Designs, Codes and Cryptography (2022) 90:2753–2762
https://doi.org/10.1007/s10623-021-00941-0

On Pless symmetry codes, ternary QR codes, and related Hadamard matrices and designs

Vladimir D. Tonchev[1]

Received: 22 February 2021 / Revised: 28 June 2021 / Accepted: 28 August 2021 /
Published online: 11 November 2021

Abstract
It is proved that a code $L(q)$ which is monomially equivalent to the Pless symmetry code $C(q)$ of length $2q + 2$ contains the (0,1)-incidence matrix of a Hadamard 3-$(2q + 2, q + 1, (q - 1)/2)$ design $D(q)$ associated with a Paley–Hadamard matrix of type II. Similarly, any ternary extended quadratic residue code contains the incidence matrix of a Hadamard 3-design associated with a Paley–Hadamard matrix of type I. If $q = 5, 11, 17, 23$, then the full permutation automorphism group of $L(q)$ coincides with the full automorphism group of $D(q)$, and a similar result holds for the ternary extended quadratic residue codes of lengths 24 and 48. All Hadamard matrices of order 36 formed by codewords of the Pless symmetry code $C(17)$ are enumerated and classified up to equivalence. There are two equivalence classes of such matrices: the Paley–Hadamard matrix H of type I with a full automorphism group of order 19584, and a second regular Hadamard matrix H' such that the symmetric 2-(36, 15, 6) design D associated with H' has trivial full automorphism group, and the incidence matrix of D spans a ternary code equivalent to $C(17)$.

Keywords Pless symmetry code · Hadamard matrix · Hadamard 3-design · Hadamard 2-design · Paley–Hadamard matrix

Mathematics Subject Classification 05B05 · 05B20 · 94B05

1 Introduction

We assume familiarity with the basic facts and notions from error-correcting codes and combinatorial designs and Hadamard matrices [1,4,9,12]. All codes in this paper are ternary. A monomial matrix with entries from $GF(3)$ is a square matrix such that every row and every column contains exactly one nonzero entry. By an automorphism group of a ternary

This is one of several papers published in *Designs, Codes and Cryptography* comprising the "Special Issue: On Coding Theory and Combinatorics: In Memory of Vera Pless".

✉ Vladimir D. Tonchev
tonchev@mtu.edu

[1] Department of Mathematical Sciences, Michigan Technological University, Houghton 49931, USA

code we mean *monomial* automorphism group, unless specified otherwise. The *permutation* automorphism group of a code is the subgroup of its monomial automorphism group that consists of coordinate permutations only.

A Hadamard matrix of order n is an $n \times n$ matrix H of 1's and -1's such that $HH^T = nI$, where I is the identity matrix. It follows that $n = 1, 2$, or $n = 4t$ for some integer $t \geq 1$. An automorphism of a Hadamard matrix H is a pair of $\{0, 1, -1\}$-monomial matrices L, R such that $LHR = H$. Two Hadamard matrices H_1, H_2 of the same order are *equivalent* if there are monomial matrices L, R such that $LH_1R = H_2$. A Hadamard matrix H is *normalized* with respect to its ith row and jth column if all entries in row i and column j are equal to 1. If H is a Hadamard matrix of order $n = 4t$ that is normalized with respect to row i and column j, deleting the ith row and the jth column and replacing all -1's with zeros gives the $(0, 1)$-incidence matrix of a symmetric 2-$(4t - 1, 2t - 1, t - 1)$ design D called a Hadamard 2-design, while deleting the jth column of H and the jth column of $-H$ from the matrix $(H, -H)$ gives the point-by-block (± 1)-incidence matrix of a 3-$(4t, 2t, t - 1)$ design D^*, called a Hadamard 3-design obtained from H with respect to column j. The design D is the *derived* design of D^* with respect to its ith point. A Hadamard matrix H of order $n = 4t$ is *regular* if all rows of H contain the same number k of -1's. It follows that $t = m^2$ for some integer m, $k = 2m^2 \pm m$, and replacing all -1's with zeros gives the $(0, 1)$-incidence matrix of a symmetric 2-$(4m^2, 2m^2 \pm t, m^2 \pm m)$ design. For more on Hadamard matrices and related designs, see, for example, [1, Chapter 7], [9, Chapter 14], [12, Sect. 8.9].

