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On the equivalence, stabilisers, and feet of Buekenhout-Tits unitals

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Abstract
This paper addresses a number of problems concerning Buekenhout-Tits unitals in $\mathrm{PG}(2, q^2)$, where $q = 2^{2e+1}$ and $e \geq 1$. We show that all Buekenhout-Tits unitals are equivalent under $\mathrm{PGL}(3, q^2)$ [addressing an open problem in Barwick and Ebert (Unitals in Projective Planes. Springer Monographs in Mathematics. Springer, New York, 2008)], explicitly describe their stabiliser in $\mathrm{P\Gamma L}(3, q^2)$ [expanding Ebert's work in Ebert (J Algebraic Comb 6(2):133–140, 1997)], and show that lines meet the feet of points not on ℓ_∞ in at most four points. Finally, we show that feet of points not on ℓ_∞ are not always a $\{0, 1, 2, 4\}$-set, in contrast to what happens for Buekenhout-Metz unitals Abarzúa et al (Adv Geom 18(2):229–236, 2018).

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1 Introduction

1.1 Background

Let $\mathrm{PG}(2, q^2)$ denote the Desarguesian projective plane over the finite field with q^2 elements, $\mathbb{F}_{q^2}$, where q is a prime power. A *unital* U in $\mathrm{PG}(2, q^2)$ is a set of q^3+1 points such that every line of $\mathrm{PG}(2, q^2)$ meets U in 1 or $q+1$ points. Lines meeting U in 1 point are *tangent lines*

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to U, and lines meeting U in $q+1$ points are *secant lines* of U. The *classical* or *Hermitian* unital, usually denoted by $\mathcal{H}(2, q^2)$, arises by taking the absolute points of a non-degenerate Hermitian polarity. Each point P not lying on a unital U, lies on $q+1$ tangent lines to U; the $q+1$ points of U whose tangent lines contain P are called the *feet* of P, and are denoted by $\tau_P(U)$.

It is well-known that $\mathrm{PG}(2, q^2)$ can be modelled by a Desarguesian line spread of $\mathrm{PG}(3, q)$ embedded in $\mathrm{PG}(4, q)$ via the *André/Bruck-Bose (ABB)* construction. A wide class of unitals in $\mathrm{PG}(2, q^2)$, called *Buekenhout unitals*, arise as follows from the ABB construction; starting in $\mathrm{PG}(4, q)$ fixing a hyperplane Σ, and a Desarguesian spread of Σ, we take any ovoidal cone $\mathcal{C}$ such that $\mathcal{C} \cap \Sigma$ is a spread line of Σ. Then in $\mathrm{PG}(2, q^2)$, $\mathcal{C}$ gives rise to a unital U. If the base of $\mathcal{C}$ is an elliptic quadric, the unital is called a *Buekenhout-Metz unital*. The family of Buekenhout-Metz unitals contains the Hermitian unitals, but there are many non-equivalent Buekenhout-Metz unitals (see [3, 8]). If $q = 2^{2e+1}$, $e \geq 1$, and the base of $\mathcal{C}$ is a Tits ovoid, the unital is a called a *Buekenhout-Tits unital*. For more information on unitals and their constructions, see [4].

Unitals may be characterised based on the combinatorial properties of the feet of certain points. It is easy to see that for the classical unital $\mathcal{H}(2, q^2)$, the feet of a point not on the unital are always collinear. Thas [13] showed the converse, namely, that a unital U is classical if and only if for all points, not on U, the feet are collinear. This was improved by Aguglia and Ebert [2] who showed that a unital U is classical if and only if there exist two tangent lines ℓ_1, ℓ_2 such that for all points $P \in (\ell_1 \cup \ell_2) \backslash U$ the feet of P are collinear. It is known (see e.g. [4]) that if U is a non-classical Buekenhout-Metz unital, the feet of a point $P \notin U$ are collinear if and only if they lie on a distinguished tangent line ℓ_∞ to U. Furthermore, it is shown in [1] that if U is Buekenhout-Metz unital, a line meets the feet of a point $P \notin \ell_\infty$ in either 0, 1, 2, or 4 points. Ebert [9] showed for a Buekenhout-Tits unital, the feet of $P \notin U$ are collinear if and only if $P \in \ell_\infty$. It is then natural to ask how a line may meet the feet of a point $P \notin \ell_\infty$ for Buekenhout-Tits unitals. We will answer this question in Theorem 3.

Many characterisations of unitals make use of their stabilisers in PGL, resp. PΓL. In [7] it is shown that a unital is classical if its stabiliser contains a cyclic group of order $q^2 - q + 1$. Several other characterisations of unitals by their stabiliser group are listed in [4]. In [9], Ebert determined the stabiliser of a Buekenhout-Tits unital in $\mathrm{PGL}(3, q^2)$ (see Result 1). We will extend this work in this paper.

1.2 Summary of this paper

In this paper we present three main results:

1. We show that all Buekenhout-Tits unitals are equivalent under $\mathrm{P\Gamma L}(3, q^2)$ (see Theorem 1). This addresses an open problem in [4], and is alluded to in [10] (see Remark 1).
2. A description of the full stabiliser group of a Buekenhout-Tits unital in $\mathrm{P\Gamma L}(3, q^2)$ (see Theorem 2). Ebert [9] only provides a description of stabiliser of the Buekenhout-Tits unital in PGL (Result 1). The stabiliser of the classical unital in $\mathrm{P\Gamma L}(3, q^2)$ is $\mathrm{P\Gamma U}(3, q^2)$, and the stabiliser of the Buekenhout-Metz unital in $\mathrm{P\Gamma L}(3, q^2)$ is described in [8] for q even and [3] for q odd.
3. If U is a Buekenhout-Tits unital, then a line ℓ meets the feet of a point $P \notin (\ell_\infty \cup U)$ in at most 4 points. Moreover, there exists a point P and line ℓ such that the feet of P meet ℓ in exactly three points (see Theorem 3). This highlights a difference between Buekenhout-Metz unitals and Buekenhout-Tits unitals. It also solves an open problem posed by Aguglia and Ebert [2] and later listed in [4, Chapter 8].

