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Nonexistence of perfect permutation codes under the Kendall τ-metric

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Abstract
In the rank modulation scheme for flash memories, permutation codes have been studied. In this paper, we study perfect permutation codes in S_n, the set of all permutations on n elements, under the Kendall τ-metric. We answer one open problem proposed by Buzaglo and Etzion. That is, proving the nonexistence of perfect codes in S_n, under the Kendall τ-metric, for more values of n. Specifically, we present the polynomial representation of the size of a ball in S_n under the Kendall τ-metric for some radius r, and obtain some sufficient conditions of the nonexistence of perfect permutation codes. Further, we prove that there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric for some n and $t = 2, 3, 4, 5,$ or $\frac{5}{8}\binom{n}{2} < 2t + 1 \leq \binom{n}{2}$.

Keywords Flash memory · Permutation codes · Kendall τ-metric · Perfect codes

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1 Introduction

Flash memory is a non-volatile storage medium that is both electrically programmable and erasable. The rank modulation scheme for flash memories has been proposed in [7]. In this scheme, a permutation corresponds to a relative ranking of all the flash memory cells' levels. A permutation code is a nonempty subset of S_n, where S_n is the set of all the permutations over $\{1, 2, \ldots, n\}$. Permutation codes have been studied under various metrics, such as the ℓ_∞-metric [9,13,15], the Ulam metric [4], and the Kendall τ-metric [1,8,12,16].

In this paper, we will focus on permutation codes under the Kendall τ-metric. The *Kendall* τ*-distance* [15] between two permutations $\pi, \sigma \in S_n$ is the minimum number of adjacent transpositions required to obtain the permutation σ from π, where an adjacent transposition is an exchange of two distinct adjacent elements. Permutation codes under the Kendall τ-distance with minimum distance d can correct up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors. Let $A_K(n, d)$ be the maximum size of a permutation code in S_n with minimum Kendall τ-distance at least d. The bounds on $A_K(n, d)$ were proposed in [2,8,11,14]. Some t-error-correcting codes in S_n were constructed in [1,6,8,17,18]. Buzaglo and Etzion [2] proved that there does not exist a perfect single-error-correcting code in S_n, where $n > 4$ is a prime or $4 \le n \le 10$. They further [2] proposed the open problem to prove the nonexistence of perfect codes in S_n, under the Kendall τ-metric, for more values of n and/or other distances. In this paper, we give some sufficient conditions of the nonexistence of perfect permutation codes. Moreover, we prove that there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric for some n and $t = 2, 3, 4, 5,$ or $\frac{5}{8}\binom{n}{2} < 2t + 1 \le \binom{n}{2}$. Specially, we prove that there does not exist a perfect two-error-correcting code in S_n, where $n + 2 > 6$ is a prime. We also prove that there does not exist a perfect three-error-correcting code in S_n, where $n + 1 > 6$ is a prime, or $n^2 + 2n - 6$ has a prime factor $p > n$, or $4 \le n \le 33$. We further prove that there does not exist a perfect four-error-correcting code in S_n, where $n + 1 > 6$ or $n + 2 > 7$ is a prime, or $n^2 + 3n - 12$ has a prime factor $p > n$, or $5 \le n \le 19$. We prove that there does not exist a perfect five-error-correcting code in S_n, where $n \ge 16$ or $n + 7 \ge 12$ is a prime or $n^3 + 3n^2 - 6n - 28$ has a prime factor $p > n$. For $\frac{5}{8}\binom{n}{2} < 2t + 1 \le \binom{n}{2}$ and $n \ge 5$, we also prove that there does not exist a perfect t-error-correcting code in S_n except for $2t + 1 = \binom{n}{2}$.

The rest of this paper is organized as follows. In Sect. 2, we will give some basic definitions for the Kendall τ-metric and for perfect permutation codes. In Sect. 3, we determine the size of some balls with radius r in S_n under the Kendall τ-metric. In Sect. 4, we present some sufficient conditions of the nonexistence of perfect permutation codes under the Kendall τ-metric. In Sect. 5, we prove the nonexistence of a perfect t-error-correcting code in S_n for some n and $t = 2, 3, 4, 5,$ or $\frac{5}{8}\binom{n}{2} < 2t + 1 \le \binom{n}{2}$. Section 6 concludes this paper.