Let q be an odd prime power such that $q \equiv -1 \pmod 3$. The Pless symmetry code $C(q)$ [22,23] of length $n = 2q + 2$ is a ternary self-dual code with a generator matrix

$$G = (I_{q+1}, S_q), \tag{1}$$

where I_{q+1} is the identity matrix of order $q + 1$, and $S_q = (s_{i,j})$ is a $(q + 1) \times (q + 1)$ matrix defined as follows. The rows and columns of S_q are labeled by ∞ and the q elements of the finite field $GF(q)$ of order q, where $s_{\infty,\infty} = 0$, $s_{a,a} = 0$, $s_{\infty,a} = 1$ for $a \in GF(q)$, $s_{a,\infty} = 1$ if -1 is a square in $GF(q)$, and $s_{a,\infty} = -1$ if -1 is not a square in $GF(q)$ for $a \in GF(q)$, and $s_{a,b} = 1$ for $a, b \in GF(q)$ such that $a \neq b$ and $a - b$ is a square in $GF(q)$, and $s_{a,b} = -1$ for $a, b \in GF(q)$ such that $a \neq b$ and $a - b$ is not a square in $GF(q)$. For example, if $q = 5$, the rows and columns of S_5 are labeled by ∞, 0, 1, 2, 3, 4, and

$$S_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

The main property of S_q is that $S_q S_q^T = qI_{q+1}$, which implies $S_q S_q^T \equiv -I_{q+1} \pmod 3$; hence $C(q)$ is self-dual. The symmetry codes $C(5)$[1] , $C(11)$, $C(17)$, $C(23)$ and $C(29)$ are extremal self-dual codes of length n divisible by 12 and minimum distance d meeting the Mallows-Sloane upper bound $d \leq 3[n/12] + 3$ [16]; hence these codes support 5-designs by the Assmus–Mattson Theorem [3].

Pless [23] proved that in addition to the trivial monomial automorphism of order 2 corresponding to the negation of all code coordinates, the monomial automorphism group of $C(q)$ contains a subgroup of order $q(q^2 - 1)$ isomorphic to $PGL_2(q)$. In addition, it was proved in [23, Theorem 4.2] that the symmetry code $C(q)$ contains a set of $2q + 2$ codewords of

[1] The symmetry code for $q = 5$ is equivalent to the extended ternary Golay code.

weight $2q+2$ that form a Hadamard matrix of order $2q+2$. Specifically, if $q \equiv 1 \pmod 4$ the Hadamard matrix formed by codewords of weight $2q+2$ is

$$H_1(q) = \begin{pmatrix} I + S_q & -I + S_q \\ -I + S_q & -I - S_q \end{pmatrix}, \tag{2}$$

while if $q \equiv -3 \pmod 4$ the Hadamard matrix is

$$H_3(q) = \begin{pmatrix} I + S_q & I + S_q \\ I - S_q & -I + S_q \end{pmatrix}, \tag{3}$$

where I is the identity matrix of order $q+1$.

The Hadamard matrices (2), (3) are known in the combinatorial literature as Paley–Hadamard matrices of type II [7,9,21, 14.1]. If $q \equiv 3 \pmod 4$, the $(q+1) \times (q+1)$ matrix obtained by bordering the matrix $S_q - I$ with one all-one row and one all-one column is a Hadamard matrix of order $q+1$, known as a Paley–Hadamard matrix, or a Paley–Hadamard matrix of type I [9, 14.1], [7,21]. The unique (up to equivalence) Hadamard matrix of order 12 is both a Paley–Hadamard matrix of type I for $q = 11$ and a Paley–Hadamard matrix of type II for $q = 5$, and its full automorphism group modulo its center of order 2 is the Mathieu group M_{12} (Hall [8]). The full automorphism group of a Paley–Hadamard matrix of type I for $q > 11$ was determined by Kantor [13] and is of order $q(q^2-1)$, while the full automorphism group of a Paley–Hadamard matrix of type II for $q > 5$ was determined by de Launey and Stafford [7], and is of order $4fq(q^2-1)$ if $q = p^f$, where p is prime.

