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Non-existence of quasi-symmetric designs with restricted block graphs

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Abstract
Quasi-symmetric designs (QSDs) with particular block graphs are investigated. We rule out the possibility of a QSD with block graph that has the same parameters as that of the Symplectic graph $Sp(2t, q)$, where q is an odd prime power or its complement. We obtain support for Bagchi's recent conjecture, which states that 'For the existence of a quasi-symmetric 2-design with block graph $K_{m\times n}$, we must have $m \equiv n+1 \pmod{n^2}$'. Under certain conditions, we rule out the possibility of a QSD having a pseudo-Latin square or negative Latin square block graph.

Keywords Strongly regular graphs · Quasi-symmetric designs · Block graph

Mathematics Subject Classification 05B05

1 Introduction

Let X be a finite set of v elements called points, and β be a set of k-element subsets of X called blocks, such that each pair of points occur in λ blocks, then the pair $\mathbf{D} = (X, \beta)$ is 2-(v, k, λ) *design*. For a 2-(v, k, λ) design $\mathbf{D}$, the number of blocks containing α in X is r, which is independent of α. The number of blocks in $\mathbf{D}$ is denoted by b.

An integer λ_1, $0 \le \lambda_1 < k$, is an *intersection number* of $\mathbf{D}$ if there exist $B, B' \in \beta$ such that $|B \cap B'| = \lambda_1$. Symmetric designs have exactly one intersection number.

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A 2-design with two intersection numbers is a *quasi-symmetric design* (QSD). Denote these intersection numbers by λ_1 and λ_2, where $0 \leq \lambda_1 < \lambda_2 < k$. We denote $\mu = \lambda_2 - \lambda_1$, and call it as the *defect* of the QSD.

The parameters $(v, b, r, k, \lambda; \lambda_1, \lambda_2)$ are the *standard parameters* of a QSD, which are said to be *feasible* if they satisfy all necessary conditions given in the Lemma 2.1. Complete parametric classification of QSDs with $\mu = 1, 2, 3$ has been obtained in [4–6].

The *block graph* G of a QSD $\mathbf{D}$ has vertices that are blocks of $\mathbf{D}$, where two distinct blocks B, B' are adjacent if and only if $|B \cap B'| = \lambda_2$. It was shown in [3,11] that G is a *strongly regular graph* (SRG) with parameters (b, a, c, d). Here b is the number of vertices of G, i.e. the number of blocks of the design $\mathbf{D}$, a its valency, any two adjacent vertices have exactly c common neighbors and any two non-adjacent vertices have exactly d common neighbors. We assume, as is customary, that a SRG is connected but is neither the null graph nor the complete graph.

The adjacency matrix of G is the $b \times b$ matrix A, with its rows and columns indexed by the vertices of G, such that, for vertices x, y, the (x, y)-th entry $A(x, y)$ of A is 1 if x, y are adjacent in G, and $A(x, y) = 0$ otherwise. The spectrum spec(A) (i.e., the multi-set of eigenvalues of A, counting multiplicity) is also called the spectrum of G, and is denoted spec(G). A connected SRG G has exactly three distinct eigenvalues, which we shall denote by $a > \rho > \sigma$, with corresponding multiplicities $1, f, g$. Here a is the degree of G and $f + g + 1 = b$. Since trace$(A) = 0$, we have $a = -f\rho - g\sigma$. If G is non-complete then we have the following inequalities, known as the *Krein Bounds*, [10].

1. $(\rho + 1)(a + \rho + 2\rho\sigma) \leq (a + \rho)(\sigma + 1)^2$;
2. $(\sigma + 1)(a + \sigma + 2\rho\sigma) \leq (a + \sigma)(\rho + 1)^2$.

It is known to be a difficult problem to decide which SRGs are block graphs of QSDs. In [7,8], it was established that several infinite families of SRGs are not block graphs of QSDs. In the recent paper [1], Bagchi obtained several restrictions on parameters of the block graph of a QSD in terms of its spectral parameters. Efforts were made to characterize QSDs associated with a particular class of SRGs. The following conjecture related to QSDs with block graph, the complete multi-partite graph $K_{m\times n}(m \geq 2, n \geq 2)$, was made.

