)q^{m-2}$. From

$$N(q-2)q^{m-2} + (q-b+2-N)(q-1)q^{m-2} < (q-1)^2(q-b+2)q^{m-3}$$

we get that $N = q - b + 2$ that gives a third minimum weight codeword of $R_q(b, m)$ which is absurd.
If $n = 2$, then for all $x = (x_1, \cdots, x_m) \in F_q^m$, we have

$$f(x) = (x_1 - \lambda_1)(x_1 - \lambda_2) g(x)$$

with $\lambda_1, \lambda_2 \in F_q$, $\lambda_1 \neq \lambda_2$ and $g \in R_q(b-2, m)$. Then for all $i \geq 3$, $f_{\lambda_i} \in R_q(b-2, m-1)$ and $|f_{\lambda_i}| = |g_{\lambda_i}| \geq (q-b+2)q^{m-2}$. Denote by $N := \#\{i \geq 3; |f_{\lambda_i}| = (q-b+2)q^{m-2}\}$. For $\lambda \in F_q$, if $|f_\lambda| > (q-b+2)q^{m-2}$, then $|f_\lambda| \geq (q-1)(q-b+3)q^{m-3}$. Since

$$(q-2)(q-1)(q-b+3)q^{m-3} > (q-1)^2(q-b+2)q^{m-3}$$

we get that $N \geq 1$. So by Lemma 9 for all $i \geq 3$, $|f_{\lambda_i}| = (q-b+t)q^{m-2}$ where $2 \leq t \leq b-1$. Therefore we have

$$|f| \geq N(q-b+2)q^{m-2} + (q-2-N)(q-b+3)q^{m-2}$$
$$= \Big(q(q-2)(q-b+3) - Nq\Big)q^{m-3}.$$

By considering $|f| < (q-1)^2(q-b+2)q^{m-3}$, we get that $N = q-2$ that gives a third minimum weight codeword of $R_q(b, m)$ which is absurd.
From now, assume $n = 1$. Then for all $x = (x_1, \cdots, x_m) \in F_q^m$, we have

$$f(x) = (x_1 - \lambda_1)g(x)$$

with $\lambda_1 \in F_q$ and $g \in R_q(b-1, m)$. Then for all $i \geq 2$, $f_{\lambda_i} \in R_q(b-1, m-1)$ and $|f_{\lambda_i}| = |g_{\lambda_i}| \geq (q-b+1)q^{m-2}$. Denote by $N := \#\{i \geq 2; |f_{\lambda_i}| = (q-b+1)q^{m-2}\}$. For $\lambda \in F_q$, if $|f_\lambda| > (q-b+1)q^{m-2}$, then $|f_\lambda| \geq (q-1)(q-b+2)q^{m-3}$. Since

$$(q-1)(q-1)(q-b+2)q^{m-3} \geq (q-1)^2(q-b+2)q^{m-3}$$

we get that $N \geq 1$. Assume H_0 is the hyperplane with the equation $x_1 = \lambda_1$. Let $\mathcal{H} = \{H_1, \cdots, H_{b-1}\}$ be the set of $(b-1)$ other affine hyperplanes which do not meet S. Denote by A the affine subspace of codimension 2 which is included in both of H_0 and H_1. Let $\mathcal{A} = \{H_i; i \geq 1, H_i \cap H_0 = A\}$. Since $n = 1$ and $N \geq 1$, for each pair $(H, H') \in \mathcal{A} \times (\mathcal{H} - \mathcal{A})$, $H \cap H'$ is an affine subspace of codimension 2 which is included in one of $H^{(i)}$ (the hyperplane with the equation $x_1 = \lambda_i$) for $2 \leq i \leq q$. Then we have

$$|f| \geq (q-1)(q-b+1)q^{m-2} + \#\mathcal{A}(b-1-\#\mathcal{A})q^{m-2}.$$

By considering $|f| < (q-1)^2(q-b+2)q^{m-3}$, we get that $|\mathcal{A}| = b-1$ that gives a second minimum weight codeword of $R_q(b, m)$ which is absurd. □

Lemma 11 *Let $m \geq 3$, $q \geq 9$ and $4 \leq b < \frac{q+4}{3}$. If $f \in R_q((m-3)(q-1)+b, m)$ and $|f| > (q-2)(q-b+2)q$, then $|f| \geq (q-1)^2(q-b+2)$.*

Proof The case where $m = 3$ comes from Lemma 10. Assume $m \geq 4$. Let $f \in R_q((m-3)(q-1)+b, m)$ such that $|f| > (q-2)(q-b+2)q$. Assume $|f| < (q-1)^2(q-b+2)$. We denote by S the support of f.

Assume S meets all affine hyperplanes. We set an order on the elements of F_q such that $|f_{\lambda_1}| \leq \cdots \leq |f_{\lambda_q}|$. Then for all H hyperplane, $\#(S \cap H) \geq (q-b)q$ and since

$$q((q-2)(q-b+2)+1) \geq (q-1)^2(q-b+2)$$

we get that $|f_{\lambda_1}| = (q-b)q$ or $|f_{\lambda_1}| = (q-1)(q-b+1)$ or $|f_{\lambda_1}| = (q-2)(q-b+2)$. By applying an affine transformation, we can assume $(1 - x_2^{q-1})$ divides f_{λ_1}. Let $k \geq 1$ be such that for all $i \leq k$, $(1 - x_2^{q-1})$ divides f_{λ_i} and $(1 - x_2^{q-1})$ does not divide $f_{\lambda_{k+1}}$. Then by Lemma 6

$$|f| \geq (q-b)q^2 + k(q-k)q$$
$$\geq (q-b)q^2 + q(q-1)$$

we get a contradiction, since $(q-b)q^2 + q(q-1) \geq (q-1)^2(q-b+2)$.
So there exists a hyperplane H_0 which does not meet S. By applying an affine transformation, we can assume $x_1 = \alpha$, $\alpha \in F_q$ is an equation of H_0. Denote by n the number of hyperplanes parallel to H_0 which do not meet S.

If $n = q-1$, then for all $x = (x_1, \cdots, x_m) \in F_q^m$ we can write

$$f(x) = (1 - x_1^{q-1})g(x_2, \cdots, x_m)$$

where $g \in R_q((m-4)(q-1)+b, m-1)$ and $|f| = |g|$. So g has the same conditions as f with one variable less. Iterating this process we end either in the case where $a = 0$ (which is absurd by Lemma 10) or in the case where $n < q-1$.
From now, we assume $n < q-1$. By Lemma 2 since $|f| < (q-1)^2(q-b+2)$, $n \in \{1, 2, b-2, b-1, b\}$. We can write for all $x = (x_1, \cdots, x_m) \in F_q^m$,

$$f(x) = \prod_{1 \le i \le n} (x_1 - \lambda_i) g(x)$$

where $g \in R_q((m-3)(q-1)+b-n, m)$. Then for all $i \ge n+1$, $f_{\lambda_i} \in R_q((m-3)(q-1)+b-n, m-1)$ and $|f_{\lambda_i}| = |g_{\lambda_i}| \ge q(q-b+n)$.
Assume $n = b$. For $\lambda \in F_q$ if $|g_\lambda| > q^2$ then $|g_\lambda| \ge 2(q-1)q$. Denote by $N := \#\{i \ge b+1; |g_{\lambda_i}| = q^2\}$. Since

$$(q-b)2(q-1)q \ge (q-1)^2(q-b+2)$$

for $b < \frac{q+4}{3}$, we get $N \ge 1$. Furthermore, since

$$(q-b)q^2 < (q-2)(q-b+2)q$$

$N \le q-b-1$.
Assume $|f_{\lambda_{b+N+1}}| \ge (N+1)q^2$. Then

$$Nq^2 + (q-b-N)(N+1)q^2 \le |f| < (q-1)^2(q-b+2).$$

Therefore

$$Nq^2(q-b-N) \le 2bq - 3q - b + 2 < (2b-3)q$$

which gives $N(q-b-N) < 1$ for $b < \frac{q+4}{3}$ and this is absurd since $1 \le N \le q-b-1$. If $|f_{\lambda_{b+N+1}}| = Nq^2$ then

$$Nq^2 + (q-b-N)Nq^2 \le |f| < (q-1)^2(q-b+2).$$

So

$$Nq^2(q-b) - N(N-1)q^2 - q^2(q-b) < 2bq - 3q - b + 2 < (2b-3)q$$

which gives

$$(N-1)(q-b-N)q^2 < (2b-3)q.$$

So the only possibility such that $|f_{\lambda_{b+N+1}}| = Nq^2$ is $N = 1$ or $N = q-b$ which are absurd by definition of N and inequality $N \le q-b-1$, respectively.