If $q = 5$, 11, or 23, the number of all codewords of full weight $2q+2$ in the symmetry code $C(q)$ is exactly $4q+4$ [23]. These codewords span the code and consist of the rows of the Hadamard matrix $H_1(q)$ from (2) (resp. $H_3(q)$ from (3)) and its negative, or $2H_1(q)$ (resp. $2H_2(q)$); hence the full monomial automorphism group of $C(q)$ coincides with the full automorphism group of $H_1(q)$ (resp. $H_3(q)$) [23].

The symmetry code $C(17)$ of length 36 contains 888 codewords of weight 36; hence it is not clear whether the automorphism group of the Hadamard matrix $H_1(17)$ (2) is the full automorphism group of $C(17)$. In Sect. 2, we prove that the full automorphism group of $C(17)$ coincides with the full automorphism group of $H_1(17)$, the latter being a Paley–Hadamard matrix of type II; hence its order is $4 \cdot 17(17^2-1) = 19584$. In addition, we classify all Hadamard matrices of order 36 having as rows codewords of $C(17)$ of weight 36, and show that up to equivalence, there are exactly two such matrices: $H_1(17)$, and a second Hadamard matrix H' having the property that all Hadamard 3-(36, 18, 8) designs associated with H' are isomorphic and have trivial full automorphism group of order 1. The full automorphism group of H' is of order 72 and is transitive on the set of 72 rows (as well as the set of 72 columns) of H' and $-H'$. The 3-rank of H' is 18; thus $C(17)$ is the row space of H'. The Hadamard matrix H' is regular, and the symmetric 2-(36, 15, 6) design with ± 1-incidence matrix H' has trivial full automorphism group.

In Sect. 3 we discuss Paley–Hadamard matrices of Type I and Hadamard 3-designs arising from extended ternary quadratic residue codes.

2 Hadamard matrices and designs arising from symmetry codes

The sum (over $GF(3)$) of all rows of the generator matrix (1) of the symmetry code $C(q)$ is a vector v of full Hamming weight $2q+2$, with all components equal to 1 if -1 is not a square in $GF(q)$, (that is, v is the constant all-one vector $\bar{1} = (1, \ldots, 1)$), and v has $2q+1$

components equal to 1, and the position labeled by ∞ is equal to -1 whenever -1 is a square in $GF(q)$.

Next, we consider a code which is monomially equivalent to the Pless symmetry code $C(q)$, and always contains the all-one vector, namely the code $L(q)$ with a generator matrix G' given by

$$G' = (I_{q+1}, U_q), \tag{4}$$

where U_q is a $(q+1) \times (q+1)$ matrix obtained from S_q by replacing every nonzero entry in the column labeled by ∞ with -1. Clearly, the generator matrix G' (4) is identical with the generator matrix G (1) if -1 is not a square in $GF(q)$, and is obtained by negating one column of G if -1 is a square in $GF(q)$. Thus, the matrices (1) and (4) generate monomially equivalent ternary codes.

Using (4), we obtain a parity check P matrix of $L(q)$ given by

$$P = (-U_q^T, I_{q+1}). \tag{5}$$

Note that since $L(q)$ is self-dual, the rows of P are codewords of $L(q)$. It is easy to check that the matrix H given by

$$H = \begin{pmatrix} G' + P \\ G' - P \end{pmatrix} = \begin{pmatrix} I_{q+1} - U_q^T & U_q + I_{q+1} \\ I_{q+1} + U_q^T & U_q - I_{q+1} \end{pmatrix} \tag{6}$$

is a Hadamard matrix of order $2q+2$, with rows being codewords of $L(q)$.