1.3 Coordinates for a Buekenhout-Tits unital

In [9], Ebert derives coordinates for a Buekenhout-Tits unital $\mathcal{U}_{BT}$ in $\mathrm{PG}(2, q^2)$, $q = 2^{2e+1}$. Pick $\epsilon \in \mathbb{F}_{q^2}$ such that $\epsilon^q = \epsilon + 1$, and $\epsilon^2 = \epsilon + \delta$ for some $1 \neq \delta \in \mathbb{F}_q$ with absolute trace equal to one. Then the following set of points in $\mathrm{PG}(2, q^2)$ is a Buekenhout-Tits unital,

$$\mathcal{U}_{BT} = \{(0, 0, 1)\} \cup \{P_{r,s,t} = (1, s + t\epsilon, r + (s^{\sigma+2} + t^{\sigma} + st)\epsilon) \mid r, s, t \in \mathbb{F}_q\}, \qquad (1)$$

where $\sigma = 2^{e+1}$ has the property that σ^2 induces the automorphism $x \mapsto x^2$ of $\mathbb{F}_q$. In addition, it can be verified that $\sigma + 1, \sigma + 2, \sigma - 1$, and $\sigma - 2$ all induce permutations of $\mathbb{F}_q$ with inverses induced by $\sigma - 1, 1 - \sigma/2, \sigma + 1$ and $-(\sigma/2 + 1)$ respectively.

The following theorem describes the group of projectivities (that is, elements of $\mathrm{PGL}(3, q^2)$) stabilising $\mathcal{U}_{BT}$.

Result 1 [9, Theorem 4 and Corollary] Let $G = \mathrm{PGL}(3, q^2)_{\mathcal{U}_{BT}}$, $q = 2^{2e+1}$, be the group of projectivities stabilising the Buekenhout-Tits unital $\mathcal{U}_{BT}$. Then G is an abelian group of order q^2, consisting of the projectivities induced by the matrices

$$M_{u,v} = \left\{ \begin{bmatrix} 1 & u\epsilon & v + u^{\sigma}\epsilon \\ 0 & 1 & u + u\epsilon \\ 0 & 0 & 1 \end{bmatrix} \middle| \, u, v \in \mathbb{F}_q \right\}, \qquad (2)$$

where $\sigma = 2^{e+1}$ and matrices act on the homogeneous coordinates of points by multiplication from the right. The group G has $q^2 - q$ orbits of length q^2 on points in $\mathrm{PG}(2, q^2) \setminus (\mathcal{U}_{BT} \cup \ell_\infty)$, where $\ell_\infty : x = 0$.

2 On the projective equivalence of Buekenhout-Tits unitals

In this section, we show that all Buekenhout-Tits unitals are equivalent under $\mathrm{PGL}(3, q^2)$ to the unital $\mathcal{U}_{BT}$ given in Eq. (1).

Remark 1 The authors of [10] give this result without proof and state it can be derived by the same techniques employed by Ebert in [9]. Ebert however, lists the equivalence of Buekenhout-Tits unitals as an open problem in [4] which appeared about ten years after his original paper [9].

It is easy to see that the Buekenhout-Tits unital $\mathcal{U}_{BT}$ is tangent to the line $\ell_\infty : x = 0$ at the point $P_\infty = (0, 0, 1)$. From the ABB construction it follows that P_∞ has the following property with respect to $\mathcal{U}_{BT}$.

Property 1 Given any unital U, a point $P \in U$ is said to have Property 1 if all secant lines through P meet U in Baer sublines.

It is shown in [5] that if two different points of U have Property 1, then U is classical. Hence, the point P_∞ is the unique point of $\mathcal{U}_{BT}$ admitting this property. We will count all Buekenhout-Tits unitals tangent to ℓ_∞ at a point P_∞ having Property 1.

Lemma 1 *There are $q^4(q^2 - 1)^2$ unitals equivalent under* $\mathrm{PGL}(3, q^2)$ *to* $\mathcal{U}_{BT}$ *in* $\mathrm{PG}(2, q^2)$ *with tangent line $\ell_\infty : x = 0$ and containing the point $P_\infty = (0, 0, 1)$ having Property 1.*

Proof Let U be a unital tangent to ℓ_∞, and containing the point P_∞ with Property 1, that is equivalent under $\mathrm{PGL}(3, q^2)$ $\mathcal{U}_{BT}$ to $\mathrm{PG}(2, q^2)$. Then, the point P_∞ is the unique point in