2 Preliminaries

In this section we give some definitions and notations for the Kendall τ-metric and perfect permutation codes. In addition, we summarize some important known facts.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Let S_n be the set of all the permutations over $[n]$. We denote by $\pi \triangleq [\pi(1), \pi(2), \ldots, \pi(n)]$ a *permutation* over $[n]$. For two permutations $\sigma, \pi \in S_n$, their multiplication $\pi \circ \sigma$ is denoted by the composition of σ on π, i.e., $\pi \circ \sigma(i) = \sigma(\pi(i))$, for all $i \in [n]$. Under this operation, S_n is a noncommutative *group* of size $|S_n| = n!$. Denote by $\epsilon_n \triangleq [1, 2, \ldots, n]$ the identity permutation of S_n. Let π^{-1} be the *inverse* element of π, for any $\pi \in S_n$. For an unordered pair of distinct numbers $i, j \in [n]$, this pair forms an inversion in a permutation π if $i < j$ and simultaneously $\pi(i) > \pi(j)$.

Given a permutation $\pi = [\pi(1), \pi(2), \ldots, \pi(i), \pi(i+1), \ldots \pi(n)] \in S_n$, an adjacent transposition is an exchange of two adjacent elements $\pi(i), \pi(i+1)$, resulting in the permutation $[\pi(1), \pi(2), \ldots, \pi(i+1), \pi(i), \ldots \pi(n)]$ for some $1 \leq i \leq n-1$. For any two permutations $\sigma, \pi \in S_n$, the Kendall τ-distance between two permutations π, σ, denoted by $d_K(\pi, \sigma)$, is the minimum number of adjacent transpositions required to obtain the permutation σ from π. The expression for $d_K(\pi, \sigma)$ [8] is as follows:

$$d_K(\sigma, \pi) = |\{(i, j) : \sigma^{-1}(i) < \sigma^{-1}(j) \wedge \pi^{-1}(i) > \pi^{-1}(j)\}|. \tag{1}$$

For $\pi \in S_n$, the Kendall τ-weight of π, denoted by $w_K(\pi)$, is defined as the Kendall τ-distance between π and the identity permutation ϵ_n. Clearly, $w_K(\pi)$ is the number of inversions in the permutation π. The Kendall τ-metric is right invariant [3] as follows. For any three permutations $\pi, \sigma, \beta \in S_n$, we have

$$d_K(\pi, \sigma) = d_K(\pi \circ \beta, \sigma \circ \beta). \tag{2}$$

Definition 1 For $1 \leq d \leq \binom{n}{2}$, $C \subseteq S_n$ is an (n, d)-*permutation code under the Kendall* τ-*metric*, if $d_K(\sigma, \pi) \geq d$ for any two distinct permutations $\pi, \sigma \in C$.

For a permutation $\pi \in S_n$, the Kendall τ-ball of radius r centered at π, denoted as $B_K^n(\pi, r)$, is defined by $B_K^n(\pi, r) \triangleq \{\sigma \in S_n | d_K(\sigma, \pi) \leq r\}$. For a permutation $\pi \in S_n$, the Kendall τ-sphere of radius r centered at π, denoted as $S_K^n(\pi, r)$, is defined by $S_K^n(\pi, r) \triangleq \{\sigma \in S_n | d_K(\sigma, \pi) = r\}$. The size of a Kendall τ-ball or a τ-sphere of radius r does not depend on the center of the ball or sphere under the Kendall τ-metric. Thus, we denote the size of $B_K^n(\pi, r)$ and $S_K^n(\pi, r)$ as $B_K^n(r)$ and $S_K^n(r)$, respectively. We denote the largest size of an (n, d)-permutation code under the Kendall τ-metric as $A_K(n, d)$. The sphere-packing bound for permutation codes under the Kendall τ-metric is as follows:

Proposition 1 [8, Theorems 17 and 18]

$$A_K(n, d) \leq \frac{n!}{B_K^n(\lfloor \frac{d-1}{2} \rfloor)}.$$

When $d = 2r+1$, an $(n, 2r+1)$-permutation code C under the Kendall τ-metric is called a perfect permutation code under the Kendall τ-metric if it attains the sphere-packing bound, i.e., $|C| \cdot B_K^n(r) = n!$. That is, the balls with radius r centered at the codewords of C form a partition of S_n. A perfect $(n, 2r+1)$-permutation code under the Kendall τ-metric is also called a perfect r-error-correcting code under the Kendall τ-metric.