Conjecture 1.1 Bagchi, [1] *For the existence of a quasi-symmetric 2-design with block graph $K_{m\times n}$, we must have $m \equiv n + 1 \pmod{n^2}$.*

In support of the Conjecture 1.1, we rule out the possibility of QSD's with block graph $K_{m\times n}$, if $n = h\alpha$, $m = h\beta + 1$ for positive integers $\alpha \geq 2$, β such that $\gcd(n, m - 1) = h$, and $1 \leq h < 4\alpha$ or $\gcd(h, \alpha - \beta) = 1$.

In [1], several algebraic conditions are given on q, for Symplectic graphs $Sp(2t, q)$, where $q > 2$ a prime power and $t \geq 2$ to be a block graph of QSDs. It was hinted that for each fixed prime power $q > 2$, the graph $Sp(2t, q)$ is a block graphs of QSDs for at most finitely many values of t. We continue with the proof of Corollary 2.6 of [1] and rule out the possibility of QSDs with block graph that has the same parameters as that of the Symplectic graph $Sp(2t, q)$. To rule out the possibility of QSDs with associated graph that has the same parameters as that of the complement of Symplectic graph $Sp(2t, q)$, we use the technique developed in [8].

Under certain conditions, we rule out the possibility of a QSD having a pseudo-Latin square or negative Latin square block graph.

We follow [1] for definitions and terminology. Symbolic calculations were made easy with the help of Mathematica [13].

2 Preliminaries

Lemma 2.1 [9,12] *Let* $\mathbf{D}$ *be a QSD with standard parameter set* $(v, b, r, k, \lambda; \lambda_1, \lambda_2)$. *Then the following relations hold:*

1. $vr = bk$ *and* $\lambda(v-1) = r(k-1)$.
2. $k(r-1)(\lambda_1 + \lambda_2 - 1) - \lambda_1\lambda_2(b-1) = k(k-1)(\lambda - 1)$.
3. $\mu = \lambda_2 - \lambda_1$ *divides* $k - \lambda_1$ *and* $r - \lambda$.
4. $-\sigma = \dfrac{k-\lambda_1}{\mu} > 0$ *and* $\rho - \sigma = \dfrac{r-\lambda}{\mu} > 0$.

Theorem 2.2 [[8], Theorem 6 (iii)] *Let* $\mathbf{D}$ *be a QSD with parameters* $(v, b, r, k, \lambda; \lambda_1, \lambda_2)$ *and* G *be the strongly regular block graph with parameters* (b, a, c, d) *of* $\mathbf{D}$. *Then, we have*

$$\mu = \frac{(-a + c - d - \sigma - b\sigma)\,(b - s)\,s}{b\,(c - d - 2\sigma)\,(-a + \sigma - b\sigma)} \tag{2.1}$$

for a positive integer $s = \dfrac{(-a + \sigma - b\sigma)\mu}{(\lambda_1 - \sigma\mu)}$.

Corollary 2.3 [[1], Corollary 2.6.] *Let* $q > 2$ *be a prime power and let* $t \geq 2$ *be an integer. Then, a quasi-symmetric* 2*-design of defect* μ *with block graph* $Sp(2t, q)$ *is parametrically feasible if and only if* t *is even,* $q \equiv 3 \pmod 8$, $\mu = (q^t - q + 2)/8$, *and the pair* (q, t) *satisfies*

$$\left(\frac{q^t - 1}{q - 1}\right)^2 - q^t\left(\frac{q^{t-1} - 1}{q - 1}\right) = x^2 \tag{2.2}$$

for some integer x.