U with Property 1. Thus, any projectivity mapping $\mathcal{U}_{BT}$ to U is contained in the group H of projectivities fixing P_∞, and fixing ℓ_∞ line-wise. The elements of H are induced by all matrices of the following form,

$$\begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix},$$

where $x_{22}x_{33} \neq 0$ and matrices act on homogeneous coordinates by multiplication on the right. It follows that $|H| = (q^2-1)^2 q^6$. Furthermore, from the description of $G = \mathrm{PGL}(3, q^2)_{\mathcal{U}_{BT}}$ in Result 1, we know that the stabiliser $H_{\mathcal{U}_{BT}}$ in H of $\mathcal{U}_{BT}$ coincides with G. Hence, the stabiliser $H_{\mathcal{U}_{BT}}$ has order q^2. By the orbit-stabiliser theorem, we find that there are $(q^2-1)^2 q^4$ unitals in the orbit of $\mathcal{U}_{BT}$ under H. □

Consider $\mathrm{PG}(2, q^2)$ modelled by the ABB construction with fixed hyperplane Σ_∞. Let p_∞ be the spread line corresponding to P_∞. Then any Buekenhout-Tits unital U tangent to ℓ_∞ at P_∞ with Property 1 corresponds uniquely to an ovoidal cone $\mathcal{C}$ meeting Σ_∞ at p_∞.

Lemma 2 *There are $q^4(q^2-1)^2$ ovoidal cones $\mathcal{C}$ in $\mathrm{PG}(4, q)$ with base a Tits ovoid, such that $\mathcal{C}$ meets Σ_∞ in the spread element p_∞.*

Proof Let V be a point on the line p_∞, and $\Sigma \neq \Sigma_\infty$ a hyperplane not containing V. Then, Σ meets Σ_∞ in a plane containing a point $R \in p_\infty \backslash \{V\}$. Any ovoidal cone $\mathcal{C}$ with vertex V and base a Tits ovoid, such that $\mathcal{C}$ meets Σ_∞ precisely in p_∞, meets Σ in a Tits ovoid tangent to $\Sigma \cap \Sigma_\infty$ at the point R. We will count all cones of this form, for all $V \in p_\infty$.

Consider the pairs of planes Π and Tits ovoids $\mathcal{O}$, $(\Pi, \mathcal{O})$, where $\Pi, \mathcal{O} \subset \Sigma$ and Π is tangent to $\mathcal{O}$. On the one hand, there are $|\mathrm{PGL}(4, q)|/|\mathcal{O}_{\mathrm{PGL}(4,q)}| = (q+1)^2 q^4 (q-1)^2 (q^2+q+1)$ Tits ovoids in $\mathrm{PG}(3, q)$, and each has q^2+1 tangent planes. On the other hand, $\mathrm{PGL}(4, q)$ is transitive on hyperplanes of $\mathrm{PG}(3, q)$, so each plane is tangent to the same number of Tits ovoids. It thus follows, that there are

$$\frac{(q+1)^2 q^4 (q-1)^2 (q^2+q+1)(q^2+1)}{q^3+q^2+q+1} = (q-1)^2 q^4 (q+1)(q^2+q+1)$$

Tits ovoids tangent to $\Sigma \cap \Sigma_\infty$ contained in Σ.

Furthermore, since $\mathrm{PGL}(4, q)_{\Sigma \cap \Sigma_\infty}$ is transitive on points of $\Sigma \cap \Sigma_\infty$, each point of $\Sigma \cap \Sigma_\infty$ is contained in the same number of Tits ovoids $\mathcal{O}$, so it follows that the number of Tits ovoids tangent to $\Sigma \cap \Sigma_\infty$ at $R = p_\infty \cap \Sigma$ is $(q-1)^2 q^4 (q+1)$. Hence, there is an equal number of ovoidal cones with base a Tits ovoid, vertex V, and meeting Σ_∞ at p_∞. As the choice of V was arbitrary, and there are $q+1$ points on p_∞, there are $(q^2-1)^2 q^4$ ovoidal cones with base a Tits ovoid, and meeting Σ_∞ at p_∞. □

Theorem 1 *All Buekenhout-Tits unitals in $\mathrm{PG}(2, q^2)$ are equivalent under $\mathrm{PGL}(3, q^2)$.*

Proof From Lemmas 1 and 2, we see that the number of ovoidal cones with base a Tits ovoid, tangent to Σ_∞ at p_∞ is equal to the number of Buekenhout-Tits unitals that are equivalent under $\mathrm{PGL}(3, q^2)$ to $\mathcal{U}_{BT}$ and tangent to l_∞ at P_∞ with Property 1. The result follows. □

Corollary 1 *Let U be a Buekenhout-Tits unital, then the projectivity group stabilising U is isomorphic to the group G in Result 1.*

Since we have shown that all Buekenhout-Tits unitals are equivalent under $\mathrm{PGL}(3, q^2)$, we may use $\mathcal{U}_{BT}$ to verify statements about general Buekenhout-Tits unitals.

3 On the stabiliser of the Buekenhout-Tits unital

We now describe the stabiliser of the Buekenhout-Tits unital $\mathcal{U}_{BT}$ in $\mathrm{P\Gamma L}(3, q^2)$.

Lemma 3 *Let $M_{u,v}$, $M_{s,t}$ be matrices inducing collineations of G as defined in Result 1, then* $M_{u,v}M_{s,t} = M_{u+s,t+v+su\delta}$.