In [2], Buzaglo and Etzion proved that there does not exist a perfect one-error-correcting code under the Kendall τ-metric if $n > 4$ is a prime or $4 \leq n \leq 10$. Based on the above definitions and notations, we will prove the nonexistence of a perfect t-error-correcting code in S_n under the Kendall τ-metric for some n and $t = 2, 3, 4, 5$, or $\frac{5}{8}\binom{n}{2} < 2t+1 \leq \binom{n}{2}$ by using the sphere-packing upper bound and the properties of $B_K^n(r)$ and $S_K^n(r)$ in the following sections.

3 The size of a ball or a sphere with radius r in S_n under the Kendall τ-metric

In this section, we compute the size of a ball or a sphere with radius r in S_n under the Kendall τ-metric and give polynomial representations of $B_K^n(r)$ and $S_K^n(r)$ for some r, respectively. Since $B_K^n(r)$ does not depend on the center of the ball, we consider the ball $B_K^n(\epsilon_n, r)$ which is a ball with radius r centered at the identity permutation ϵ_n and denote by $S_K^n(\epsilon_n, r) \triangleq \{\sigma \in S_n | d_K(\sigma, \epsilon_n) = w_k(\sigma) = r\}$ the sphere centered at ϵ_n and of radius r.

3.1 The size of a sphere of radius r in S_n under the Kendall τ-metric

In order to give the polynomial representation of $S_K^n(r)$, we require some notations and lemmas as follows. For a permutation $\pi = [\pi(1), \pi(2), \ldots, \pi(n)] \in S_n$, the *reverse* of π is the permutation $\pi^r \triangleq [\pi(n), \pi(n-1), \ldots, \pi(2), \pi(1)]$. For all $\pi \in S_n$, we have $w_K(\pi) \leq \binom{n}{2}$. For convenience, we denote $S_K^n(r) = 0$ for $r \geq \binom{n}{2} + 1$ or $r < 0$.

Lemma 1 [2, Lemma 1] *For every* $\pi \in S_n$,

$$w_K(\pi) + w_K(\pi^r) = d_K(\pi, \pi^r) = \binom{n}{2}. \tag{3}$$

By Lemma 1, we can obtain the following lemma.

Lemma 2 *For any* $0 \leq i \leq \binom{n}{2}$,

$$S_K^n(i) = S_K^n\left(\binom{n}{2} - i\right). \tag{4}$$

Proof Let $m = \binom{n}{2}$. We just need to prove that $|S_K^n(\epsilon_n, i)| = |S_K^n(\epsilon_n, m-i)|$. First we define a function $f : S_K^n(\epsilon_n, i) \to S_K^n(\epsilon_n, m-i)$, where $f(\pi) = \pi^r$ for any $\pi \in S_K^n(\epsilon_n, i)$.

If $\pi \in S_K^n(\epsilon_n, i)$, then $w_K(\pi) = i$. By (3), $w_K(\pi^r) = \binom{n}{2} - i = m - i$. Hence, $f(\pi) \in S_K^n(\epsilon_n, m-i)$. Moreover, we can easily prove that the function f is reasonable and bijection. Thus, $S_K^n(i) = S_K^n\left(\binom{n}{2} - i\right)$. □

Lemma 3 *For any* $0 \leq r \leq \binom{n}{2}$, $S_K^n(r)$ *is the number of permutations with* r *inversions in* S_n.

Proof Since $S_K^n(r) = |S_K^n(\epsilon_n, r)| = |\{\sigma \in S_n | d_K(\sigma, \epsilon_n) = w_k(\sigma) = r\}|$, by (1), we clearly obtain that $S_K^n(r)$ is the number of permutations with r inversions in S_n. □

When $i = 0$ or 1, $S_K^n(0) = 1$ and $S_K^n(1) = n - 1$. By Lemma 3, it is known that $S_K^n(r)$ is the Triangle of Mahonian numbers which has some recursive structure in the following lemma. For the recursive structure of Mahonian numbers, we can see the Mahonian numbers sequence $A008302$ [10].

Lemma 4 *[10, FORMULA] For all* $1 \leq n$ *and* $1 \leq i \leq \binom{n}{2}$,

$$S_K^n(i) = \sum_{j=\max\{0, i-(n-1)\}}^{i} S_K^{n-1}(j). \tag{5}$$

Moreover, for all $0 \le i \le \binom{n}{2}$*, we have*

$$S_K^n(i) = S_K^n(i-1) + S_K^{n-1}(i) - S_K^{n-1}(i-n),$$

where $S_K^n(r) = 0$ *for* $r < 0$.