Theorem 2.1 *The code $L(q)$ contains a set of $4q+2$ (0,1)-codewords of weight $q+1$ that form the block-by-point incidence matrix of a Hadamard 3-$(2q+2, q+1, (q-1)/2)$ design $D(q)$ associated with a Paley–Hadamard matrix of type II.*

Proof All entries in the first row of the Hadamard matrix H (6) are equal to 1; that is, H is normalized with respect to its first row, and consequently, all entries in the first row of $-H$ are equal to -1. Adding the constant codeword $\bar{2} = (2, \ldots, 2)$ with all entries equal to 2 to every row of the matrix

$$\begin{pmatrix} H \\ -H \end{pmatrix}$$

gives a (0, 1)-matrix M with all-zero first row, and all-one row labeled by the first row of $-H$. Deleting the all-zero row and the all-one row from M gives a $(4q+2) \times (2q+2)$ (0, 1)-matrix A, being the block-by-point incidence matrix of a Hadamard 3-$(2q+2, q+1, (q-1)/2)$ design associated with the first row of H. Clearly, H is equivalent to the corresponding matrix (2) or (3); hence H is equivalent to a Paley–Hadamard matrix of type II. □

Theorem 2.2 *If $q = 5, 11, 17, 23$, the code $L(q)$ contains exactly $4q+2$ (0,1)-codewords of weight $q+1$, and every such codeword is the incidence vector of a block of the Hadamard 3-design $D(q)$ from Theorem 2.1.*

Proof Let m denote the total number of (0,1)-codewords of weight $q+1$ in $L(q)$. By Theorem 2.1, $m \geq 4q+2$. If $v \in L(q)$ is a (0,1)-codeword of weight $q+1$ then $v + \bar{1}$ is a codeword of full weight $2q+2$, having $q+1$ components equal to 1, and $q+1$ components equal to 2. Adding the codewords $\bar{1}$ and $\bar{2} = 2 \cdot \bar{1}$ gives $m+2 \geq 4q+4$ codewords of weight $2q+2$. Since $C(q)$ contains exactly $4q+4$ codewords of weight $2q+2$ for $q = 5, 11, 23$ [23], the statement is true in these cases.

The case $q = 17$ needs additional analysis because the symmetry code $C(17)$, as well as its equivalent code $L(17)$, contains 888 codewords of weight 36 [23]. The set of all codewords of weight 36 is easily computed with Magma [5]. This set comprises of the following codewords:

- the 36 rows of the Hadamard matrix H (6), one of the rows being $\bar{1}$, and 35 rows with 18 components equal to 1, and 18 components equal to -1 (note that $-1 \equiv 2 \pmod 3$);
- the 36 rows of $2H$ that include $\bar{2}$ and 35 rows with 18 components equal to 1, and 18 components equal to 2;
- a set T of 408 codewords having 15 components equal to 1 and 21 components equal to 2;
- a set $2T$ of 408 codewords obtained by multiplying every codeword from T by 2.

Note that adding $\bar{2}$ to any (0, 1)-codeword of weight 18 gives a codeword of weight 36 with 18 1's and 18 2's; hence the code $L(17)$ contains exactly 70 (0, 1)-codewords of weight 18 obtained by adding the codeword $\bar{2}$ to the rows of H and $2H$, and these 70 (0, 1)-codewords form the incidence matrix of the 3-design $D(17)$ from Theorem 2.1. □

Note 1 *The code $L(29)$ contains 19606 (0,1)-codewords of weight 30. It is an open question whether this set contains the incidence matrices of any Hadamard 3-$(60, 30, 14)$ designs that are not isomorphic to $D(29)$. The number of codewords of weight 60 in $L(29)$ is 41184. It seems likely that there may be Hadamard matrices of order 60 formed by codewords of weight 60 that are not equivalent to the Paley–Hadamard matrix of type II.*

Corollary 2.3 *If $q = 5, 11, 17, 23$, the full permutation automorphism group of $L(q)$ coincides with the full automorphism group of the Hadamard 3-design $D(q)$ from Theorem 2.1.*

Proof The results of De Launey and Stafford [7] and Norman [20] imply that the full automorphism group of $D(q)$ has order $q(q-1)$ if $q > 5$ is prime. The full automorphism group of $D(5)$ has order 7920.