Proof Using Eq. (2), we find

$$M_{u,v}M_{s,t} = \begin{bmatrix} 1 & (s+u)\epsilon & (t+v+su\delta)+(s+u)^{\sigma} \\ 0 & 1 & (u+s)+(u+s)\epsilon \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, we have $M_{u,v}M_{s,t} = M_{u+s,t+v+su\delta}$. □

Corollary 2 *The order of any collineation of G induced by a matrix $M_{u,v}$ as defined in Result 1 is four if and only if $u \neq 0$, and two if and only if $u = 0$ and $v \neq 0$.*

Proof Firstly note that $M_{0,0} = I$. Direct calculation shows that $M_{u,v}^2 = M_{0,u^2\delta}$, $M_{u,v}^3 = M_{u,v+u^2\delta}$ and $M_{u,v}^4 = M_{0,0}$. □

Corollary 3 *The stabiliser group G as defined in Result 1 is isomorphic to $(C_4)^{2e+1}$.*

Proof Recall from Result 1 that $|G| = q^2 = 2^{4e+2}$. From Corollary 2, we have that $G \equiv (C_4)^k(C_2)^l$ for some integers k, l such that $2^{2k+l} = |G| = 2^{4e+2}$, and hence,

$$l = 2(2e+1-k).$$

Furthermore, we see that the number of elements of order four in G is $q^2 - q$ as they correspond to all matrices $M_{u,v}$ with $u, v \in \mathbb{F}_q$ and $u \neq 0$. The number of elements of order four in a group isomorphic to $(C_4)^k(C_2)^l$ is $(4^k - 2^k)2^l$. Thus,

$$(4^k - 2^k)2^l = 4^{2e+1} - 2^{2e+1}. \tag{3}$$

Using Eq. (3), we find that $k = 2e+1$, and therefore $G \equiv (C_4)^{2e+1}$. □

Theorem 2 *Let $q = 2^{2e+1}$, then the stabiliser of $\mathcal{U}_{BT}$ in $\mathrm{P\Gamma L}(3, q^2)$ is the order $q^2(4e+2)$ group GK, where $G = \mathrm{PGL}(3, q^2)_{\mathcal{U}_{BT}}$ as described in Result 1, and K is a cyclic subgroup of order $16e+8$ generated by*

$$\psi : \mathbf{x} \mapsto \mathbf{x}^2 \begin{bmatrix} 1 & 1 & \epsilon \\ 0 & \delta^{\sigma/2}(1+\epsilon) & \delta^{\sigma/2}(1+\epsilon) \\ 0 & 0 & \delta^{\sigma+1} \end{bmatrix}.$$

(Here, $\mathbf{x}$ denotes the row vector containing the three homogeneous coordinates of a point, and $\mathbf{x}^2$ denotes its elementwise power.)

Proof From Lemma 2, the number of Buekenhout-Tits unitals tangent to $\ell_\infty : x = 0$ at a point $P_\infty = (0, 0, 1)$ with Property 1 is $q^4(q^2-1)^2$. By the arguments of Lemma 1, all of these unitals are equivalent under $\mathrm{PGL}(3, q^2)$ to $\mathcal{U}_{BT}$ under the stabiliser groups $\mathrm{PGL}(3, q^2)_{\{\ell_\infty, P_\infty\}}$ and $\mathrm{P\Gamma L}(3, q^2)_{\{\ell_\infty, P_\infty\}}$ fixing P_∞ and stabilising ℓ_∞. Any collineation stabilising $\mathcal{U}_{BT}$ must stabilise P_∞ and ℓ_∞, so $\mathrm{P\Gamma L}(3, q^2)_{\mathcal{U}_{BT}} < \mathrm{P\Gamma L}(3, q^2)_{\{\ell_\infty, P_\infty\}}$. Therefore, the orbit of $\mathcal{U}_{BT}$ under $\mathrm{P\Gamma L}(3, q^2)_{\{\ell_\infty, P_\infty\}}$ has size $q^4(q^2-1)^2$, that is

$$|\mathrm{P\Gamma L}(3, q^2)_{\mathcal{U}_{BT}}| = \frac{|\mathrm{P\Gamma L}(3, q^2)_{\{\ell_\infty, P_\infty\}}|}{q^4(q^2-1)^2}.$$

We can now see that $\mathrm{P\Gamma L}(3, q^2)_{\mathcal{U}_{BT}}$ must have order $q^2(4e+2)$.

Direct calculation shows that ψ stabilises $\mathcal{U}_{BT}$. Because $\mathbf{x}^{2^{4e+2}} = \mathbf{x}^{q^2} = \mathbf{x}$, the collineation ψ^{4e+2} is a linear map stabilising $\mathcal{U}_{BT}$, and so $\psi^{4e+2} \in G$. Therefore, we deduce that $|\psi| = (4e+2)|\psi^{4e+2}|$. From Corollary 2, it follows that $|\psi^{4e+2}| \in \{1, 2, 4\}$, with $|\psi^{4e+2}| = 4$ if and only if ψ^{4e+2} is induced by $M_{u,v}$ for some $u \neq 0$. Hence, $|\psi^{4e+2}| = 4$ if and only if $\psi^{4e+2}(0, 1, 0) \neq (0, 1, 0)$ as $(0, 1, 0)M_{u,v} = (0, 1, u + u\epsilon)$. Consider the point $(0, 1, z)$ for some arbitrary $z \in \mathbb{F}_q$. Direct calculation shows that $\psi(0, 1, z) = (0, 1, 1 + \mu z^2)$, where $\mu = \frac{\delta^{\sigma+1}}{\delta^{\sigma/2}(1+\epsilon)} = \delta^{\sigma/2}\epsilon$. Thus,

$$\psi^k(0, 1, z) = \left(0, 1, \sum_{i=0}^{k} \mu^{2^i-1} + zg(z)\right)$$

for some polynomial $g(z)$ depending on k. If $z = 0$ and $k = 4e+2$ we thus find

$$\begin{aligned}\psi^{4e+2}(0, 1, 0) &= \left(0, 1, \sum_{i=0}^{4e+2} \mu^{2^i-1}\right)\\ &= \left(0, 1, \frac{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mu)}{\mu}\right).\end{aligned}$$