By Lemma 4, we clearly obtain that $S_K^n(i)$ is an increasing sequence for $0 \le i \le \frac{1}{2}\binom{n}{2}$. Moreover, we give the formula of $S_K^n(i)$ by using some $S_K^m(j)$ for $m < n$ and $j < i$ in the following lemma.

Lemma 5 *For all* $4 \le n$ *and* $3 \le i \le n-1$*, there exists a unique integer* t *such that* $\binom{t-1}{2} < i \le \binom{t}{2}$*. Then, we have*

$$S_K^n(i) = S_K^t(\binom{t}{2} - i) + \sum_{l=t}^{i-1}\sum_{j=i-l}^{i-1} S_K^l(j) + \sum_{l=i}^{n-1}\sum_{j=0}^{i-1} S_K^l(j).$$

Proof When $3 \le n$ and $2 \le i \le n-1$, by (5), we have

$$S_K^n(i) - S_K^{n-1}(i) = \sum_{j=0}^{i-1} S_K^{n-1}(j). \tag{6}$$

In (6), we set n to $i+1, \ldots, n$ and obtain $n-i$ equations, respectively. Then by summing all the equations, we have

$$S_K^n(i) - S_K^i(i) = \sum_{l=i}^{n-1}\sum_{j=0}^{i-1} S_K^l(j). \tag{7}$$

For j and i such that $0 \le j < i \le l \le n-1$, if $S_K^l(j)$ and $S_K^i(i)$ are known, then by (7) we can compute $S_K^n(i)$. In the following, we will compute $S_K^i(i)$. When $5 \le i$, then $i \le \binom{i-1}{2}$. Hence, by (5), for $5 \le i$, we obtain that

$$S_K^i(i) - S_K^{i-1}(i) = \sum_{j=1}^{i-1} S_K^{i-1}(j).$$

For $5 \le i$, we can find an integer t such that $\binom{t-1}{2} < i \le \binom{t}{2}$ and $t < i$. We also obtain

$$0 \le \binom{t}{2} - i < i. \tag{8}$$

Similarly, when $5 \le i$, in (5), we set n to $t+1, \ldots, i$ and obtain $i-t$ equations, respectively. By summing all the equations, we have

$$S_K^i(i) - S_K^t(i) = \sum_{l=t}^{i-1}\sum_{j=i-l}^{i-1} S_K^l(j). \tag{9}$$

Combining (4), (8), and (9), we have

$$S_K^i(i) = S_K^t(\binom{t}{2} - i) + \sum_{l=t}^{i-1}\sum_{j=i-l}^{i-1} S_K^l(j), \tag{10}$$

for $5 \le i$.

When $i = 4$, $S_K^4(4) = S_K^4(\binom{4}{2} - 4) = S_K^4(2)$. When $i = 3$, we have $S_K^3(3) = S_K^3(\binom{3}{2} - 3) = S_K^3(0) = 1$.

Therefore, if $i \in \{3, 4\}$, we choose $t = i$, and the second term of Equation (10) is zero. Thus, in these cases, we still have the representation in (10).

By (7) and (10), we can obtain the expression of $S_K^n(i)$ in the above lemma. □

Specifically, we give the polynomial representations of $S_K^n(2)$ and $S_K^n(3)$ for all $3 \le n$ as follows.

Lemma 6 *For all* $3 \le n$*, we have*

$$S_K^n(2) = \frac{n(n-1)}{2} - 1,$$
$$S_K^n(3) = \frac{n^3 - 7n}{6}.$$

Proof When $i = 2$, by (7), we have

$$S_K^n(2) - S_K^2(2) = \sum_{l=2}^{n-1}\sum_{j=0}^{1} S_K^l(j). \tag{11}$$

Since $S_K^n(0) = 1$, $S_K^n(1) = n - 1$ and $S_K^2(2) = 0$, by (11), we have

$$S_K^n(2) = \sum_{l=2}^{n-1}\sum_{j=0}^{1} S_K^l(j) = \sum_{l=2}^{n-1} l = \frac{n(n-1)}{2} - 1. \tag{12}$$