Any derived design with respect to a point of a Hadamard 3-$(2q+2, q+1, (q-1)/2)$ design D is a symmetric Hadamard 2-$(2q+1, q, (q-1)/2)$ design D' of order $q-(q-1)/2 = (q+1)/2$. Since $q \equiv -1 \pmod 3$, 3 divides $(q+1)/2$. If 9 does not divide the order $(q+1)/2$ (which is true if $q = 5$, 11, or 23), the rank of the incidence matrix if D' over $GF(3)$ (or the 3-rank of D') is equal to $q+1$ (see, for example, Assmus and Key [1,2]), hence the 3-rank of D is $q+1$ and the code $L(q)$ is spanned by the incidence matrix of D. If $q = 17$, a direct computation shows that the 3-rank of $D(17)$ is 18, hence $D(17)$ spans the code $L(17)$. □

Theorem 2.4 *(i) The code $L(17)$ contains two equivalence classes of Hadamard matrices of order 36 having as rows codewords of weight 36, with representatives the Hadamard matrix H (6), which is equivalent to a Paley–Hadamard matrix of type II and has full automorphism group of order* 19584*, and a second Hadamard matrix H', being a regular Hadamard matrix such that the symmetric 2-$(36, 15, 6)$ design D' with $(0, 1)$-incidence matrix $(H' + J)/2$, where J is the 36×36 all-one matrix, has a trivial automorphism group.*

(ii) The row span of the incidence matrix of the 2-$(36, 15, 6)$ design D' is an extremal ternary [36, 18, 12] *code equivalent to the symmetry code $C(17)$.*

(iii) The full automorphism group of the code $L(17)$ coincides with the full automorphism group H.

Proof (i) In the context of Hadamard matrices, we consider the element 2 of $GF(3)$ as -1. Using the notation from the proof of Theorem 2.2, we define a graph Γ having as vertices the

408 codewords from T, where two codewords $u, v \in T$ are adjacent in Γ if and only if the Hamming distance between u and v is 18, or equivalently, the intersection of the supports of the (0,1)-vectors $\bar{2} - u$ and $\bar{2} - v$ is of size 6. Replacing all entries equal to 2 by zero in every vector from T gives a set $T(0, 1)$ of (0,1)-vectors of weight 15. Using the restricted Johnson bound, it is easy to verify that the maximum number codewords in a binary constant weight code of length 36 with codewords of weight 15 and minimum distance 18, is 36. Every set K of 36 vectors from $T(0, 1)$ that meets the Johnson bound corresponds to a clique of size 36 in Γ, and the 36×36 matrix having as rows the vectors from K is the incidence matrix N of a symmetric 2-(36, 15, 6) design (see [30, Theorem 2.4.12, p. 99] or [31, Sect. 3]). Replacing all zeros in N with -1's gives a regular Hadamard matrix of order 36. Using the clique finding algorithm Cliquer [19], a quick computer search shows that the graph Γ contains exactly 272 cliques of size 36, or in other words, there are 272 collections of 36 codewords from T that form a Hadamard matrix of order 36. Further analysis with Magma shows that all 272 Hadamard matrices are equivalent to a matrix H' with a full monomial automorphism group of order 72 that acts transitively on the set of size 72 being the union of the rows of H' and the rows of $-H'$.

The incidence matrix of the symmetric 2-(36, 15, 6) design D' obtained by replacing all -1-entries of H' with zeros is listed in the Appendix. The design D' has a trivial full automorphism group of order 1.

(ii) The 3-rank of the incidence matrix of D' is 18, and its row span over $GF(3)$ is a ternary [36, 18, 12] code equivalent to the Pless symmetry code.

Parts (iii) was verified by computer using Magma. The full automorphism group of $L(17)$ partitions the set of the 888 codewords of weight 36 into two orbits, of length 72 and 816 respectively, the orbit of length 72 comprised of the rows of H (6) and $-H$. Thus, the full automorphism group of the code $L(17)$ coincides with the full automorphism group of H, and is of order[2] $19584 = 2^7 \cdot 3^2 \cdot 17$. □

Note 2 Up to equivalence, there are exactly 11 Hadamard matrices of order 36 with automorphism groups of order divisible by 17 (Tonchev [28]). Each of these matrices spans a ternary self-dual code of length 36, but only the symmetry code $C(17)$ spanned the Paley–Hadamard matrix of type II is extremal, that is, has minimum distance 12, and supports 5-designs. A stronger characterization of the Pless symmetry code $C(17)$ was proved by Huffman [11], namely that up to equivalence, $C(17)$ is the only extremal ternary self-dual code of length 36 that admits a monomial automorphism of order 17.