Recall that $\epsilon^q = \epsilon + 1$, so $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\epsilon) = 1$. Therefore, we have $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_2}(\delta^{\sigma/2}\epsilon) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\delta^{\sigma/2}\epsilon)) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\delta^{\sigma/2}\,\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\epsilon)) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\delta^{\sigma/2}) = 1$. Hence, we see $\psi((0, 1, 0)) \neq (0, 1, 0)$, so $|\psi^{4e+2}| = 4$ and $|\psi| = 16e + 8$. Let $K = \langle\psi\rangle$, because $|K \cap G| = 4$, it follows that $|GK| = q^2(4e+2)$ and thus $GK = \mathrm{P\Gamma L}(3, q^2)_{\mathcal{U}_{BT}}$. □

4 On the feet of the Buekenhout-Tits unital

Recall that the feet $\tau_P(U)$ of a point P not on a unital U is the set of all points on tangent lines to U through P. The feet of the Buekenhout-Tits unital $\mathcal{U}_{BT}$ (as coordinatised in 1) for points $P \notin \mathcal{U}_{BT}$ are first described by Ebert in [9]. He shows that the feet of a point $P = (1, y_1 + y_2\epsilon, z_1 + z_2\epsilon)$ is the following set of points:

$$\begin{aligned}\tau_P(\mathcal{U}_{BT}) = \{&(1, s + t\epsilon, s^2 + t^2\delta + st + y_1 s + y_1 t + y_2\delta t + z_1 + (s^{\sigma+2} + t^\sigma + st)\epsilon)\\ &\mid s, t \in \mathbb{F}_q,\ s^{\sigma+2} + t^\sigma + st = y_2 s + y_1 t + z_2\}. \quad (4)\end{aligned}$$

If the line ℓ has Equation $\alpha x + y = 0$, where $\alpha \in \mathbb{F}_{q^2}$, Ebert shows that $|\ell \cap \tau_P(\mathcal{U}_{BT})| \leq 1$. Otherwise, ℓ has equation $(a_1 + a_2\epsilon)x + (b_1 + b_2\epsilon)y + z = 0$ and Ebert shows that ℓ meets $\tau_P(\mathcal{U}_{BT})$ in the points $P_{r,s,t} \in \mathcal{U}_{BT}$, where $r = s^2 + t^2\delta + st + y_1 s + y_1 t + y_2\delta t + z_1$ and s, t satisfy

$$s^2 + \delta t^2 + st + (y_1 + b_1)s + (y_1 + y_2\delta + b_2\delta)t + z_1 + a_1 = 0, \quad (5)$$

$$s^{\sigma+2} + t^\sigma + st = b_2 s + (b_1 + b_2)t + a_2, \quad (6)$$

$$y_2 s + y_1 t + z_2 = b_2 s + (b_1 + b_2)t + a_2. \quad (7)$$

We will show that for all choices of points $P \notin \ell_\infty$ and lines ℓ, $|\tau_P(\mathcal{U}_{BT}) \cap \ell| \leq 4$.

Recall that the group G as described in Result 1 has $q^2 - q$ orbits of $\mathrm{PG}(2, q^2) \setminus (\mathcal{U}_{BT} \cup \ell_\infty)$ of size q^2. Here we give a set of $q^2 - q$ representatives for these orbits.

Lemma 4 *Let G be the group of projectivities stabilising $\mathcal{U}_{BT}$ as described in Result 1. Then, the set of q^2-q points $\{P_{a,b}=(1,a,b\epsilon)\,|\,a,b\in\mathbb{F}_q,\ b\neq a^{\sigma+2}\}$ are points from q^2-q distinct point orbits of size q^2 under G.*

Proof Suppose there exists a collineation of G induced by a matrix $M_{u,v}$ such that $P_{a,b}M_{u,v}=P_{c,d}$. Then,

$$(1,a,b\epsilon)\begin{bmatrix}1 & u\epsilon & v+u^{\sigma}\epsilon\\ 0 & 1 & u+u\epsilon\\ 0 & 0 & 1\end{bmatrix}=(1,c,d\epsilon)\,.$$

However, it is clear that $P_{a,b}M_{u,v}=(1,a+u\epsilon,v+u^{\sigma}\epsilon+a\,(u+u\epsilon)+b\epsilon)$, so $a+u\epsilon=c$. Therefore, $a=c$ and $u=0$. If $u=0$, then $v+b\epsilon=d\epsilon$, and we have $b=d$. Hence, $P_{a,b}=P_{c,d}$ and the lemma follows. □

There are $q^4-q^3=q^2(q^2-q)$ points of $\mathrm{PG}(2,q^2)$ not on ℓ_∞ or $\mathcal{U}_{BT}$. By Lemma 4, each of these points lies in the orbit of a point of the form $(1,a,b\epsilon)$. Therefore, in order to study the feet of a point P, we may assume that the point $P=(1,y_1,z_2\epsilon)$.

The following lemma shows that the feet of a point $P=(1,y_1,z_2\epsilon)$, with $y_1^{\sigma+2}\neq z_2$ meets almost all lines in at most 2 points.