By Lemma 8 for all $b+1 \le i \le N+b$, $g_{\lambda_{b+1}} = g_{\lambda_i}$. So we can write for all $x = (x_1, \cdots, x_m) \in F_q^m$

$$\begin{aligned} f(x) &= \prod_{1 \le i \le b} (x_1 - \lambda_i) \Big(g_{\lambda_{b+1}}(x_2, \cdots, x_m) + \prod_{b+1 \le i \le N+b} (x_1 - \lambda_i) h(x) \Big) \\ &= \prod_{1 \le i \le b} (x_1 - \lambda_i) \Big(\alpha f_{\lambda_{b+1}}(x_2, \cdots, x_m) + \prod_{b+1 \le i \le N+b} (x_1 - \lambda_i) h(x) \Big) \end{aligned}$$

where $h \in R_q((m-3)(q-1)-N, m)$ and $\alpha \in F_q^*$.

Then for all $(x_2, \cdots, x_m) \in F_q^{m-1}$,

$$f_{\lambda_{b+N+1}}(x_2, \cdots, x_m) = \beta f_{\lambda_{b+1}}(x_2, \cdots, x_m) + \gamma h_{\lambda_{b+N+1}}(x_2, \cdots, x_m)$$

which is absurd by Lemma 7.

Now, assume $n \in \{1, 2, b-2, b-1\}$.

Applying argument as in the beginning of the proof of this lemma, we can assume that $(1 - x_2^{q-1})$ does not divide f.

Since $n \geq 1$, $f_{\lambda_1} = 0$. So $(1 - x_2^{q-1})$ divides f_{λ_1}. Then since $(1 - x_2^{q-1})$ does not divide f, there exists $k \in \{1, 2, \cdots, q-1\}$ such that for all $i \leq k$, $(1 - x_2^{q-1})$ divides f_{λ_i} and $(1 - x_2^{q-1})$ does not divide $f_{\lambda_{k+1}}$. For $\lambda \in F_q$, if $|f_\lambda| > q(q-b+n)$, then

$$|f_\lambda| \geq W_2 = \begin{cases} q^2 & \text{if } n = b-1, \\ (q-1)(q-b+2) & \text{if } n = 1, \\ (q-1)(q-b+3) & \text{if } n = 2, \\ (q-1)^2 & \text{if } n = b-2. \end{cases}$$

Denote by $N := \#\{i \geq n+1; |f_{\lambda_i}| = q(q-b+n)\}$. In all cases, $(q-n)W_2 \geq (q-1)^2(q-b+2)$. So $N \geq 1$. Furthermore, for all $n \in \{1, 2, b-2, b-1\}$, $(q-n)q(q-b+n) \leq (q-2)(q-b+2)q$. So $N \leq q-n-1$.
Then $|f_{\lambda_{n+1}}| = q(q-b+n)$ and $f_{\lambda_{n+1}}$ is a minimal weight codeword of $R_q((m-3)(q-1)+b-n, m-1)$. So by applying an affine transformation we can assume $(1 - x_2^{q-1})$ divides $f_{\lambda_{n+1}}$. Then $k \geq n+1 \geq 2$.

If $N \geq 2$ and $n+1 \leq k \leq n+N-1$, then $|f_{\lambda_{k+1}}| = q(q-b+n) < q(q-b+k)$. If $N = 1$ and $n+1 \leq k \leq q-1$ or $N \geq 2$ and $n+N \leq k \leq q-1$, assume $|f_{\lambda_{k+1}}| \geq q(q-b+k)$. Then

$$\begin{aligned} |f| &\geq Nq(q-b+n) + (k-n-N)W_2 + (q-k)q(q-b+k) \\ &\geq (q-1)^2(q-b+2) \end{aligned}$$

which is absurd.

Since for all $n+1 \leq i \leq k$, $(1 - x_2^{q-1})$ divides f_{λ_i}, it divides g_{λ_i} too. Then for all $x = (x_1, \cdots, x_m) \in F_q^m$ we can write

$$f(x) = \prod_{1 \leq i \leq n} (x_1 - \lambda_i) \left(\prod_{n+1 \leq i \leq k} (x_1 - \lambda_i) h(x_1, \cdots, x_m) + (1 - x_2^{q-1}) l(x_1, x_3, \cdots, x_m) \right)$$

with $deg(h) \leq (m-3)(q-1)+b-k$. Then for all $x = (x_1, \cdots, x_m) \in F_q^m$

$$f_{\lambda_{k+1}}(x_2, \cdots, x_m) = \alpha h_{\lambda_{k+1}}(x_2, \cdots, x_m) + \beta(1 - x_2^{q-1}) l_{\lambda_{k+1}}(x_3, \cdots, x_m))$$

Therefore, we get a contradiction by Lemma 5, since $k \geq 2$ and $|f_{\lambda_{k+1}}| < q(q-b+k)$. □

Lemma 12 *Let $q \geq 4$, $m \geq 3$. If $f \in R_q(3, m)$ and $|f| > (q-1)^3 q^{m-3}$ then, $|f| \geq ((q-1)^3 + 1)q^{m-3}$.*

Proof We prove this lemma by induction on m. The case where $m = 3$ is an immediate result. Suppose that for some $m \geq 4$, if $f \in R_q(3, m-1)$ is such that $|f| > (q-1)^3 q^{m-4}$ then $|f| \geq ((q-1)^3 + 1)q^{m-4}$.
Let $f \in R_q(3, m)$ such that $|f| > (q-1)^3 q^{m-3}$. Assume $|f| < ((q-1)^3 + 1)q^{m-3}$. Denote by S the support of f.

Assume S meets all affine hyperplanes. Then for all H hyperplanes $\#(S \cap H) \geq (q-3)q^{m-2}$. Suppose that there exists H_1 such that $\#(S \cap H_1) = (q-3)q^{m-2}$. By applying an affine transformation, we can assume $x_1 = \alpha$ is an equation of H_1. Set an order on the elements of F_q such that $|f_{\lambda_1}| \leq \cdots \leq |f_{\lambda_q}|$. Then f_{λ_1} is a minimum weight codeword of $R_q(3, m-1)$. By applying an affine transformation, we can assume f_{λ_1} depends only on x_2. Let $k \geq 1$ be such that f_{λ_i} depends only on x_2 for all $i \leq k$ but $f_{\lambda_{k+1}}$ does not depend only on x_2. If $k > 3$, we can write for all $x = (x_1, \cdots, x_m) \in F_q^m$

$$f(x) = \sum_{i=0}^{3} f^{(i)}_{\lambda_{i+1}}(x_2, \cdots, x_m) \prod_{1 \leq j \leq i} (x_1 - \lambda_j)$$

Since for $i \leq 4$, f_{λ_i} depends only on x_2, then f depends only on x_1, x_2, Then $|f| \equiv 0 \pmod{q^{m-2}}$. Since $|f| > (q-1)^3 q^{m-3}$, then $|f| \geq ((q-1)^3+1)q^{m-3}$ which is absurd. So $k \leq 3$. Since $f_{\lambda_1}, \cdots, f_{\lambda_k}$ depend only on x_2, we can write for all $x_1, x_2 \in F_q$ and $\widehat{x} \in F_q^{m-2}$

$$f(x_1, x_2, \widehat{x}) = g(x_1, x_2) + \prod_{i=1}^{k} (x_1 - \lambda_i) h(x_1, x_2, \widehat{x})$$

where $deg(h) \leq 3 - k$. Then

$$f_{\lambda_{k+1}}(x_2, \widehat{x}) = g_{\lambda_{k+1}}(x_2) + \alpha h_{\lambda_{k+1}}(x_2, \widehat{x})$$

where $\alpha \in F_q^*$. So by Lemma 4 since $f_{\lambda_{k+1}}$ does not depend only on x_2, $|f_{\lambda_{k+1}}| \geq (q-3+k)q^{m-2}$. So

$$|f| \geq k(q-3)q^{m-2} + (q-k)(q-3+k)q^{m-2} = (q-3)q^{m-1} + k(q-k)q^{m-2}.$$

By considering $|f| < ((q-1)^3+1)q^{m-3}$, we get a contradiction.