3 QSDs with complete multipartite graphs

The complete multi-partite graph $K_{m\times n}(m \geq 2, n \geq 2)$ has mn vertices partitioned into m parts of size n each, where two vertices are adjacent if and only if they are in different parts. In other words, $K_{m\times n}$ is the complement of mK_n (the disjoint union of m copies of the n-vertex complete graph K_n). In this section we rule out the possibility of QSDs with complete multi-partite block graph under certain conditions, which provide support to the Conjecture 1.1.

Theorem 3.1 *Let* $\mathbf{D}$ *be a QSD whose block graph has the same parameters as that of the complete multipartite graphs with* $m \geq 2$ *classes of size* $n \geq 2$, *i.e.,*

$$(nm, n(m-1), n(m-2), n(m-1)).$$

If $\gamma = \gcd(nm - m + 1, n^2)$, *then* $\gamma \geq 4(n-1)$.

Proof We find $\sigma = -n$. We use Eq. (2.1) to find

$$\mu = \frac{s(nm - s)(nm - m + 1)}{(n-1)n^2m^2}. \tag{3.1}$$

If $\gamma = \gcd(nm - m + 1, n^2)$ then $(n-1)n^2m^2$ divides $s(nm - s)\gamma$, write $s(nm - s)\gamma = e(n-1)n^2m^2$ for some positive integer e and observe that the discriminant of this quadratic in s is $n^2m^2\gamma(\gamma - 4e(n-1))$, which is a perfect square. As $e \geq 1$, we get $\gamma \geq 4(n-1)$. □

Remark 3.2 1. As an application of the Theorem 2.3 of [1] one can show that $m-1 \geq n$, where equality holds if and only if $\mu = 1$ and $\mathbf{D}$ is a 2-$(n^2, n, 1)$ design, known as affine plane of order n.

2. The truthfulness of Conjecture 1.1 implies n divides $m-1$.
3. Suppose $m-1 > n$ and $n = h\alpha, m = h\beta + 1$ for positive integers α, β such that $\gcd(n, m-1) = h$. Then

$$\begin{aligned}\gamma &= \gcd(n^2, nm - m + 1)\\ &= \gcd(h^2\alpha^2, h(h\beta\alpha + \alpha - \beta))\\ &= h\gcd(h\alpha^2, h\beta\alpha + \alpha - \beta)\\ &= h\gcd(h, h\beta\alpha + \alpha - \beta) \text{ as } \gcd(\alpha, \beta) = 1\\ &= h\gcd(h, \alpha - \beta).\end{aligned}$$

We take $\gamma = hh'$, with $h' = \gcd(h, \alpha - \beta)$. As $\gamma \geq 4(n-1)$ and $hh' \geq 4(h\alpha - 1)$. Hence $h' \geq 4(\alpha - 1)$. If $h' < 4$ then $\alpha = 1$, which implies n divides $m-1$.

In support of the Conjecture 1.1, we give the following results.

Corollary 3.3 *There is no QSD whose block graph has the same parameters as that of the complete multipartite graph with $m \geq 2$ classes of size $n \geq 2$, with* $\gcd(n, m-1) = 1$.

Proof We give proof with same notations as used in the Corollary 2.4 of [1]. Since $m = tn^2/\alpha + n + 1$, this means that $\gcd(tn^2/\alpha, n) = 1$; also, since $n = l + l^* + 2\alpha$, $\alpha \leq n/2$. Since n^2 divides $(tn^2/\alpha)(\alpha)$ and is co-prime to the first factor, it follows that n^2 divides α, and so $\alpha \geq n^2$, which is a contradiction.

Alternately, if $\gcd(n, m-1) = 1$, then $\gamma = \gcd(nm - m + 1, n^2) = 1$, which contradicts Theorem 3.1. □

If $m \not\equiv n + 1 \pmod{n^2}$, and $n = h\alpha, m = h\beta + 1$ for positive integers α, β such that $\gcd(n, m-1) = h$, then $\alpha \geq 2$.