Lemma 5 *Let $\ell:\alpha x+\beta y+z=0$ be a line in $\mathrm{PG}(2,q^2)$, where $\alpha=a_1+a_2\epsilon$, $\beta=b_1+b_2\epsilon$ and $a_1,a_2,b_1,b_2\in\mathbb{F}_q$. Let $P=(1,y_1,z_2\epsilon)$, with $y_1,z_2\in\mathbb{F}_q$ such that $z_2\neq y_1^{\sigma+2}$. Unless $b_2=0$, $y_1=b_1$ and $a_2=z_2$, we have $|\tau_P(\mathcal{U}_{BT})\cap\ell|\leq 2$.*

Proof From the description given in Eq. (4), we see that the points $P_{r,s,t}\in\tau_P(\mathcal{U}_{BT})$ satisfy

$$s^{\sigma+2}+t^{\sigma}+st=y_1t+z_2,\tag{8}$$

and this equation has $q+1$ solutions. Substituting Eqs. (8) into (5) and combining Eqs. (6) and (7), it follows that the points $P_{r,s,t}\in\tau_P(\mathcal{U}_{BT})\cap\ell$ have s,t satisfying

$$s^{\sigma+2}+t^{\sigma}+st+y_1t+z_2=0.$$

$$s^2+\delta t^2+st+(y_1+b_1)\,s+(y_1+b_2\delta)\,t+a_1=0\tag{9}$$

$$b_2s+(y_1+b_1+b_2)\,t+a_2+z_2=0\tag{10}$$

We will now count the solutions to this system, by considering the geometry of these equations in the solution space $\mathrm{AG}(2,q)$ with coordinates (s,t). Recall that the points $(1,s,t,s^{\sigma+2}+t^{\sigma}+st)$, where $s,t\in\mathbb{F}_q$ are the q^2 affine points of a Tits ovoid in $\mathrm{PG}(3,q)$ [14]. Because $\tau_P(\mathcal{U}_{BT})$ has $q+1$ points, the Eq. 8 must have $q+1$ solutions (s,t) in the solution space. Hence the $q+1$ points (s,t) in $\mathrm{AG}(2,q)$ satisfying 8 are a translation oval.

Unless $b_2=0$ and $y_1=b_1$, Eq. (10) represents a line in the solution space $\mathrm{AG}(2,q)$. A line meets the oval defined by Eq. 8 in at most two points, so we have at most two solutions to the system. If $b_2=0$, $y_1=b_1$, and $a_2\neq z_2$, then Eq. (10) has no solutions. □

Remark 2 Lemma 5 is a refinement of [4, Theorem 4.33], where Barwick and Ebert rework Ebert's earlier proof in [9] that the feet of a point $P\notin(\ell_\infty\cup\mathcal{U}_{BT})$ are not collinear. This reworked proof asserts that the feet cannot be collinear because the line given by Eq. (10) and the conic from Eq. (9) cannot have $q+1$ common solutions. However, we can see that this logic is not complete, and leaves an interesting case to examine when Eq. (10) vanishes. Ebert's original proof in [9] does not contain this error, instead arguing that Eqs. (9) and 8 cannot have $q+1$ common solutions.

It follows from Lemma 5 that the feet of a point $P \notin (\ell_\infty \cup \mathcal{U}_{BT})$ is a set of $q+1$ points such that every line meets $\tau_P(\mathcal{U}_{BT})$ in at most two points except for a set of q concurrent lines.

To investigate the latter case, assume that $b_2 = 0$, $y_1 = b_1$ and $a_2 = z_2$. In this case, Eq. (10) vanishes. The system describing $\ell \cap \tau_P(\mathcal{U}_{BT})$ is thus

$$s^2 + \delta t^2 + st = y_1 t + a_1 \tag{11}$$

$$s^{\sigma+2} + t^\sigma + st = y_1 t + z_2. \tag{12}$$

The lines that produce these cases are the lines with dual coordinates $[a_1 + z_2\epsilon, y_1, 1]$. These lines are concurrent at the point $(0, 1, y_1)$ which lies on ℓ_∞. We will show in Corollary 4 that these latter lines meet $\tau_P(\mathcal{U}_{BT})$ in at most four points.

Recall that an affine section of a Tits ovoid in $\mathrm{PG}(3, q)$ contains $q+1$ points equivalent under $\mathrm{PGL}(3, q^2)$ to the translation oval [14]

$$\mathcal{D}_\sigma = \left\{(1, t, t^\sigma) \mid t \in \mathbb{F}_q\right\} \cup \{(0, 0, 1)\}.$$

For a reference on translation ovals, see [11, pp. 182–186]. We require the following lemma, which adapts arguments found in [6, Lemma 2.1].

Lemma 6 *Let $\mathcal{O}$ be a translation oval in $\mathrm{PG}(2, q)$ projectively equivalent to $\mathcal{D}_\sigma$, and let $\mathcal{C}$ be a non-degenerate conic. If the nucleus of $\mathcal{O}$ is also the nucleus of $\mathcal{C}$, then $|\mathcal{O} \cap \mathcal{C}| \le 4$.*