So for all H hyperplanes, $\#(S \cap H) \geq (q-1)(q-2)q^{m-3}$. By induction hypothesis and considering q parallel hyperplanes there exists an affine hyperplane H_0 such that $\#(S \cap H_0) = (q-1)(q-2)q^{m-3}$ or $\#(S \cap H_0) = (q-1)^3 q^{m-4}$. In both cases, we get that there exists A an affine subspace of codimension 2 included in H_0 which does not meet S. Considering all hyperplanes through A, since for all H hyperplanes, $\#(S \cap H) \geq (q-1)(q-2)q^{m-3}$, we get

$$(q+1)(q-1)(q-2)q^{m-3} < ((q-1)^3+1)q^{m-3}$$

and this is absurd. So there exists an affine hyperplane H_1 which does not meet S. Denote by n the number of hyperplanes parallel to H_1 which do not meet S.

By applying an affine transformation, we can assume $x_1 = \lambda_1$ is an equation of H_1. We have $n \leq 3$.

If $n = 3$, then for all $x = (x_1, \cdots, x_m) \in F_q^m$ we can write

$$f(x) = (x_1 - \lambda_1)(x_1 - \lambda_2)(x_1 - \lambda_3)g(x)$$

where $\lambda_i \in F_q$, $\lambda_i \neq \lambda_j$ for all $i \neq j$, $deg(g) = 0$. So $|f| = (q-3)q^{m-1}$ that gives a minimum weight codeword of $R_q(3, m)$ which is absurd.

If $n = 2$, then for all $x = (x_1, \cdots, x_m) \in F_q^m$ we can write

$$f(x) = (x_1 - \lambda_1)(x_1 - \lambda_2)g(x)$$

where $\lambda_i \in F_q$, $\lambda_1 \neq \lambda_2$, $deg(g) \leq 1$. If $deg(g) = 0$, $|f| = (q-2)q^{m-1}$. If $deg(g) = 1$, $|f| = (q-2)(q-1)q^{m-2}$. We get a contradiction in both cases.

From now, assume $n = 1$. Then for all $x = (x_1, \cdots, x_m) \in F_q^m$ we have

$$f(x) = (x_1 - \lambda_1)g(x)$$

where $deg(g) \leq 2$. Then for $i \geq 2$, $deg(f_{\lambda_i}) \leq 2$ and so either $|f_{\lambda_i}| = (q-2)q^{m-2}$ or $|f_{\lambda_i}| = (q-1)^2 q^{m-3}$ or $|f_{\lambda_i}| \geq (q^2 - q - 1)q^{m-3}$ (see [13]). Since

$$(q-1)(q^2 - q - 1)q^{m-3} \geq ((q-1)^3 + 1)q^{m-3},$$

is a contradiction, there exists $i \geq 2$ such that $|f_{\lambda_i}| = (q-2)q^{m-2}$ or $|f_{\lambda_i}| = (q-1)^2 q^{m-3}$. Denote by H' a hyperplane such that $\#(S \cap H') = (q-2)q^{m-2}$ ($\#(S \cap H') = (q-1)^2 q^{m-3}$). Then there exist P_1 and P_2 two parallel affine subspaces of codimension 2 (two affine subspaces of codimension 2 intersect in an affine subspace of codimension 3) included in H' not in S. Consider P an affine subspace of codimension 2 included in H' which intersect P_1 and P_2 (in two different subspace of codimension 3). Then $\#(S \cap P) = (q-2)q^{m-3}$. Then for all H hyperplane through P, $\#(S \cap H) \geq (q-1)(q-2)q^{m-3}$. We can apply the same argument to all affine subspaces of codimension 2 included in H' parallel to P. Now, consider a hyperplane through P and the $q-1$ parallel hyperplanes to this hyperplane. Since $|f| < ((q-1)^3 + 1)q^{m-3}$, by induction hypothesis one of these hyperplanes say H'' meets S either in $(q-2)(q-1)q^{m-3}$ or $(q-1)^3 q^{m-4}$ points.

Denote by $(A_i)_{1 \leq i \leq 3}$ the 3 affine subspaces of codimension 2 included in H'' which do not meet S. Suppose that S meets all hyperplanes through A_i and consider H one of them. If all hyperplanes parallel to H meet S then as in the beginning of the proof of this lemma, we get that $\#(S \cap H) \geq (q-1)(q-2)q^{m-3}$. If there exists a hyperplane parallel to H which does not meet S then $\#(S \cap H) \geq (q-2)q^{m-2}$. In all cases we get a contradiction since $(q+1)(q-1)(q-2)q^{m-3} \geq ((q-1)^3 + 1)q^{m-3}$.

Then there exist three hyperplanes H_1 (with the equation $x_1 = \lambda_1$), H_2 and H_3 which do not meet S. Since $n = 1$, the intersection of H_2 and H_3 is an affine subspace of codimension 2 say $A_{2,3}$. There are three following cases:

If $A_{2,3}$ is contained in the hyperplane H_1, then for all $i \geq 2$ $|f_{\lambda_i}| = (q-2)q^{m-2}$. So $|f| = (q-1)(q-2)q^{m-2}$ which is absurd.

If $A_{2,3}$ is contained in one of the hyperplanes $x_1 = \lambda_j$ for $j \geq 2$, then $|f_{\lambda_j}| = (q-1)q^{m-2}$ and $|f_{\lambda_i}| = (q-2)q^{m-2}$ for $i \geq 2$ and $i \neq j$. So

$$\begin{aligned}|f| &= (q-2)(q-2)q^{m-2} + (q-1)q^{m-2} \\ &= (q^2 - 3q + 3)q^{m-2} \\ &= ((q-1)^3 + 1)q^{m-3},\end{aligned}$$

we get a contradiction, since $|f| < ((q-1)^3 + 1)q^{m-3}$.

If $A_{2,3}$ meets the hyperplane $x_1 = \lambda_i$ in an affine subspace P_i of codimension 3 for all i, then $|f_{\lambda_i}| = (q-1)^2 q^{m-3}$. So $|f| = (q-1)(q-1)^2 q^{m-3} = (q-1)^3 q^{m-3}$ which is absurd. □

In what follows, let

$$\tilde{c}_b = \begin{cases} (q-2)(q-b+2)q & \text{if } 4 \leq b < \frac{q+4}{3}, \quad q \geq 9 \\ (q-1)^3 & \text{if } b = 3, \quad q \geq 7 \end{cases}$$

and

$$\tilde{d}_b = \begin{cases} (q-1)^2(q-b+2) & \text{if } 4 \leq b < \frac{q+4}{3}, \quad q \geq 9 \\ (q-1)^3 + 1 & \text{if } b = 3, \quad q \geq 7 \end{cases}$$

Proposition 1 *Let $m \geq 3$, $q \geq 9$, $0 \leq a \leq m-3$, $3 \leq b < \frac{q+4}{3}$ and $f \in R_q(a(q-1)+b, m)$. If $|f| > \tilde{c}_b q^{m-a-3}$ then $|f| \geq \tilde{d}_b q^{m-a-3}$.*

Proof The cases where $a = 0$ or $m = 3$ have been considered in Lemmas 10 and 12. Assume $m \geq 4$ and $a \geq 1$. We prove the result by induction on $m - a$. The case where $a = m - 3$ is an immediate result for $b = 3$ and follows from Lemma 11 for $b \geq 4$. If $m = 4$, the only possibilities are $a = 0$ and $a = 1$ that have been considered before. So, from now suppose that $m \geq 5$. Let $1 \leq a \leq m - 4$. Assume if $f \in R_q((a+1)(q-1)+b, m)$ is such that $|f| > \tilde{c}_b q^{m-a-4}$ then, $|f| \geq \tilde{d}_b q^{m-a-4}$.

Let $f \in R_q(a(q-1)+b, m)$ such that $|f| > \tilde{c}_b q^{m-a-3}$. Assume $|f| < \tilde{d}_b q^{m-a-3}$. Denote by S the support of f.

Suppose that S meets all affine hyperplanes. We set an order on the elements of F_q such that $|f_{\lambda_1}| \leq \cdots \leq |f_{\lambda_q}|$. Since $|f| < \tilde{d}_b q^{m-a-3}$, by induction hypothesis, f_{λ_1} is either a minimal weight codeword or second weight codeword or third weight codeword of $R_q(a(q-1)+b, m-1)$. In all cases, by applying an affine transformation we can assume $(1 - x_2^{q-1})$ divides f_{λ_1}. Let $k \geq 1$ be such that for all $i \leq k$, $(1 - x_2^{q-1})$ divides f_{λ_i} but $(1 - x_2^{q-1})$ does not divide $f_{\lambda_{k+1}}$. Then, by Lemma 6,

$$|f| \geq (q-b)q^{m-a-1} + k(q-k)q^{m-a-2} \geq (q-b)q^{m-a-1} + (q-1)q^{m-a-2}$$

which is absurd, since $(q-b)q^{m-a-1} + (q-1)q^{m-a-2} \geq \tilde{d}_b q^{m-a-3}$.