Similarly, when $i = 3$, by (7), we have

$$S_K^n(3) - S_K^3(3) = \sum_{l=3}^{n-1}\sum_{j=0}^{2} S_K^l(j). \tag{13}$$

Since $S_K^n(0) = 1$, $S_K^n(1) = n - 1$, $S_K^n(2) = \frac{n(n-1)}{2} - 1$, and $S_K^3(3) = 1$, by (13), we have

$$S_K^n(3) = S_K^3(3) + \sum_{l=3}^{n-1}\sum_{j=0}^{2} S_K^l(j) = 1 + \sum_{l=3}^{n-1} \frac{l^2 + l - 2}{2} = \frac{n^3 - 7n}{6}. \tag{14}$$

According to (12) and (14), we can obtain the expressions of $S_K^n(2)$ and $S_K^n(3)$, respectively. □

Here, we easily obtain $S_K^2(0) = S_K^2(1) = 1$. By Lemma 6, when $n = 3$, we have $S_K^3(0) = 1$, $S_K^3(1) = 2$, $S_K^3(2) = 2$, and $S_K^3(3) = 1$. By Lemma 6 and Lemma 2, we have $S_K^4(0) = 1$, $S_K^4(1) = 3$, $S_K^4(2) = 5$, $S_K^4(3) = 6$, $S_K^4(4) = 5$, $S_K^4(5) = 3$, and $S_K^4(6) = 1$.

If all the $S_K^n(j)$ for all n and $j \le i - 1$ are known, by Lemma 5, we can compute $S_K^n(i)$ for $4 \le n$ and $4 \le i \le n - 1$. Next we present an example to compute $S_K^n(i)$ in Lemma 5.

Example 1 When $i = 4$, $\binom{3}{2} < 4 \le \binom{4}{2}$. Then, we obtain $t = 4$ in Lemma 5. Furthermore, we have

$$S_K^n(4) = S_K^4(\binom{4}{2} - 4) + \sum_{l=4}^{3}\sum_{j=i-l}^{i-1} S_K^l(j) + \sum_{l=4}^{n-1}\sum_{j=0}^{3} S_K^l(j).$$

By Lemma 6, we have $S_K^4(\binom{4}{2}-4)=S_K^4(2)=5$. Thus,

$$S_K^n(4)=5+\sum_{l=4}^{n-1}\left(1+(l-1)+\frac{l(l-1)}{2}-1+\frac{l^3-7l}{6}\right)=\frac{n(n+1)(n^2+n-14)}{24}.$$

In the following, we also give the formula of $S_K^n(i)$ for all $5\leq n$ and $n\leq i\leq \binom{n-1}{2}$.

Lemma 7 *For all $5\leq n$ and $n\leq i\leq \binom{n-1}{2}$, there exists a unique integer t such that $\binom{t-1}{2}<i\leq\binom{t}{2}$ and $t\geq 4$. Then, we have*

$$S_K^n(i)=S_K^t(\binom{t}{2}-i)+\sum_{l=t}^{i-1}\sum_{j=i-l}^{i-1}S_K^l(j)-\sum_{l=n}^{i-1}\sum_{j=i-l}^{i-1}S_K^l(j). \tag{15}$$

Proof When $5\leq n$ and $n\leq i\leq\binom{n-1}{2}$, in (5), we set n to $n+1,\ldots,i$, respectively. Then we obtain $n-i$ equations and sum all the equations. Thus, we have

$$S_K^i(i)-S_K^n(i)=\sum_{l=n}^{i-1}\sum_{j=i-l}^{i-1}S_K^l(j). \tag{16}$$

By (10) and (16), we have

$$S_K^n(i)=S_K^t(\binom{t}{2}-i)+\sum_{l=t}^{i-1}\sum_{j=i-l}^{i-1}S_K^l(j)-\sum_{l=n}^{i-1}\sum_{j=i-l}^{i-1}S_K^l(j).$$

When $i=n$, the third term (i.e., $\sum_{l=n}^{i-1}\sum_{j=i-l}^{i-1}S_K^l(j)$) is zero. □

Example 2 When $i=5$ and $n=5$, we have $\binom{3}{2}<5\leq\binom{4}{2}$. Then, we obtain $t=4$ in Lemma 7. Furthermore, by (15), we have

$$S_K^5(5)=S_K^4(\binom{4}{2}-5)+\sum_{l=4}^{4}\sum_{j=5-l}^{4}S_K^l(j).$$

Thus, $S_K^5(5)=S_K^4(1)+\sum_{j=1}^4 S_K^4(j)=3+(3+5+6+5)=22$.