Proof Without loss of generality we may take $\mathcal{O} = \mathcal{D}_\sigma$, so that the nucleus of $\mathcal{O}$ is $N = (0, 1, 0)$. If N is also the nucleus of $\mathcal{C}$, then $\mathcal{C}$ is a conic of the following form,

$$a_1 x^2 + a_2 y^2 + a_3 z^2 + xz = 0,$$

for some $a_1, a_2, a_3 \in \mathbb{F}_q$ with $a_2 \neq 0$. Suppose that $(0, 0, 1) \notin \mathcal{C}$. Then $a_3 \neq 0$, and the point $(1, t, t^\sigma) \in \mathcal{C}$ if and only if t satisfies

$$a_1 + a_2 t^2 + a_3 t^{2\sigma} + t^\sigma = 0, \tag{13}$$

hence,

$$0 = \left(a_1 + a_2 t^2 + a_3 t^{2\sigma} + t^\sigma\right)^{\sigma/2} = a_1^{\sigma/2} + a_2^{\sigma/2} t^\sigma + a_3^{\sigma/2} t^2 + t.$$

Therefore,

$$t^\sigma = \left(\frac{a_3}{a_2}\right)^{2^e} t^2 + \frac{1}{a_2^{2^e}} t + \left(\frac{a_1}{a_2}\right)^{2^e}. \tag{14}$$

and substituting Eqs. (14) into (13), we find that Eq. (13) has at most four solutions. If instead $(0, 0, 1) \in \mathcal{C}$, then $a_3 = 0$ and arguing as above we find that Eq. (13) has at most two solutions, so $|\mathcal{O} \cap \mathcal{C}| \le 3$. □

Corollary 4 *The feet of a point $P \notin (\ell_\infty \cup \mathcal{U}_{BT})$ meet a line ℓ in at most four points.*

Proof From Lemma 5, we know we can restrict ourselves to the case $b_2 = 0$, $y_1 = b_1$, $a_2 = z_2$ which means we are looking at the points $P_{r,s,t} \in \tau_P(\mathcal{U}_{BT}) \cap \ell$ have s, t satisfying

$$s^2 + \delta t^2 + st = y_1 t + a_1 \tag{15}$$

$$s^{\sigma+2} + t^\sigma + st = y_1 t + z_2, \tag{16}$$

where Eq. (15) represents a conic $\mathcal{C}$, and Eq. (16) represents an oval $\mathcal{O}$ in AG$(2, q)$. If the conic is degenerate, the oval and conic have at most four points in common. So we may assume that the conic is non-degenerate. The nucleus of $\mathcal{C}$ is $N = (y_1, 0, 1)$. We now show that N is the nucleus of the oval $\mathcal{O}$. The line $t = 0$ goes through N and meets the oval $\mathcal{O}$ when $s^{\sigma+2} = z_2$, which has one solution as $\sigma + 2$ is a permutation of $\mathbb{F}_q$. The line $s + y_1 = 0$ through N meets the oval $\mathcal{O}$ when $t^\sigma = y^{\sigma+2} + z_2$ which has one solution for t. Therefore, N is the nucleus, as it is the intersection of two tangent lines to the oval. It now follows from Lemma 6 that Eqs. (15) and (16) have at most four common solutions. □

We now show the existence of a point $P \notin (\mathcal{U}_{BT} \cup \ell_\infty)$ and a line ℓ such that $|\ell \cap \tau_P(\mathcal{U}_{BT})| = 3$, and demonstrate our bound is sharp.

Lemma 7 *Consider the Equation $s^{\sigma+2} + t^\sigma + st = y_1 t + z_2$, whose solutions (s, t) are a translation oval of* AG$(2, q)$. *If $y_1 = 0$, then the points of the oval given by Eq. (16) are*

$$\left\{ P_u = \left(\frac{z_2^{1-\sigma/2} u^\sigma}{1 + u + u^\sigma}, \frac{z_2^{\sigma/2}(1 + u^\sigma)}{1 + u + u^\sigma} \right) \middle| u \in \mathbb{F}_q \right\} \cup \left\{ \left(z_2^{1-\sigma/2}, z_2^{\sigma/2} \right) \right\}.$$

Proof If $y_1 = 0$, then Eq. (16) reduces to

$$s^{\sigma+2} + t^\sigma + st + z_2 = 0. \tag{17}$$

Using the properties of σ described in Sect. 1.3, one can show the point $(z_2^{1-\sigma/2}, z_2^{\sigma/2})$ satisfies Eq. (17). Furthermore, the points $\overline{P_u} = (z_2^{1-\sigma/2} u^\sigma, z_2^{\sigma/2}(1+u^\sigma), 1+u+u^\sigma)$, where $u \in \mathbb{F}_q$, are projective points satisfying the following homogeneous equation

$$x^{\sigma+2} + y^\sigma z^2 + xyz^\sigma + z_2 z^{\sigma+2} = 0.$$

Because $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(u+u^\sigma) = 0$, and $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(1) = 1$ when $q = 2^{2e+1}$, we have $u^\sigma + u + 1 \neq 0$ for all $u \in \mathbb{F}_q$. Thus, normalising so $z = 1$, the points $\overline{P_u}$ have the form $(s, t, 1)$ where s and t satisfy Eq. (17). □

Corollary 5 *Let $y_1 = 0$ and consider the points P_u as described in Lemma 7. A point P_u lies on the conic given by Eq. (15), if and only if u is a root of the following polynomial*

$$a_1^{\sigma/2} u^\sigma + (z_2^{\sigma-1} + \delta^{\sigma/2} z_2 + z_2^{\sigma/2} + a_1^{\sigma/2}) u^2 + z_2^{\sigma/2} u + \delta^{\sigma/2} z_2 + a_1^{\sigma/2}. \tag{18}$$

Proof By directly substituting P_u into Eq. (15) we have

$$(z_2^{2-\sigma} + \delta z_2^\sigma + z_2 + a_1) u^{2\sigma} + z_2 u^\sigma + a_1 u^2 + (\delta z_2^\sigma + a_1) = 0. \tag{19}$$