So there exists a hyperplane H_0 which does not meet S. By applying an affine transformation we can assume $x_1 = \alpha$, $\alpha \in F_q$, is an equation of H_0. We denote by n the number of hyperplanes parallel to H_0 which do not meet S.

If $n = q - 1$, then we can write for all $x = (x_1, \cdots, x_m) \in F_q^m$

$$f(x) = (1 - x_1^{q-1})g(x_2, \cdots, x_m)$$

where $g \in R_q((a-1)(q-1)+b, m-1)$ and $|f| = |g|$. Then g has the same conditions as f. Iterating this process, we end either in the case where $a = 0$ (which gives a contradiction by Lemma 10 and 12) or in the case where $n < q - 1$. So from now we assume $n < q - 1$. Since $|f| < \tilde{d}_b q^{m-a-3}$, by Lemma 2 the only possibilities are $n \in \{1, 2, b-2, b-1, b\}$. We can write for all $x = (x_1, \cdots, x_m) \in F_q^m$

$$f(x) = \prod_{i=1}^{n}(x_1 - \lambda_i)g(x)$$

where $g \in R_q(a(q-1)+b-n, m)$. Then for all $i \geq n+1$, $f_{\lambda_i} \in R_q(a(q-1)+b-n, m-1)$ and $|f_{\lambda_i}| = |g_{\lambda_i}| \geq (q-b+n)q^{m-a-2}$.

Assume $n = b$. For $\lambda \in F_q$, if $|g_\lambda| > q^{m-a-1}$, then $|g_\lambda| \geq 2(q-1)q^{m-a-2}$. We denote By $N = \#\{i \geq b+1 : |g_{\lambda_i}| = q^{m-a-1}\}$. Since for $i \geq b+1$ $|f_{\lambda_i}| = |g_{\lambda_i}|$ and $(q-b)2(q-1)q^{m-a-2} > \tilde{d}_b q^{m-a-3}$, $N \geq 1$. On the other hand since $(q-b)q^{m-a-1} < \tilde{c}_b q^{m-a-3} < |f|$, $N \leq q - b - 1$.

Assume $|f_{\lambda_{b+N+1}}| \geq (N+1)q^{m-a-1}$. Then

$$Nq^{m-a-1} + (q-b-N)(N+1)q^{m-a-1} \leq |f| < \tilde{d}_b q^{m-a-3}$$

which gives

$$(q-b)Nq^2 - N^2q^2 < \tilde{d}_b - (q-b)q^2$$

therefore

$$N(q-b-N)q^2 < (2b-3)q.$$

which is absurd since $N(q-b-N) \geq 1$ and $b < \frac{q+4}{3}$.

If $|f_{\lambda_{b+N+1}}| = Nq^{m-a-1}$, then

$$Nq^{m-a-1} + (q-b-N)Nq^{m-a-1} \leq |f| < \tilde{d}_b q^{m-a-3}$$

which gives

$$Nq^2(q-b-N+1) < \tilde{d}_b < q^2(q-b+1).$$

So we have

$$(N-1)(q-b+1) < N^2$$

which is absurd for $2 \leq N \leq q-b-1$. So the only possibility such that $|f_{\lambda_{b+N+1}}| = Nq^{m-a-1}$ is the case where $N = 1$ that is a contradiction by definition of N.

By Lemma 8, for all $b+1 \leq i \leq N+b$, $g_{\lambda_{b+1}} = g_{\lambda_i}$. So we can write for all $x = (x_1, \cdots, x_m) \in F_q^m$

$$\begin{aligned} f(x) &= \prod_{1 \leq i \leq b} (x_1 - \lambda_i)\left(g_{\lambda_{b+1}}(x_2, \cdots, x_m) + \prod_{b+1 \leq i \leq N+b} (x_1 - \lambda_i) h(x)\right) \\ &= \prod_{1 \leq i \leq b} (x_1 - \lambda_i)\left(\alpha f_{\lambda_{b+1}}(x_2, \cdots, x_m) + \prod_{b+1 \leq i \leq N+b} (x_1 - \lambda_i) h(x)\right) \end{aligned}$$

where $h \in R_q(a(q-1)-N, m)$ and $\alpha \in F_q^*$.

Then, for all $(x_2, \cdots, x_m) \in F_q^{m-1}$,

$$f_{\lambda_{b+N+1}}(x_2, \cdots, x_m) = \beta f_{\lambda_{b+1}}(x_2, \cdots, x_m) + \gamma h_{\lambda_{b+N+1}}(x_2, \cdots, x_m).$$

This is a contradiction by Lemma 7.

From now, assume $n \in \{1, 2, b-2, b-1\}$.

Applying argument as in the begining of the proof of this proposition, we can assume that $(1 - x_2^{q-1})$ does not divide f.

Since $n \geq 1$, $f_{\lambda_1} = 0$. So, $(1 - x_2^{q-1})$ divides f_{λ_1}. Since $(1 - x_2^{q-1})$ does not divide f, there exists $k \in \{1, \cdots, q-1\}$ such that for all $i \leq k$, $(1 - x_2^{q-1})$ divides f_{λ_i} and $(1 - x_2^{q-1})$ does not divide $f_{\lambda_{k+1}}$. For $i \geq n+1$, if $|f_{\lambda_i}| > (q-b+n)q^{m-a-2}$ then

$$|f_{\lambda_i}| \geq W_2 = \begin{cases} q^{m-a-1} & \text{if } n = b-1, \\ (q-1)(q-b+2)q^{m-a-3} & \text{if } n = 1, \\ (q-1)(q-b+3)q^{m-a-3} & \text{if } n = 2 \text{ and } b \neq 3, \\ (q-1)^2 q^{m-a-3} & \text{if } n = b-2. \end{cases}$$

We denote by $N = \#\{i \geq n+1 : |f_{\lambda_i}| = (q-b+n)q^{m-a-2}\}$. Since for $n \in \{1, 2, b-2, b-1\}$ if $(n, b) \neq (1, 3)$ $(q-n)W_2 \geq \tilde{d}_b q^{m-a-3}$, $N \geq 1$. Furthermore, in all cases, $(q-n)(q-b+n)q^{m-a-2} \leq \tilde{c}_b q^{m-a-3} < |f|$. So $N \leq q-n-1$.

Assume $(n, b) = (1, 3)$. We denote by $N = \#\{i \geq 2 : |f_{\lambda_i}| = (q-2)q^{m-a-2}$ *or* $|f_{\lambda_i}| = (q-1)^2 q^{m-a-3}\}$. For $i \geq 2$, if $|f_{\lambda_i}| > (q-1)^2 q^{m-a-3}$ then $|f_{\lambda_i}| \geq W_3 = (q^2-q-1)q^{m-a-3}$. Since $(q-1)(q^2-q-1)q^{m-a-3} > ((q-1)^3+1)q^{m-a-3}$, $N \geq 1$. Also, since $(q-1)(q-1)^2 q^{m-a-3} < |f|$, $N \leq q-2$.

The above arguments show that for $3 \leq b < \frac{q+4}{3}$ and $n \in \{1, 2, b-2, b-1\}$ except possibly $(n, b) = (1, 3)$ $f_{\lambda_{n+1}}$ is a minimum weight codeword of $R_q(a(q-1)+b-n, m-1)$ and if $(n, b) = (1, 3)$ $f_{\lambda_{n+1}}$ is either a minimum or second minimum weight codeword of $R_q(a(q-1)+2, m-1)$. So by applying an affine transformation we can assume $(1-x_2^{q-1})$ divides $f_{\lambda_{n+1}}$. Thus $k \geq n+1 \geq 2$.

If $N \geq 2$ and $n+1 \leq k \leq n+N-1$, then $k+1 \in N$ and so $|f_{\lambda_{k+1}}| < (q-b+k)q^{m-a-2}$. If $N = 1$ and $n+1 \leq k \leq q-1$ or $N \geq 2$ and $n+N \leq k \leq q-1$, assume that $|f_{\lambda_{k+1}}| \geq (q-b+k)q^{m-a-2}$. We get

$$|f| \geq N(q-b+n)q^{m-a-2} + (k-n-N)W_2 + (q-k)(q-b+k)q^{m-a-2}$$

which is absurd since $|f| < \tilde{d}_b q^{m-a-3}$ and $1 \leq N \leq q-n-1$.