Theorem 3.4 *Let G be a SRG that has the same parameters as that of the complete multipartite graph with $m \geq 2$ classes of size $n \geq 2$. Suppose $n = h\alpha, m = h\beta + 1$ for positive integers $\alpha \geq 2, \beta$ such that* $\gcd(n, m-1) = h$. *If* $1 < h < 4\alpha$ *or* $\gcd(h, \alpha - \beta) = 1$, *then there is no QSD whose block graph is G.*

Proof We use Eq. (2.1) to find

$$\mu = \frac{s(h\beta\alpha + \alpha - \beta)\left(\alpha\beta h^2 + \alpha h - s\right)}{h\alpha^2(h\alpha - 1)(h\beta + 1)^2}.$$

As $\gcd(\alpha, \beta) = 1$, we get $\gcd(\alpha^2(h\alpha - 1)(h\beta + 1)^2, h\beta\alpha + \alpha - \beta) = 1$, hence

$$\frac{s\left(\alpha\beta h^2 + \alpha h - s\right)}{\alpha^2(h\alpha - 1)(h\beta + 1)^2} = e, \tag{3.2}$$

for some positive integer e. We consider the Eq. (3.2) as a quadratic in s and find the discriminant

$$\Delta = \alpha^2(h\beta + 1)^2\left(h^2 - 4e\alpha h + 4e\right).$$

Observe that $h^2 - 4e\alpha h + 4e \leq h(h - 4\alpha) + 4$, which is negative for $1 < h < 4\alpha$ and $\alpha \geq 2$. This a contradiction as s is a positive integer.

If $\gcd(h, \alpha - \beta) = 1$, then we get $\gcd(h\alpha^2(h\alpha - 1)(h\beta + 1)^2, h\beta\alpha + \alpha - \beta) = 1$, hence

$$\frac{s\left(\alpha\beta h^2 + \alpha h - s\right)}{h\alpha^2(h\alpha - 1)(h\beta + 1)^2} = e, \tag{3.3}$$

for some positive integer e. As before, we consider the Eq. (3.3) as a quadratic in s and find the discriminant

$$\Delta = h\alpha^2(h\beta + 1)^2(h + e(4 - 4h\alpha)).$$

Observe that $h + e(4 - 4h\alpha) \leq 4 - h(4\alpha - 1) < 0$, as $\alpha \geq 2$ and $h > 1$, which is a contradiction. □

4 QSDs with Symplectic graphs

Let $t \geq 2$ and let q be a prime power. Take a $(2t)$-dimensional vector space V over the field of order q with a non-degenerate symplectic bilinear form $< \cdot, \cdot >$ (such a form is unique up to linear isomorphisms). Let $P(V) = PG(2t - 1, q)$ be the corresponding projective space. For non-zero vectors $x \in V$, let $[x]$ denote the point in $P(V)$ with homogeneous co-ordinates x. The symplectic graph $Sp(2t, q)$ has the points of $PG(2t - 1, q)$ as its vertices. Two points $[x]$, $[y]$ are adjacent in $Sp(2t, q)$ if $< x, y > \neq 0$.

Theorem 4.1 *Let $q > 2$ be a prime power. Then there does not exist a QSD with block graph that has the same parameters as that of the Symplectic graph $Sp(2t, q)$, with $t \geq 2$.*

Proof We show that the Eq. (2.2) have no integer solution and use Corollary 2.3 to complete the proof. The condition $q \equiv 3 \pmod 8$ obtained in the Corollary 2.3 implies q must be an odd prime power. We assume the Eq. (2.2) has integer solutions and rewrite it as follows:

$$\left(q^t - 1\right)^2 - q^t\,(q - 1)\left(q^{t-1} - 1\right) = y^2,$$

where $y = x(q - 1)$. Observe that $q^t\left(q^{t-1} + q - 3\right) = y^2 - 1$, which implies q^t divides either $y - 1$ or $y + 1$, as q is an odd prime power. We take $y = uq^t + 1$ and $y = uq^t - 1$, for positive integer u to observe that