Raising both sides of Eq. (19) to the power of $\sigma/2$ yields our result. □

Theorem 3 *Let U be a Buekenhout-Tits unital in* PG$(2, q^2)$. *The feet of a point $P \notin (\ell_\infty \cup U)$ meet a line ℓ in at most four points. Moreover, there exists a line ℓ and point P such that $|\ell \cap \tau_P(U)| = k$ for each $k \in \{0, 1, 2, 3, 4\}$.*

Proof By Theorem 1 we may assume that $U = \mathcal{U}_{BT}$. The first part of the proof comes from Corollary 4. Let $P = (1, y_1, z_2\epsilon)$. All lines through P meet $\tau_P(U)$ in at most one point by definition, so it is clear that there exists lines ℓ such that $|\ell \cap \tau_P(U)|$ is zero or one. Because the points of $\tau_P(U)$ are not collinear, there exists a pair of points $Q, R \in \tau_P(U)$ such that the line QR does not contain $(0, 1, y_1)$. Because QR does not contain $(0, 1, y_1)$ it cannot have dual coordinates of the form $[a_1 + z_2\epsilon, y_1, 1]$ for any $a_1 \in \mathbb{F}_q$, and so Lemma 5 applies to QR. Hence, the line QR meets $\tau_P(U)$ in precisely two points.

Now consider a line ℓ with Equation $(\delta+\epsilon)x+z=0$ and let P be the point $(1,0,\epsilon)$ (that is, $a_1=\delta, a_2=1, b_1=b_2=y_1=0, z_2=1$). The number of points of $\ell\cap\tau_P(U)$ is the same as the number of solutions to Eqs. (11) and (12). By Lemma 7 the points P_u satisfying Eq. (12) lie on the conic determined by Eq. (11) when

$$\delta^{\sigma/2}u^\sigma+u=u(\delta^{\sigma/2}u^{\sigma-1}+1)=0. \tag{20}$$

Equation (20) has exactly two solutions as $\sigma-1$ is a permutation of $\mathbb{F}_q$: $u=0$ and the unique solution to $u^{\sigma-1}=\frac{1}{\delta^{\sigma/2}}$. It can also be shown that $\left(z_2^{1-\sigma/2}, z_2^{\sigma/2}\right)=(1,1)$ satisfies both equations. Hence, the intersection of the feet of the point $(1,0,\epsilon)$ and ℓ has exactly three points.

Finally, consider the point $P(1,0,\frac{1}{\delta^\sigma}\epsilon)$ and the line ℓ with dual coordinates $\left[\frac{1}{\delta}+\frac{1}{\delta^2}\epsilon,0,1\right]$. By Corollary 5, the number of feet of P on the line ℓ is the number of roots of the polynomial (18), where $a_1=\frac{1}{\delta}$ and $z_2=\frac{1}{\delta^\sigma}$. Substituting $a_1=\frac{1}{\delta}$ and $z_2=\frac{1}{\delta^\sigma}$ yields

$$\frac{1}{\delta^{\sigma/2}}u^\sigma+\left(\frac{1}{\delta^{2-\sigma}}+\frac{1}{\delta}\right)u^2+\frac{1}{\delta}u=0. \tag{21}$$

Since Eq. (21) describes the roots of a $\mathbb{F}_2$-linearised polynomial, and there are at most 4 roots, we have that the polynomial (18) has 1, 2, or 4 roots. We will show that, under the condition $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\delta)=1$, it has four roots. Multiplying Eq. (21) by δ yields $\delta^{1-\sigma/2}u^\sigma+(\delta^{\sigma-1}+1)u^2+u=0$ and now substituting $a=\delta^{\sigma-1}+1$ gives

$$(a^{\sigma/2}+1)u^\sigma+au^2+u=0. \tag{22}$$

We find that $u=0$ and $u=\frac{1}{a^{1+\sigma/2}}$ are solutions to Eq. (22). Now consider

$$u^\sigma+au^2+1=0. \tag{23}$$

Any solution to Eq. (23) also satisfies $(u^\sigma+au^2+1)^{\sigma/2}+u^\sigma+au^2+1=0$ which is precisely Eq. (22). Multiply Eq. (23) with $a^{\sigma+1}$, then we find $(a^{\sigma/2+1}u)^\sigma+(a^{\sigma/2+1}u)^2+a^{\sigma+1}=0$, and letting $z=(a^{\sigma/2+1}u)^2$,

$$z^{\sigma/2}+z+a^{\sigma+1}=0, \tag{24}$$

which is known (see [12]) to have solutions if and only if $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a^{\sigma+1})=0$. As $z=0$ and $z=1$ are not solutions of Eq. (24), no solutions of Eq. (24) correspond to the solutions $u=0$ or $u=\frac{1}{a^{1+\sigma/2}}$ of Eq. (21). Furthermore, recall that Eq. (21) has 1, 2 or 4 solutions and that we have assumed that $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\delta)=1$. Since $\delta^{\sigma-1}=a+1$, it follows that $\delta=(a+1)^{\sigma+1}$ and $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\delta)=\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a^{\sigma+1}+a^\sigma+a+1)=\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a^{\sigma+1})+\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(1)=\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a^{\sigma+1})+1$. Hence, the conditions $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\delta)=1$ and $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a^{\sigma+1})=0$ are equivalent, and we find exactly four solutions to Eq. (21). □

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