Since $(1-x_2^{q-1})$ divides f_{λ_i} for all $i \leq k$, it divides g_{λ_i} too. Therefore, we can write for all $x = (x_1, x_2, \cdots, x_m) \in F_q^m$

$$f(x) = \prod_{1 \leq i \leq n} (x_1 - \lambda_i)\Bigg(\prod_{n+1 \leq i \leq k} (x_1 - \lambda_i) h(x_1, x_2, \cdots, x_m) + (1-x_2^{q-1}) l(x_1, x_3, \cdots, x_m)\Bigg)$$

with $deg(h) \leq a(q-1)+b-k$ and $l \in R_q((a-1)(q-1)+b-n, m-1)$. Then for all $(x_2, \cdots, x_m) \in F_q^{m-1}$,

$$f_{\lambda_{k+1}}(x_2, \cdots, x_m) = \alpha h_{\lambda_{k+1}}(x_2, \cdots, x_m) + \beta(1-x_2^{q-1}) l_{\lambda_{k+1}}(x_3, \cdots, x_m).$$

We get a contradiction by Lemma 5, since $k \geq 2$ and $|f_{\lambda_{k+1}}| < (q-b+k)q^{m-a-2}$ □

4 An upper bound on the fourth weight

Theorem 5 *Let $q \geq 3$, $m \geq 2$, $0 \leq a \leq m-1$, $1 \leq b \leq q-1$, then if W_4 is the fourth weight of $R_q(a(q-1)+b, m)$, we have*

(1) *If $b = 1$ then,*

- *for $q = 3$, $m \geq 3$ and $1 \leq a \leq m-2$, $W_4 \leq 4.3^{m-a-1}$,*
- *for $q = 4$, $m \geq 3$ and $1 \leq a \leq m-2$, $W_4 \leq 6.4^{m-a-1}$,*
- *for $q = 3$ and $a = m-1$ or $q = 4$ and $a = m-1$, $W_4 \leq 2q$,*
- *for $q \geq 5$ and $1 \leq a \leq m-1$, $W_4 \leq 2(q-1)q^{m-a-1}$,*

(2) *If $2 \leq b \leq q-1$*

- *for $q \geq 5$, $m \geq 3$, $0 \leq a \leq m-3$, and $4 \leq b \leq \lfloor \frac{q}{2}+2 \rfloor$, $W_4 \leq (q-1)^2(q-b+2)q^{m-a-3}$,*
- *for $q \geq 7$, $0 \leq a \leq m-2$, and $\lfloor \frac{q}{2}+2 \rfloor \leq b \leq q-1$, $W_4 \leq (q-2)(q-b+2)q^{m-a-2}$,*
- *for $q \geq 4$, $m \geq 3$, $0 \leq a \leq m-3$ and $b = 3$, $W_4 \leq ((q-1)^3+1)q^{m-a-3}$,*
- *for $q \geq 4$, $0 \leq a \leq m-2$ and $b = 2$, $W_4 \leq q^{m-a}$.*
- *for $q = 3$, $1 \leq a \leq m-1$ and $b = 2$, $W_4 \leq 2.3^{m-a-1}$.*

Proof (1) – For $q = 3$, $m \geq 3$ and $1 \leq a \leq m-2$, define for $x = (x_1, \cdots, x_m) \in F_q^m$,

$$f(x) = \prod_{i=1}^{a-1}(1 - x_i^2)(x_a - u)(x_{a+1} - v)$$

with $u, v \in F_q$. Then, $f \in R_3(2a, m)$ and $|f| = 4.3^{m-a-1} \geq 3^{m-a}$.

– For $q = 4$, $m \geq 3$ and $1 \leq a \leq m-2$, define for $x = (x_1, \cdots, x_m) \in F_q^m$,

$$f(x) = \prod_{i=1}^{a-1}(1 - x_i^3)(x_a - u)(x_a - v)(x_{a+1} - w)$$

with $u, v, w \in F_q$ and $u \neq v$. Then, $f \in R_4(3a, m)$ and $|f| = 6.4^{m-a-1} \geq 18.4^{m-a-2}$.

– For $q = 3$ and $a = m-1$ define for $x = (x_1, \cdots, x_m) \in F_3^m$

$$f(x) = \prod_{i=1}^{m-2}(1 - x_i^2)(x_{m-1} - u)$$

with $u \in F_q$. Then, $f \in R_q(2m-3, m) \subset R_q(2(m-1)+1, m)$ and $|f| = 6 > 4 \geq W_3$.

– For $q = 4$ and $a = m-1$ define for $x = (x_1, \cdots, x_m) \in F_4^m$

$$f(x) = \prod_{i=1}^{m-3}(1 - x_i^3)(x_{m-2} - u_1)(x_{m-2} - u_2)(x_{m-1} - u_3)$$
$$(x_{m-1} - u_4)(x_m - u_5)(x_m - u_6)$$

with $u_i \in F_q$ and $u_{2i-1} \neq u_{2i}$ for $i = 1, 2, 3$. Then, $f \in R_q(3m-3, m) \subset R_q(3(m-1)+1, m)$ and $|f| = 8 > 6 \geq W_3$.

– For $q \geq 5$ and $1 \leq a \leq m-1$ define for $x = (x_1, \cdots, x_m) \in F_q^m$

$$f(x) = \prod_{i=1}^{a-1}(1 - x_i^{q-1}) \prod_{j=1}^{q-2}(x_a - b_j)(x_{a+1} - u)$$

with $u, b_j \in F_q$ and $b_i \neq b_j$ for $i \neq j$. Then $f \in R_q(a(q-1), m) \subset R_q(a(q-1)+1, m)$ and $|f| = 2(q-1)q^{m-a-1} > 2(q-2)q^{m-a-1} \geq W_3$.

(2) For $q \geq 5$, $0 \leq a \leq m-3$ and $4 \leq b \leq \lfloor \frac{q}{2} + 2 \rfloor$, define for $x = (x_1, \cdots, x_m) \in F_q^m$,

$$f(x) = \prod_{i=1}^{a}(1 - x_i^{q-1}) \prod_{j=1}^{b-2}(x_{a+1} - b_j)(x_{a+2} - c)(x_{a+3} - d)$$

with $b_j \in F_q$, $b_j \neq b_k$ for $j \neq k$ and $c, d \in F_q$. Then, $f \in R_q(a(q-1)+b, m)$ and $|f| = (q-1)^2(q-b+2)q^{m-a-3} > (q-2)(q-b+2)q^{m-a-2}$.

– For $q \geq 7$, $0 \leq a \leq m-2$, $\lceil \frac{q}{2} + 2 \rceil \leq b \leq q-1$, define for $x = (x_1, \cdots, x_m) \in F_q^m$,

$$f(x) = \prod_{i=1}^{a}(1 - x_i^{q-1}) \prod_{j=1}^{b-2}(x_{a+1} - b_j)(x_{a+2} - c)(x_{a+2} - d)$$

with $b_j \in F_q$, $b_j \neq b_k$ for $j \neq k$, $c, d \in F_q$ and $c \neq d$. Then, $f \in R_q(a(q-1)+b, m)$ and $|f| = (q-2)(q-b+2)q^{m-a-2} > W_3$.

– For $q \geq 4$, $m \geq 3$, $0 \leq a \leq m-3$ and $b = 3$,

$$f(x) = \prod_{i=1}^{a}(1 - x_i^{q-1})(x_{a+1} - c)(x_{a+2} - d)(\alpha x_{a+1} + \beta x_{a+2} - e)$$

with $c, d, e \in F_q, \alpha, \beta \in F_q^*$ and $e \neq \alpha c + \beta d$. Then, $f \in R_q(a(q-1)+3, m)$ and $|f| = ((q-1)^3 + 1)q^{m-a-3} > (q-1)^3 q^{m-a-3}$.

– For $q \geq 4$, $0 \leq a \leq m-2$ and $b = 2$, define for $x = (x_1, \cdots, x_m) \in F_q^m$,

$$f(x) = \prod_{i=1}^{a}(1 - x_i^{q-1})$$

then, $f \in R_q(a(q-1), m) \subset R_q(a(q-1)+2, m)$ and $|f| = q^{m-a} > (q-1)q^{m-a-1} \geq W_3$.

– For $q = 3$, $1 \leq a \leq m-1$ and $b = 2$, define for $x = (x_1, \cdots, x_m) \in F_q^m$,

$$f(x) = \prod_{i=1}^{a}(1 - x_i^2)(x_{a+1} - u)$$

with $u \in F_q$. Then, $f \in R_3(2a+1, m) \subset R_3(2a+2, m)$ and $|f| = 2.3^{m-a-1} > 16.3^{m-a-3} \geq W_3$. □

5 Fourth weight in the case where $m \geq 3$

By combining the results in Sections 3 and 4, we have the following results.