$$\begin{aligned}\left(q^t - 1\right)^2 - q^t\,(q - 1)\left(q^{t-1} - 1\right) - y^2 &= -q^{t-1}\left(-q^t - q^2 + 3q + u^2q^{t+1} + 2uq\right)\\ &\leq -q^{t-1}\left(q^{t+1} - q^t - q^2 + 5q\right)\\ &< 0,\end{aligned}$$

and

$$\begin{aligned}\left(q^t - 1\right)^2 - q^t\,(q - 1)\left(q^{t-1} - 1\right) - y^2 &= -q^{t-1}\left(-q^t + u^2q^{t+1} - q^2 - 2uq + 3q\right)\\ &\leq -(q - 1)q^{t-1}\left(q^t - q\right)\\ &< 0,\end{aligned}$$

respectively, which is a contradiction. □

Theorem 4.2 *There is no QSD whose block graph parameters are*

$$\left(\frac{q^{2t} - 1}{q - 1}, \frac{q\left(q^{2t-2} - 1\right)}{q - 1}, \frac{\left(q^{2t-4} - 1\right)q^2}{q - 1} + q - 1, \frac{\left(q^{2t-4} - 1\right)q^2}{q - 1} + q + 1\right),$$

which are same as that of the complement of Symplectic graph $Sp(2t, q)$, where q is an odd prime power and $t \geq 2$.

Proof We find $\sigma = -q^{t-1} - 1$. We use Eq. (2.1) to find

$$\begin{aligned}\mu &= \frac{\left(q^t + q - 2\right) s \left(q^{2t} - 1 - sq + s\right)}{2q^t \left(q^t - 1\right)^2 \left(q^{t-1} + 1\right)} \\ &= \frac{\left((q^t - 1) + (q - 1)\right) s \left(\left(q^t - 1\right)\left(q^t + 1\right) - (q - 1)s\right)}{2q^t \left(q^t - 1\right)^2 \left(q^{t-1} + 1\right)} \\ &= \frac{(q^{t-1} + q^{t-2} + \cdots + q + 2)((q^{t-1} + q^{t-2} + \cdots + q + 1)(q^t + 1) - s)s}{2q^t (q^{t-1} + q^{t-2} + \cdots + q + 1)^2 \left(q^{t-1} + 1\right)}\end{aligned}$$

Observe the following:

1. $q^t - 1 = (q - 1)(q^{t-1} + q^{t-2} + \cdots + q + 1)$;
2. $q^t + q - 2 = (q^t - 1) + (q - 1) = (q - 1)(q^{t-1} + q^{t-2} + \cdots + q + 2)$;
3. $\gcd(q^{t-1} + q^{t-2} + \cdots + q + 2, q^t) = 1$;
4. $\gcd(q^{t-1} + q^{t-2} + \cdots + q + 2, q^{t-1} + q^{t-2} + \cdots + q + 1) = 1$;
5. $\left(q^{t-1} + q^{t-2} + \cdots + q + 2\right) - \left(q^{t-1} + 1\right) = q^{t-2} + \cdots + q + 1$;
6. $\left(q^{t-1} + 1\right) - (q^{t-2} + \cdots + q + 1)(q - 1) = 2$;
7. $\gcd(q^{t-1} + q^{t-2} + \cdots + q + 2, q^{t-1} + 1) = \begin{cases} 1 & \text{if } t \text{ is an even integer;} \\ 2 & \text{if } t \text{ is an odd integer.} \end{cases}$

As before we get

$$\frac{2((q^{t-1} + q^{t-2} + \cdots + q + 1)(q^t + 1) - s)s}{q^t (q^{t-1} + q^{t-2} + \cdots + q + 1)^2 \left(q^{t-1} + 1\right)} = e,$$

for some positive integer e.

We take $\alpha = q^{t-1} + q^{t-2} + \cdots + q + 1$ and observe that

$$\frac{2(\alpha(q^t + 1) - s)s}{q^t \alpha^2 \left(q^{t-1} + 1\right)} = e,$$

for some positive integer e.