Note 3 Hadamard matrices and designs are used for the construction of self-orthogonal and self-dual codes over other finite fields. A classical example is the extended binary Golay code generated by a bordered incidence matrix of a symmetric Hadamard 2-(23, 11, 5) design associated with a Paley–Hadamard matrix of type I. Hadamard matrices of order 28 with an automorphism of order 7 [27] were used by Pless and Tonchev [24] for the classification of self-orthogonal codes over $GF(7)$. The Paley–Hadamard matrix of type II of order 28 is the only Hadamard matrix of this order that admits an automorphism of order 13 and yields an extremal binary self-dual code of length 56 [26,29]. More extremal binary self-dual codes derived from Hadamard matrices of order 28 were found in [6].

[2] This is the order of the Paley–Hadamard matrix of Type II for $q = 17$ [7].

3 Hadamard matrices and designs arising from ternary QR codes

The symmetry codes $C(11)$, $C(23)$, and $C(29)$ have siblings with the same parameters and weight distribution, being ternary extended quadratic-residue codes that support 5-designs by the Assmus-Mattson theorem. If $q \equiv 3 \pmod 4$ is a prime power, a quadratic residue (QR) code of length q is a code spanned by the (0,1)-incidence matrix A of a symmetric Hadamard 2-$(q, (q-1)/2, (q-3)/4)$ design obtained from the Paley–Hadamard matrix of type I, and its extended code is spanned by a matrix obtained by adding one all-one column to A. If, in addition, $q \equiv -1 \pmod 3$, that is, q is of the form $q = 12s + 11$ for some integer $s \geq 0$, the ternary extended QR code is self-dual.

Theorem 3.1 *Let $q = 12s + 11$ be a prime power, and let QR_q be the ternary extended QR code of length $q + 1$.*

(i) QR_q contains a Paley–Hadamard matrix of type I having as rows codewords of weight $q + 1$.

(ii) QR_q contains a set of $2q$ (0,1)-codewords of weight $(q+1)/2$ that form the incidence matrix of a Hadamard 3-$(q + 1, (q + 1)/2, (q - 3)/4)$ design associated with the Paley–Hadamard matrix of type I of order $q + 1$.

(iii) If $q = 11$, 23 or 47, QR_q contains exactly $2q$ (0,1)-codewords of weight $(q + 1)/2$, and the permutation automorphism group of the code coincides with the full automorphism group of the Hadamard 3-$(q + 1, (q + 1)/2, (q - 3)/4)$ design from part (ii).

Proof (i) The statement (i) is implicit in [3]. The column sum of the (0,1)-incidence A of the Hadamard 2-$(q, (q-1)/2, (q-3)/4)$ design obtained from the Paley–Hadamard matrix of type I is

$$(q-1)/2 = (12s + 10)/2 \equiv -1 \pmod 3,$$

and the sum of all q entries of the all-one column is $q \equiv -1 \pmod 3$, hence the sum over $GF(3)$ of all rows of the $q \times (q+1)$-matrix B obtained by bordering A with one all-one column, is equal to the constant vector $\bar{2}$. Since QR_q is the row span of B, the constant vectors $\bar{1}$ and $\bar{2}$ belong to the code. Let E be the $(q+1) \times (q+1)$ matrix obtained from B by adding one extra all-one row. The matrix $H_{q+1} = 2J - E$, where J is the $(q+1) \times (q+1)$ all-one matrix, is a Paley–Hadamard matrix of type I. Every row of H_{q+1} is the difference of the codeword $\bar{2}$ and a row of B, hence the rows of H_{q+1} belong to the code QR_q.

(ii) Adding the codeword $\bar{2}$ to every row of

$$\begin{pmatrix} H_{q+1} \\ -H_{q+1} \end{pmatrix} \tag{7}$$

gives a $(2q+2) \times (q+1)$ matrix with one all-zero row, one all-one row, and $2q$ (0,1)-rows of weight $(q+1)/2$ that form the incidence matrix of a Hadamard 3-$(q+1, (q+1)/2, (q-3)/4)$ design associated with H_{q+1}.

(iii) The proof is similar to that of Corollary 2.3. □

Note 4 The number of codewords of weight 60 in QR_{59} is 41184. It seems likely that there may be Hadamard matrices of order 60 formed by codewords of weight 60 that are not equivalent to the Paley–Hadamard matrux of type I from Theorem 3.1.