Theorem 6 *Let $m \geq 3$, $q \geq 9$, $0 \leq a \leq m-3$ and $4 \leq b < \frac{q+4}{3}$. The fourth weight of $R_q(a(q-1)+b, m)$ is $W_4 = (q-1)^2(q-b+2)q^{m-a-3}$.*

Theorem 7 *Let $m \geq 3$, $q \geq 7$ and $0 \leq a \leq m-3$. The fourth weight of $R_q(a(q-1)+3, m)$ is $W_4 = ((q-1)^3+1)q^{m-a-3}$.*

Proof By Proposition 1 we have

$$W_4 \geq \begin{cases} (q-1)^2(q-b+2)q^{m-a-3} & \text{if } 4 \leq b < \frac{q+4}{3}, \\ ((q-1)^3+1)q^{m-a-3} & \text{if } b = 3. \end{cases}$$

By Theorem 5 Part (2), there exists $g \in R_q(b, 3)$ such that $|g| = \tilde{d}_b$. For $x = (x_1, \cdots, x_m) \in F_q^m$, we define

$$f(x) = \prod_{i=1}^{a}(1 - x_i^{q-1})g(x_{a+1}, x_{a+2}, x_{a+3}).$$

Then $f \in R_q(a(q-1)+b, m)$ and $|f| = |g|q^{m-a-3}$ which proves both of theorems. □

6 Fourth weight in the case where $m = 2$

In this section, we determine the fourth weight and the fourth weight codewords of $R_q(b, 2)$.

Proposition 2 *For $q \geq 11$, the fourth weight of $R_q(4,2)$ is $W_4 = (q-2)^2 + 1$. Furthermore, if $f \in R_q(4,2)$ is such that $|f| = (q-2)^2+1$ then up to affine transformation for all $(x, y) \in F_q^2$ either*

$$f(x, y) = \prod_{j=1}^{2}(a_1x + b_1y + c_j)(a_2x + b_2y)(a_3x + b_3y)$$

with $(a_i, b_i) \in F_q^2 \setminus \{(0,0)\}$ are such that $a_ib_j - a_jb_i \neq 0$ for $i \neq j$ and $c_j \in F_q^*$ for $j = 1, 2$

or

$$f(x, y) = \prod_{j=1}^{3}(a_jx + b_jy)(a_4x + b_4y + c)$$

with $(a_i, b_i) \in F_q^2 \setminus \{(0,0)\}$ are such that $a_ib_j - a_jb_i \neq 0$ for $i \neq j$ and $c \in F_q^*$.

Proof The third weight in this case is $(q-2)^2$. So if there exists $f \in R_q(4,2)$ such that $|f| = (q-2)^2+1$, $W_4 = (q-2)^2+1$ and f is a fourth weight codeword of $R_q(4,2)$. Since $(q-2)^2+1 < (q-3)q$ and $(q-3)q$ is the minimum weight of $R_q(3,2)$, $deg(f) = 4$. We prove first that f is the product of 4 affine factors. Let p be a point of F_q^2 which is not in S and l be a line in F_q^2 such that $p \in l$. Then either l does not meet S or l meets S in at least $q-4$ points. If any line through p meets S then,

$$(q+1)(q-4) \leq |f| = (q-2)^2 + 1$$

which is absurd for $q \geq 11$. So there exists a line through p which does not meet S. By applying the same argument to all points not in S, we get that f is the product of affine factors.

Denote by Z the set of zeros of f. We have just proved that Z is the union of 4 lines in F_q^2. If the 4 lines are parallel then, f is the minimum weight codeword of $R_q(4,2)$ which is absurd. If 3 of these lines are parallel or the 4 lines intersect in one common point, f is a second weight codeword of $R_q(4,2)$ which is absurd. Assume 2 of these lines are parallel. If the 2 other lines are parallel or intersect in a point which is included in one of the parallel lines then, f is a third weight codeword of $R_q(4,2)$ which is absurd. If the 2 other lines intersect in a point which is not included in any of the parallel lines then we are in the first case of the proposition. Finally, assume all of 4 lines intersect pairwise. They can not intersect in one point. If 3 of 4 lines intersect in a point then, we are in the second case of the proposition. Otherwise $|Z| = 4q-6 < 4q-5$ which is absurd. □

Proposition 3 *For $q \geq 13$, the fourth weight of $R_q(5,2)$ is $W_4 = (q-2)(q-3)+1$. Furthermore, if $f \in R_q(5,2)$ is such that $|f| = (q-2)(q-3)+1$ then up to affine transformation for all $(x, y) \in F_q^2$ either*

$$f(x, y) = \prod_{j=1}^{3}(a_1x + b_1y + c_j)(a_2x + b_2y)(a_3x + b_3y)$$

with $(a_i, b_i) \in F_q^2 \setminus \{(0,0)\}$ are such that $a_ib_j - a_jb_i \neq 0$ for $i \neq j$ and $c_j \in F_q^*$ for $j = 1, 2, 3$

or

$$f(x,y)=\prod_{i=1}^{2}(a_1x+b_1y+c_i)\prod_{j=1}^{2}(a_2x+b_2y+d_j)(\alpha(a_1x+b_1y+c_i)$$
$$+\beta(a_2x+b_2y+d_j)),\quad i,j\in\{1,2\}$$

with $(a_i,b_i)\in F_q^2\backslash\{(0,0)\}$, $a_1b_2-a_2b_1\neq 0$, $c_i,d_i\in F_q$ *are such that* $c_1\neq c_2$ *and* $d_1\neq d_2$ *and* $\alpha,\beta\in F_q^*$

or

$$f(x,y)=\prod_{\substack{i=1\\ j=2,3}}^{3}(a_ix+b_iy)(a_1x+b_1y+c)(\alpha(a_jx+b_jy)+\beta(a_1x+b_1y+c)),$$

with $(a_i,b_i)\in F_q^2\backslash\{(0,0)\}$, $a_ib_j-a_jb_i\neq 0$ *for* $i\neq j$ *and* $c,\alpha,\beta\in F_q^*$

or

$$f(x,y)=\prod_{j=1}^{4}(a_jx+b_jy)(a_5x+b_5y+c)$$

with $(a_j,b_j)\in F_q^2\setminus\{(0,0)\}$ *are such that* $a_ib_j-a_jb_i\neq 0$ *for* $i\neq j$ *and* $c\in F_q^*$.

Proof The third weight in this case is $(q-2)(q-3)$. So if there exists $f\in R_q(5,2)$ such that $|f|=(q-2)(q-3)+1$, $W_4=(q-2)(q-3)+1$ and f is a fourth weight codeword of $R_q(5,2)$. Let $f\in R_q(5,2)$ such that $|f|=(q-2)(q-3)+1$. Since $(q-2)(q-3)+1<(q-4)q$ and $(q-4)q$ is the minimum weight of $R_q(4,2)$, $deg(f)=5$. We prove first that f is the product of 5 affine factors. Let p be a point of F_q^2 which is not in S and l be a line in F_q^2 such that $p\in l$. Then either l does not meet S or l meets S in at least $q-5$ points. If any line through p meets S then

$$(q+1)(q-5)\leq|f|=(q-3)(q-2)+1$$

which is absurd for $q\geq 13$. So there exists a line through p which does not meet S. By applying the same argument to all points not in S, we get that f is the product of affine factors.

Denote by Z the set of zeros of f. We have just proved that Z is the union of 5 lines in F_q^2. If the 5 lines are parallel then f is a minimum weight codeword of $R_q(5,2)$ which is absurd. If 4 of these lines are parallel or the 5 lines intersect in one common point, then f is a second minimum weight codeword of $R_q(5,2)$ which is absurd. Assume that 3 of these lines are parallel. Consider all possibilities:

1. If the 2 other lines are parallel or intersect in one point which is included in one of the parallel lines then, f is a third minimum weight codeword of $R_q(5,2)$ which is absurd.
2. If the 2 other lines intersect in one point which is not included in any of the parallel lines then we are in the first case of the proposition.

Assume 2 of these lines are parallel. Consider all of cases.

1. If an other pair of lines are parellel and the fifth line meets the four other lines in two points then, f is a third minimum weight codeword of $R_q(5,2)$ which is absurd. If the fifth line meets the four other lines in three points then, we are in the second case of the proposition. Otherwise $\#Z=5q-8<5q-7$ which is absurd.

2. If the 3 other lines intersect in one common point which is included in one of the parallel lines then, f is a third minimum weight codeword of $R_q(5,2)$ which is absurd. If 2 of the three other lines intersect in a point which is included in one of 2 parallel lines and the fifth line meets the four other lines in 3 points then, we are in the third case of the proposition. Otherwise $\#Z \leq 5q-8 < 5q-7$ which is absurd.