Discriminant of the above quadratic is $4\alpha^2\Delta$, where $\Delta = \left(q^t + 1\right)^2 - 2eq^t \left(q^{t-1} + 1\right)$. Observe that Δ must be a perfect square. We take $\Delta = x^2$ for some positive integer x and, as q is an odd prime, observe that q^t divides either $x - 1$ or $x + 1$.

If $x = q^t u + 1$, for some positive integer u, then $\Delta - x^2 = q^t \delta(u)$, where $\delta(u) = -2eq^{t-1} + q^t - 2e + 2 - u\left(uq^t + 2\right)$, which is a decreasing function of u. Hence $\delta(u) \le \delta(1) = -2e\left(q^{t-1} + 1\right) < 0$, a contradiction.

If $x = q^t u - 1$, for some positive integer u, then $\Delta - x^2 = q^t \delta(u)$, where $\delta(u) = -2eq^{t-1} + q^t - 2e + 2 - u\left(q^t u - 2\right)$, which is a decreasing function of u. Hence $\delta(u) \le \delta(1) = -2\left(eq^{t-1} + e - 2\right) < 0$, a contradiction. □

5 QSDs with pseudo-Latin square and negative Latin square graphs

Given $m - 2$ mutually orthogonal Latin squares of order n with $m - 1 < n$, the vertices of a Latin square graph $LS_m(n)$ are the n^2 cells; two vertices are adjacent if and only if they lie in the same row or column or they have same entry in one of the Latin squares. This graph is a SRG, with parameters $(n^2, m(n-1), n+m(m-3), m(m-1))$. A SRG with these parameters is known as a pseudo-Latin square graph, denoted by $L_m(n)$. A Negative Latin square graph $NL_m(n)$, is obtained by replacing m and n by their negatives in the parameters of a $LS_m(n)$. Hence $NL_m(n)$ has parameters $(n^2, m(n+1), m^2+3m-n, m(m+1))$, where $n \leq m^2+3m$, and equality holds if and only if the Krein bounds are met. In [2], Cameron, Goethals and Seidel characterized SRG's attaining the Krein bounds in terms of Negative Latin square graph $NL_e(e^2+3e)$. In [8], the possibility of QSD's whose block graph is $NL_e(e^2+3e)$, with $2 \leq e$ or its complement was ruled out.

Theorem 5.1 *Suppose $m = h\alpha + 1, n = h\beta$, for positive integers α and β such that* $\gcd(n, m-1) = h$. *If $1 \leq h^2 < 4\alpha$ or* $\gcd(h, \beta-\alpha) = 1$, *then there is no QSD whose block graph is a pseudo-Latin Square graph $L_m(n)$.*

Proof For a pseudo-Latin Square graph $\sigma = -m$. As before, we use Eq. (2.1) to find

$$\mu = \frac{(h\beta\alpha + \beta - \alpha)\left(h^2\beta^2 - s\right)s}{h^2(h\alpha+1)\beta^3(h\beta-1)}.$$

As $\gcd(\alpha, \beta) = 1$, we get the following:

1. $\gcd(h\beta\alpha + \beta - \alpha, \beta^3) = 1$;
2. $\gcd(h\beta\alpha + \beta - \alpha, (h\beta - 1)) = 1$;
3. $\gcd(h\beta\alpha + \beta - \alpha, (h\alpha + 1)) = 1$.

As before we get

$$\frac{\left(h^2\beta^2 - s\right)s}{(h\alpha+1)\beta^3(h\beta-1)} = e,$$

for some positive integer e.

Discriminant of above quadratic is $\beta^3\Delta$, where

$$\begin{aligned}\Delta &= \beta h^4 + e(4 - 4h(h\beta\alpha - \alpha + \beta))\\ &\leq \beta h^4 + (4 - 4h(h\beta\alpha - \alpha + \beta))\\ &= h\left(4(\alpha-\beta) + h\left(h^2 - 4\alpha\right)\beta\right) + 4.\end{aligned}$$

If $1 \leq h^2 < 4\alpha$ then as $\alpha < \beta$, we get $\Delta < 0$, which is contradiction.