4 Concluding remarks

The extended ternary Golay code of length 12, the Pless symmetry codes $C(q)$ ($q = 11, 17, 23$ and 29), of lengths 24, 36, 48 and 60, the extended ternary QR codes of lengths 24, 48 and 60, and an extremal code of length 60 discovered by Nebe and Villard [18] as an analogue of the Pless symmetry code $C(29)$, are the only known extremal ternary self-dual codes of length divisible by 12 that support 5-designs. It is known that the symmetry code of length 84 ($q = 41$), as well as the extended QR code of this length are not extremal. Extremal ternary self-dual codes of length n divisible by 12 do not exist for $n = 72, 96, 120$, and all $n \geq 144$, because then the extremal Hamming weight enumerator contains a negative coefficient [25].

All ternary self-dual codes of length 24 have been classified up to equivalence (Harada and Munemasa [10]), and the symmetry code $C(11)$ and the extended QR code are the only extremal codes of this length. Nine of the self-dual ternary codes of length 24 are spanned by Hadamard matrices of order 24 [14,15], but only two codes, QR_{23} and $C(11)$, that are spanned by the Paley–Hadamard matrices of type I and II respectively, are extremal.

It is an interesting open question whether the Pless symmetry codes of length 36, 48, and 60, the extended QR codes of lengths 48 and 60, and the extremal code of length 60 found by Nebe and Villard [18] are the only extremal self-dual codes of these lengths. The results from Sect. 2 show that the symmetry code of length 36 can be obtained from a Hadamard matrix that is not a Paley–Hadamard matrix of type II, and a natural question that arises is whether any other extremal codes of length 36, 48, or 60 can be obtained form Hadamard matrices that are not of Paley type.

The extremal ternary self-dual codes of lengths $n \geq 36$ have not been classified up to equivalence. A partial classification of such codes of length $n \leq 40$ admitting automorphisms of prime order $p \geq 5$ was given by Huffman [11]. In addition, it was proved by Nebe [17] that, up to equivalence, the only extremal ternary self-dual codes of length 48 that admit an automorphism of a prime order $p \geq 5$, are the Pless symmetry code and the extended QR code.

Acknowledgements The author thanks Cary Huffman for reading a preliminary version of this paper and making several useful suggestions that led to an improvement of the text.

Appendix

110110001101001100001010100000110001
000010101000001100100111001011001101
011000100010001100011001110011010010
010010000000010011110100110010110101
001011110100100010011111100000000001
011111000110011000000110000110000110
000000001111110000110110100001011010
000010010111101001000000111111000001
010001000101111110010001001001100100
100011000110000110111000001110011000

000111111000000000010000101101110110
000111101011110000101001010010100000
010100110010101110100100100100101000
100101000000101001011100000011101011
101001001001001010101100110101000100
001001100011001011100010001000110011
010001111100011001001100011000011000
100100010010011001111011101000000100
100010101000111111010010010100000010
000100010001100110001110011010010110
111110001010100011000101001001010000
001010000001010101001101101100101010
111001011000110100100000101010000011
110100100101000000110101011100000011
000101001110000111000001110000001111
101000111111000101010100000010100100
010000011100000011101011000111100010
001100100100110101101000000101010101
001100011001011010010001000110011001
011000001010100000011010011100101101
100001010010010100000111010101110001
101010010100101000100001010000111110
011111010001000101110010010001001000
101100100100010010000010111011101000
110010110011010010001000000001001111
110001100001100001000011100110011100