Assume all lines intersect pairwise. They can not intersect in one point. If 4 of 5 lines intersect in one common point and the fifth line meets the other lines in different points of that point then, we are in the last case of the proposition. Otherwise $\#Z \leq 5q-8 < 5q-7$ which is absurd. □

Proposition 4 *For $q \geq 16$, the fourth weight of $R_q(6,2)$ is $W_4 = (q-2)(q-4)+1$. Furthermore, if $f \in R_q(6,2)$ is such that $|f| = (q-2)(q-4)+1$ then up to affine transformation for all $(x,y) \in F_q^2$ either*

$$f(x,y) = \prod_{j=1}^{4}(a_1x+b_1y+c_j)(a_2x+b_2y)(a_3x+b_3y)$$

with $(a_i,b_i) \in F_q^2 \setminus \{(0,0)\}$ are such that $a_ib_j - a_jb_i \neq 0$ for $i \neq j$ and $c_j \in F_q^*$ for $j = 1,2,3,4$

or

$$f(x,y) = \prod_{i=1}^{3}(a_1x+b_1y+c_i)\prod_{j=1}^{3}(a_2x+b_2y+d_j)$$

with $(a_i,b_i) \in F_q^2 \setminus \{(0,0)\}$ are such that $a_1b_2 - a_2b_1 \neq 0$, $c_i, d_i \in F_q$ and $c_i \neq c_j$, $d_i \neq d_j$ for $i \neq j$

or

$$f(x,y) = \prod_{i=1}^{3}(x-a_i)\prod_{j=1}^{2}(y-b_j)\Big((a_i-a_k)y+(b_1-b_2)x+a_kb_2-a_ib_1\Big) \qquad i \neq k$$

with $a_i, b_j \in F_q$, $b_1 \neq b_2$ and $a_i \neq a_j$ for $i \neq j$

or

$$f(x,y) = (a_1x+b_1y)\prod_{i=1}^{2}(a_1x+b_1y+c_i)\prod_{j=2}^{4}(a_jx+b_jy)$$

with $(a_i,b_i) \in F_q^2 \setminus \{(0,0)\}$ are such that $a_ib_j - a_jb_i \neq 0$, $c_1, c_2 \in F_q^*$ and $c_1 \neq c_2$

or

$$f(x,y) = \prod_{i=1}^{3}(x-a_i)(y-b_1)(\alpha x+\beta y-\alpha a_1-\beta b_1)\Big(\alpha(a_1-a_2)(x-a_3) + \beta(a_3-a_2)(y-b_1)\Big)$$

with $b_1, a_i \in F_q$, $a_i \neq a_j$ for $i \neq j$ and $\alpha, \beta \in F_q^*$

or

$$f(x,y) = \prod_{i=1}^{2}(x-a_i)\prod_{j=1}^{2}(y-b_j)\Big((b_2-b_1)x+(a_2-a_1)y+a_1b_1-a_2b_2\Big)$$

$$\Big((b_2 - b_1)x + (a_2 - a_1)y + a_1b_2 + a_2b_1 - 2a_2b_2\Big)$$

with $a_i, b_i \in F_q$ *are such that* $a_1 \neq a_2$ *and* $b_1 \neq b_2$ *and* $a_1b_1 + a_2b_2 \neq a_1b_2 + a_2b_1$
or

$$f(x, y) = \prod_{i=1}^{2}(x - a_i)\prod_{j=1}^{2}(y - b_j)\Big((b_2 - b_1)x + (a_2 - a_1)y + a_1b_1 - a_2b_2\Big)$$
$$\Big((b_1 - b_2)x + (a_2 - a_1)y + b_2a_1 - b_1a_2\Big)$$

with $a_i, b_i \in F_q$ *are such that* $a_1 \neq a_2$ *and* $b_1 \neq b_2$
or

$$f(x, y) = \prod_{j=1}^{5}(a_jx + b_jy)(a_6x + b_6y + c)$$

with $(a_j, b_j) \in F_q^2 \setminus \{(0, 0)\}$ *are such that* $a_ib_j - a_jb_i \neq 0$ *for* $i \neq j$ *and* $c \in F_q^*$.

Proof The third weight in this case is $(q - 2)(q - 4)$. So if there exists $f \in R_q(6, 2)$ such that $|f| = (q-2)(q-4)+1$, $W_4 = (q-2)(q-4)+1$ and f is a fourth weight codeword of $R_q(6, 2)$. Let $f \in R_q(6, 2)$ such that $|f| = (q - 2)(q - 4) + 1$. Since $(q - 2)(q - 4) + 1 < q(q - 5)$ and $q(q - 5)$ is the minimum weight of $R_q(5, 2)$, $deg(f) = 6$. We prove first that f is the product of 6 affine factors. Let p be a point of F_q^2 which is not in S and l be a line in F_q^2 such that $p \in l$. Then either l does not meet S or l meets S in at least $q - 6$ points. If any line through p meets S then

$$(q + 1)(q - 6) \leq |f| = (q - 2)(q - 4) + 1$$

which is absurd for $q \geq 16$. So there exists a line through p which does not meet S. By applying the same argument to all points not in S, we get that f is the product of affine factors.

Denote by Z the set of zeros of f. We have just proved that Z is the union of 6 lines in F_q^2. If the 6 lines are parallel then, f is the minimum weight codeword of $R_q(6, 2)$ which is absurd. If 5 of these lines are parallel or the 6 lines intersect in one common point then, f is a second weight codeword of $R_q(6, 2)$ which is absurd. If 4 of these lines are parallel, the following cases will happen:

1. If the two other lines are parallel or intersect in one point which is included in one of the parallel lines then, f is a third weight codeword of $R_q(6, 2)$ which is absurd.
2. If the two other lines intersect in one point which is not included in any of the parallel lines then, we are in the first case of the proposition (see Fig. 1a).

If 3 of these lines are parallel, the following cases will happen:

1. If the three other lines are parallel then, we are in the second case of the proposition (see Fig. 1b).
2. Assume that two of three other lines are parallel. If the last line meets the five other lines in 3 points then, we are in the third case of the proposition (see Fig. 1c). Otherwise $\#Z \leq 6q - 10 < 6q - 9$ which is absurd.
3. If the three other lines intersect in one common point which is included in one of the parallel lines then, we are in the fourth case of the proposition (see Fig. 1d).

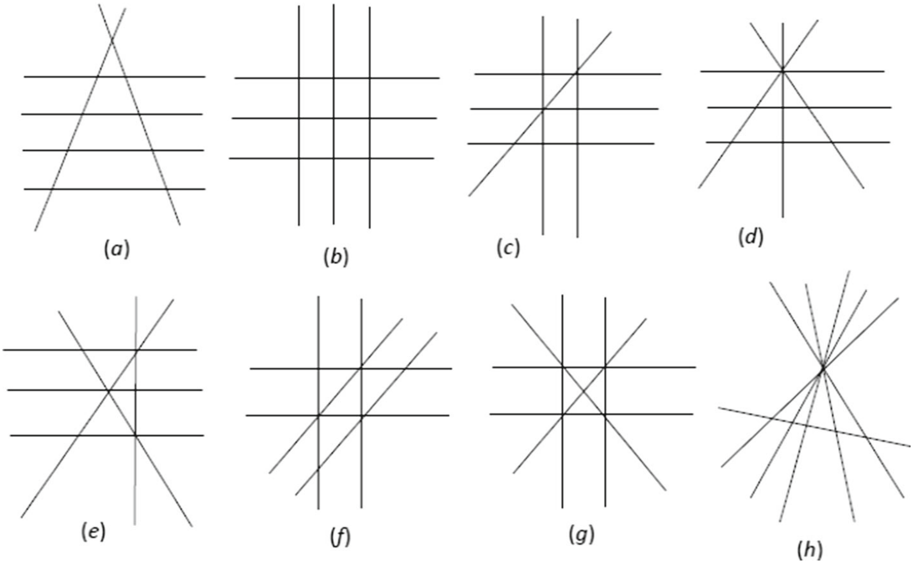

Fig. 1 Possible sets of zeros of fourth weight codewords in $R_q(6, 2)$ for $q \geq 16$

4. If the three other lines intersect pairwise in three different points which are included in the parallel lines then, we are in the fifth case of the proposition (see Fig. 1e). Otherwise $\#Z \leq 6q - 10 < 6q - 9$ which is absurd.