For every $6\leq n$, due to $i=5\leq n-1$, $S_K^n(5)$ can be computed by Lemma 5.

Hence, if $S_K^n(j)$ are known for all $1\leq j\leq i-1$ and n, we will compute $S_K^n(i)$ for all n in the next two steps. For $5\leq i$, there exists a unique integer t such that $\binom{t-1}{2}<i\leq\binom{t}{2}$. Then, for every $2\leq l\leq t-1$, $S_K^l(i)=0$. First, when $t\leq l\leq i$, if $i>\left\lfloor\binom{l}{2}/2\right\rfloor$, we have $S_K^l(i)=S_K^l(\binom{l}{2}-i)$ where $\binom{l}{2}-i<i$; otherwise, by Lemma 7, we compute $S_K^l(i)$ for $i\leq\left\lfloor\binom{l}{2}/2\right\rfloor$. Second, when $i+1\leq l$, we compute $S_K^l(i)$ by Lemma 5.

When $i=5$, we can compute $S_K^n(5)$ for all n. Here, $t=4$. Then, $S_K^4(5)=S_K^4(\binom{4}{2}-5)=S_K^4(1)=3$ and $S_K^5(5)=22$ by Lemma 7 in Example 2. In the following, we will give the formula of $S_K^n(5)$ for all $6\leq n$ by Lemma 5.

Example 3 When $i=5$ and $6\leq n$, by Lemma 5 and (7), we have

$$S_K^n(5)=S_K^5(5)+\sum_{l=5}^{n-1}\sum_{j=0}^{4}S_K^l(j).$$

By Examples 1 and 2 and Lemma 6, we have

$$S_K^n(5) = 22 + \sum_{l=5}^{n-1} \left(1 + (l-1) + \frac{l(l-1)}{2} - 1 + \frac{l^3 - 7l}{6} + \frac{l(l+1)(l^2+l-14)}{24}\right)$$
$$= \frac{(n-1)(n^4 + 6n^3 - 9n^2 - 74n - 120)}{120}$$

for all $5 \le n$.

By Lemmas 2, 5, and 7, we can compute the value of $S_K^n(i)$ for all $6 \le i$ and n as follows.

Proposition 2 *When $6 \le i$, we can compute $S_K^n(i)$ for all $5 \le n$ by using Lemmas 2, 5, and 7.*

Proof For all $0 \le i \le 5$ and $3 \le n$, all the $S_K^n(i)$ are computed. We can compute $S_K^n(i)$ for all n by using $S_K^n(j)$ for all $1 \le j \le i-1$ and n.

First, we find an integer t such that $\binom{t-1}{2} < i \le \binom{t}{2}$. For every $t \le l \le i$, if $i > \left\lfloor \binom{l}{2}/2 \right\rfloor$, we have $S_K^l(i) = S_K^l(\binom{l}{2} - i)$ where $\binom{l}{2} - i < i$; else if $i \le \left\lfloor \binom{l}{2}/2 \right\rfloor$, we compute $S_K^l(i)$ by Lemma 7. Second, for every $i+1 \le l$, we compute $S_K^l(i)$ by Lemma 5. So, we can obtain $S_K^n(i)$ for all $5 \le n$ and $6 \le i$. □

3.2 The size of a ball of radius r in S_n under the Kendall τ-metric

In this subsection, we will give the polynomial representation of the size of a ball with radius r in S_n under the Kendall τ-metric by using $S_K^n(r)$. We easily obtain the following lemma about the relationship between $B_K^n(r)$ and $S_K^n(r)$.

Lemma 8 *For any $0 \le r \le \binom{n}{2}$, we have*

$$B_K^n(r) = \sum_{l=0}^{r} S_K^n(l).$$

Given $S_K^n(i)$ for all $0 \le i \le r-1$, by Lemmas 5, 7 and 8, we easily obtain the recursion formula of $B_K^n(r)$ in the following theorem.