A 2-(36, 15, 6) design associated with the Pless symmetry code of length 36

References

1. Assmus E.F. Jr., Key J.D.: Designs and Their Codes. Cambridge University Press, Cambridge (1992).
2. Assmus E.F. Jr., Key J.D.: Hadamard matrices and their designs: a coding-theoretic approach. Trans. Am. Math. Soc. **330**(1), 269–293 (1992).
3. Assmus E.F. Jr., Mattson H.F. Jr.: New 5-designs. J. Combin. Theory Ser. A **6**, 122–151 (1969).
4. Beth T., Jungnickel D., Lenz H.: Design Theory, 2nd edn Cambridge University Press, Cambridge (1999).
5. Bosma W., Cannon J.: Handbook of Magma Functions. Department of Mathematics, University of Sydney, Sydney (1994).
6. Bussemaker F.C., Tonchev V.D.: New extremal doubly-even codes of length 56 derived from Hadamard matrices of order 28. Discret. Math. **76**, 45–49 (1989).
7. de Launey W., Stafford R.M.: On the automorphisms of Palye's type II Hadamard matrix. Discret. Math. **308**, 2910–2924 (2008).
8. Hall M. Jr.: Note on the Mathieu group M_{12}. Arch. Math. **13**, 334–340 (1962).
9. Hall M. Jr.: Combinatorial Theory. Wiley, New York (1986).

10. Harada M., Munemasa A.: A complete classification of ternary self-dual codes of length 24. J. Combin. Theory Ser. A **116**, 1063–1072 (2009).
11. Huffman W.C.: On extremal self-dual ternary codes of lengths 28 to 40. IEEE Trans. Inf. Theory **38**(4), 1395–1400 (1992).
12. Huffman W.C., Pless V.: Fundamentals of Error-Correcting Codes. Cambridge University Press, Cambridge (2003).
13. Kantor W.M.: Automorphism groups of Hadamard matrices. J. Combin. Theory Ser. A **6**, 279–281 (1969).
14. Lam C.W.H., Thiel L., Pautasso A.: On ternary codes generated by Hadamard matrices of order 24. Congr. Num. **89**, 7–14 (1992).
15. Leon J.S., Pless V., Sloane N.J.A.: On ternary self-dual codes of length 24. IEEE Trans. Inf. Theory **27**(2), 176–180 (1981).
16. Mallows C.L., Sloane N.J.A.: An upper bound for self-dual codes. Inf. Control **22**, 188–200 (1973).
17. Nebe G.: On extremal self-dual ternary codes of length 48, Int. J. Combinat. https://doi.org/10.1155/2012/154281 (2012).
18. Nebe G., Villar D.: An analogue of the Pless symmetry codes. In: Seventh International Workshop on Optimal Codes and Related Topics, Albena, Bulgaria pp. 158–163 (2013).
19. Niskanen S., Östergård P.R.J.: Cliquer User's Guide, Version 1.0. Tech. Rep. T48, Communications Laboratory, Helsinki University of Technology, Espoo, Finland (2003).
20. Norman C.W.: Nonisomorphic Hadamard designs. J. Combin. Theory Ser. A **21**, 336–344 (1976).
21. Paley R.E.A.C.: On orthogonal matrices. J. Math. Phys. **12**, 311–320 (1933).
22. Pless V.: On a new family of symmetry codes and related new five-designs. Bull. Am. Math. Soc. **75**(6), 1339–1342 (1969).
23. Pless V.: Symmetry codes over $GF(3)$ and new five-designs. J. Combin. Theory Ser. A **12**, 119–142 (1972).
24. Pless V., Tonchev V.D.: Self-dual codes over $GF(7)$. IEEE Trans. Info. Theory **33**, 723–727 (1987).
25. Rains E.M., Sloane N.J.: Self-dual codes. In: Pless V.S., Huffman W.C. (eds.) Handbook of Coding Theory. Elsevier, Amsterdam (1998).
26. Tonchev V.D.: Hadamard matrices of order 28 with an automorphism of order 13. J. Combin. Theory Ser. A **35**, 43–57 (1983).
27. Tonchev V.D.: Hadamard matrices of order 28 with an automorphism of order 7. J. Combin. Theory Ser. A **40**, 62–81 (1985).
28. Tonchev V.D.: Hadamard matrices of order 36 with automorphisms of order 17. Nogoya Math. J. **104**, 163–174 (1986).
29. Tonchev V.D.: Self-orthogonal designs and extremal dobly-even codes. J. Combin. Theory Ser. A **52**, 197–205 (1989).
30. Tonchev V.D.: Combinatorial Configurations. Wiley, New York (1998).
31. Tonchev V.D.: Codes and designs. In: Pless V.S., Huffman W.C. (eds.) Handbook of Coding Theory, pp. 1229–1268. Elsevier, Amsterdam (1998).