If 2 of these lines are parallel, the following cases will happen:

1. Assume the 6 lines can be partitioned to three pair of parallel lines. If the lines of each pair meet the four other lines in 5 points, we are in the 6th case of the proposition (see Fig. 1f). Otherwise $\#Z \leq 6q - 11 < 6q - 9$ which is absurd.
2. Assume the 6 lines can be partitioned so that there are two pair of parallel lines. So if the two other lines meet the four other lines in 4 points, we are in the 7th case of the proposition (see Fig. 1g). Otherwise $\#Z \leq 6q - 10$ which is absurd.
3. Assume there is one pair of parallel lines. If the four other lines intersect in one point included in one of the parallel lines then f is a third weight codeword of $R_q(6, 2)$ which is absurd. Otherwise $\#Z \leq 6q - 10$ that is a contradiction.

If the 6 lines intersect pairwise, they can not intersect in one point. Then if 5 lines intersect in one point and the 6th line meets the other lines in different points of that point then we are in the last case of the proposition (see Fig. 1h). Otherwise $\#Z \leq 6q - 11$ which is absurd. □

Theorem 8 *For $q \geq 19$ and $7 \leq b < \frac{q}{3} + 1$, the fourth weight of $R_q(b, 2)$ is $W_4 = (q-2)(q-b+2)+1$.*
Furthermore, if $f \in R_q(b, 2)$ is such that $|f| = (q-2)(q-b+2)+1$ then up to affine transformation for all $(x, y) \in F_q^2$ either

$$f(x, y) = \prod_{j=1}^{b-2}(a_1x + b_1y + c_j)(a_2x + b_2y)(a_3x + b_3y)$$

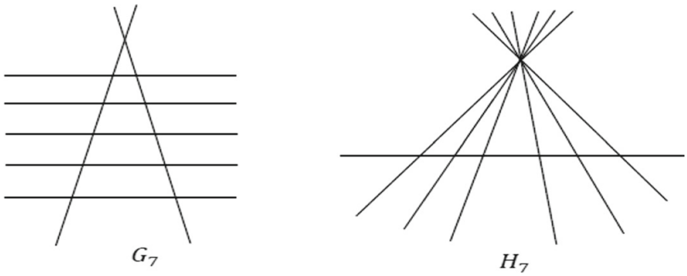

Fig. 2 Possible sets of zeros of fourth weight codewords in $R_q(7, 2)$ for $q \geq 19$

with $(a_i, b_i) \in F_q^2 \setminus \{(0,0)\}$ *are such that* $a_i b_j - a_j b_i \neq 0$ *for* $i \neq j$ *and* $c_j \in F_q^*$ *for* $j = 1, \cdots, b-2$ *or*

$$f(x, y) = \prod_{j=1}^{b-1}(a_j x + b_j y)(a_b x + b_b y + c)$$

with $(a_i, b_i) \in F_q^2 \setminus \{(0,0)\}$ *are such that* $a_i b_j - a_j b_i \neq 0$ *for* $i \neq j$ *and* $c \in F_q^*$.

Proof The third weight in this case is $(q-2)(q-b+2)$. So if there exists $f \in R_q(b, 2)$ such that $|f| = (q-2)(q-b+2)+1$, $W_4 = (q-2)(q-b+2)+1$ and f is a fourth weight codeword of $R_q(b, 2)$. Let $f \in R_q(b, 2)$ such that $|f| = (q-2)(q-b+2)+1$. Denote by S its support.
Since $(q-2)(q-b+2)+1 < q(q-b+1)$ and $q(q-b+1)$ is the minimum weight of $R_q(b-1, 2)$, $deg(f) = b$. We prove first that f is the product of b affine factors. Let p be a point of F_q^2 which is not in S and l be a line in F_q^2 such that $p \in l$. Then either l does not meet S or l meets S in at least $q-b$ points. If any line through p meets S then

$$(q+1)(q-b) \leq |f| = (q-2)(q-b+2)+1$$

which is absurd for $b < \frac{q}{3} + 1$. So there exists a line through p which does not meet S. By applying the same argument to all points not in S, we get that f is the product of affine factors.

Denote by Z the set of zeros of f. We have just proved that Z is the union of b lines in F_q^2. For each line l which does not meet S, we denote by $n_l = \#(\{l' | \, l \cap l' = \emptyset, l' \cap S = \emptyset\} \cup \{l\})$ the number of lines parallel to l and not in S plus 1. By Lemma 2 since $n_l \leq b$,

$$(q-b)q + n_l(b-n_l) \leq (q-2)(q-b+2)+1$$

we get that $n_l \in \{1, 2, b-2, b-1, b\}$.

We say that those lines are in configuration A_b if the b lines are parallel, in configuration B_b if exactly $b-1$ lines are parallel, in configuration C_b if the b lines meet in a point, in configuration D_b if $b-2$ lines are parallel and the 2 other lines are also parallel, in configuration E_b if $b-2$ lines are parallel and the 2 other lines intersect in one point included in one of the parallel lines, in configuration F_b if $b-1$ lines intersect in one point and the bth line is parallel to one of the previous, in configuration G_b if $b-2$ lines are parallel and the 2 other lines intersect in one point which is not included in any of the parallel lines, in configuration H_b if $b-1$ lines intersect in one point and the bth line meets the other lines in different points of that point (see Fig. 2) and in configuration I_b if we are in none of the previous configurations.

We prove by induction on b that Z the set of zeros of f is of type G_b or H_b. Since the number of points in such set is $bq - 2b + 3$, we get the result.

Assume $b = 7$. We have just proved that Z is the union of 7 lines in F_q^2. According to what we said $n_l \in \{1, 2, 5, 6, 7\}$. If the 7 lines are parallel (configuration A_7) then f is the minimum weight codeword of $R_q(7, 2)$ which is absurd. If 6 of these lines are parallel (configuration B_7) or the 7 lines intersect in a point (configuration C_7) then, f is a second weight codeword of $R_q(7, 2)$ which is absurd. If 5 of these lines are parallel, then if the 2 other lines are parallel (configuration D_7) or intersect in one point included in one of the parallel lines (cofiguration E_7) then, f is the third weight codeword of $R_q(7, 2)$ which is absurd. So the only possibility in this case is configuration G_7. If 2 of these lines are parallel then, if the 5 other lines intersect in one point included in one of parallel lines (configuration F_7) then, f is the third weight codeword of $R_q(7, 2)$ which is absurd. Otherwise $\#Z \leq 7q - 12 < 7q - 11$ that gives a contradiction. If all lines intersect pairwise then they cannot intersect in one point. Then if $b - 1$ lines intersect in one point and the bth line meets the other lines in different point of that point then, we are in configuration H_7. Otherwise $\#Z \leq 7q - 14 < 7q - 11$ which is absurd. This proves the result for $b = 7$.Let $7 \leq b < \frac{q}{3} + 1$. Assume if $f \in R_q(b, 2)$ and $|f| = (q - 2)(q - b + 2) + 1$ then its set of zeros is of type G_b or H_b.
Let $f \in R_q(b + 1, 2)$ such that $|f| = (q - 2)(q - b + 1) + 1$. Denote by Z the set of zeros of f. Then as in the beginning of the proof, we get that Z is the union of $b + 1$ lines in F_q^2 and as we said before for any line l in Z $n_l \in \{b + 1, b, b - 1, 2, 1\}$. Let us consider now a type I_{b+1} configuration of $b + 1$ lines. By considering the structure of the introduced configuration, we get that $n_l = 1$ or 2. So if we extract from this configuration a subset with b lines we obtain one of the following situations:

1. Z is the union of a type F_b configuration and a line l. Since Z is a configuration I_{b+1}, l cannot intersect the configuration F_b in the point where $b - 1$ lines of the configuration intersect. So, l intersects the configuration F_b in at least $b - 2$ points. We get that $\#Z \leq bq - 2b + 4 + q - b + 2 = (b + 1)q - 3b + 6 < (b + 1)q - 2b + 1$.
2. Z is the union of a type H_b configuration and a line l. Since Z is a configuration I_{b+1}, l cannot intersect the configuration H_b in the point where $b - 1$ lines of the configuration intersect. So, l intersect the configuration H_b in at least $b - 2$ points. We get that $\#Z \leq bq - 2b + 4 + q - b + 2 = (b + 1)q - 3b + 6 < (b + 1)q - 2b + 1$.
3. Z is the union of a type I_b configuration and a line l. Since for any line l $n_l = 1$ or 2 so, l meets the configuration I_b in at least 2 points. Then, by induction hypothesis, $\#Z < bq - 2b + 3 + q - 2 = bq + q - 2b + 1$.

□

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