If $\gcd(h, \beta - \alpha) = 1$, then $\gcd(h\beta\alpha + \beta - \alpha, h^2(h\alpha+1)\beta^3(h\beta-1)) = 1$. Hence

$$\frac{\left(h^2\beta^2 - s\right)s}{h^2(h\alpha+1)\beta^3(h\beta-1)} = e,$$

for some positive integer e.

Discriminant of above quadratic is $h^2\beta^3\Delta$, where

$$\begin{aligned}\Delta &= \beta h^2 + e\left(-4\alpha\beta h^2 + 4(\alpha-\beta)h + 4\right)\\ &\leq \beta h^2 + \left(-4\alpha\beta h^2 + 4(\alpha-\beta)h + 4\right)\\ &= -(4\alpha-1)\beta h^2 - 4(\beta-\alpha)h + 4.\end{aligned}$$

This implies $\Delta < 0$, which is a contradiction. □

Theorem 5.2 *Suppose $m = h\alpha, n = h\beta$ for positive integers α and β such that* $\gcd(n, m) = h$. *If $h^2 < 4(\beta-\alpha)$ or* $\gcd(h, \alpha) = 1$, *then there is no QSD whose block graph is a negative Latin Square graph $NL_m(n)$.*

Proof For a negative Latin Square graph $\sigma = -(n-m)$. As before, we use Eq. (2.1) to find

$$\mu = \frac{\left(h\beta^2 - h\alpha\beta - \alpha\right)\left(h^2\beta^2 - s\right)s}{h^2\beta^3(h\beta+1)(-h\alpha+h\beta-1)}.$$

We observe the following:

1. As $\gcd(\alpha, \beta) = 1$, $\gcd(h\beta^2 - h\alpha\beta - \alpha, \beta^3) = 1$;
2. $h\left(h\beta^2 - h\alpha\beta - \alpha\right) - (h\beta+1)(-h\alpha+h\beta-1) = 1$;
3. $\gcd(h\beta^2 - h\alpha\beta - \alpha, (h\beta+1)(-h\alpha+h\beta-1)) = 1$.

As before we have

$$\frac{s\left(h^2\beta^2 - s\right)}{\beta^3(h\beta+1)(-h\alpha+h\beta-1)} = e,$$

for some positive integer e.

Discriminant of above quadratic is $\beta^3\Delta$, where

$$\begin{aligned}\Delta &= 4(\alpha-\beta)\beta h^2 + \beta h^2 + 4(h\alpha+1)\\ &\leq \left(h^2 - 4(\beta-\alpha)\right)\beta h^2 + 4(h\alpha+1)\end{aligned}$$

If $1 < h^2 < 4(\beta - \alpha)$ then as $\alpha < \beta$ we get $\Delta < 0$, which is contradiction.

As $\gcd(h, \alpha) = 1$, $\gcd(h\beta^2 - h\alpha\beta - \alpha, h^2) = 1$ we get

$$\frac{\left(h^2\beta^2 - s\right)s}{h^2\beta^3(h\beta+1)(-h\alpha+h\beta-1)} = e;$$

for some positive integer e. The discriminant of this quadratic in s is $h^2\beta^3\Delta$, where

$$\begin{aligned}\Delta &= \beta h^2 + e\left(4(\alpha-\beta)\beta h^2 + 4(h\alpha+1)\right)\\ &\leq 4(\alpha-\beta)\beta h^2 + (4\alpha + h\beta)h + 4.\end{aligned}$$

As $h(\beta-\alpha) \geq 2$, we consider two cases (i) $h = 1$ and $\beta \geq \alpha+2$; (ii) $h \geq 2$ and $\beta \geq \alpha+1$ to observe that $\Delta < 0$, which is a contradiction. □

Remark 5.3 Results similar to the Theorems 5.1 and 5.2 can be obtained for complements of pseudo-Latin square and negative Latin square graphs as these are also pseudo-Latin square and negative Latin square graphs respectively.

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