Theorem 1 *Suppose $S_K^n(i)$ are known for all $0 \le i \le r-1$ and $5 \le n$. If $4 \le r \le \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor$, there exists a unique integer t such that $\binom{t-1}{2} < r \le \binom{t}{2}$. When $4 \le r \le n-1$, we have*

$$B_K^n(r) = \sum_{l=0}^{r-1} S_K^n(l) + S_K^t(\binom{t}{2} - r) + \sum_{l=t}^{r-1} \sum_{j=r-l}^{r-1} S_K^l(j) + \sum_{l=r}^{n-1} \sum_{j=0}^{r-1} S_K^l(j).$$

When $n \le r \le \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor$, we have

$$B_K^n(r) = \sum_{l=0}^{r-1} S_K^n(l) + S_K^t(\binom{t}{2} - r) + \sum_{l=t}^{r-1} \sum_{j=r-l}^{r-1} S_K^l(j) - \sum_{l=n}^{r-1} \sum_{j=r-l}^{r-1} S_K^l(j).$$

Specially, we have $B_K^n(0) = 1$ and $B_K^n(1) = n$. When $r = 2$, for all $n \geq 2$, we have

$$B_K^n(2) = \sum_{l=0}^{2} S_K^n(l) = (1 + n - 1 + \frac{n(n-1)}{2} - 1) = \frac{(n+2)(n-1)}{2}. \tag{17}$$

When $r = 3$, for all $n \geq 3$, we have

$$B_K^n(3) = \sum_{l=0}^{3} S_K^n(l) = (1 + n - 1 + \frac{n(n-1)}{2} - 1 + \frac{n^3 - 7n}{6}) = \frac{(n+1)(n^2+2n-6)}{6}. \tag{18}$$

Example 4 When $r = 4$ and $4 \leq n$, by Example 1 and Theorem 1, we have

$$\begin{aligned} B_K^n(4) &= \sum_{l=0}^{3} S_K^n(l) + S_K^4(\binom{4}{2} - 4) + \sum_{l=4}^{3} \sum_{j=4-l}^{4-1} S_K^l(j) + \sum_{l=4}^{n-1} \sum_{j=0}^{3} S_K^l(j) \\ &= \frac{(n+2)(n+1)(n^2+3n-12)}{24}. \end{aligned} \tag{19}$$

Moreover, when $r = 5$ and $5 \leq n$, by Example 3 and Theorem 1, we have

$$\begin{aligned} B_K^n(5) &= \sum_{l=0}^{4} S_K^n(l) + S_K^n(5) \\ &= \frac{(n+7)n(n^3+3n^2-6n-28)}{120}. \end{aligned} \tag{20}$$

When $r \geq 6$, we can compute $B_K^n(r)$ by using Proposition 2 and Theorem 1.

4 Some sufficient conditions of the nonexistence of perfect permutation codes

In this section, we will give some sufficient conditions of the nonexistence of a perfect t-error-correcting code in S_n under the Kendall τ-metric.

4.1 The first sufficient condition of the nonexistence of perfect permutation codes

In this subsection, we present a sufficient condition of the nonexistence of a perfect t-error-correcting code under the Kendall τ-metric by using the sphere-packing upper bound.

Lemma 9 *For any $0 \leq t \leq \left\lfloor \frac{\binom{n}{2}-1}{2} \right\rfloor$, if there exists a perfect t-error-correcting code C in S_n under the Kendall τ-metric. Then, we must have*

$$B_K^n(t) \cdot |C| = n!.$$

That is, the necessary condition of the existence of a perfect t-error-correcting code in S_n under the Kendall τ-metric is $B_K^n(t) | n!$.

Proof By the sphere-packing upper bound in Proposition 1, if there exists a perfect t-error-correcting code C in S_n under the Kendall τ-metric, we must have $B_K^n(t) \cdot |C| = n!$. Thus, $B_K^n(t) | n!$. So, the necessary condition of the existence of a perfect t-error-correcting code in S_n under the Kendall τ-metric is $B_K^n(t) | n!$. □

According to Lemma 9, we have the following theorem which illustrate the nonexistence of a perfect t-error-correcting code in S_n under the Kendall τ-metric.

Theorem 2 *For any* $0 \leq t \leq \left\lfloor \frac{\binom{n}{2}-1}{2} \right\rfloor$*, if* $B_K^n(t)$ *has a prime factor* $p > n$*, then there does not exist a perfect* t*-error-correcting code in* S_n *under the Kendall* τ*-metric.*

Proof By Lemma 9, the necessary condition of the existence of a perfect t-error-correcting code in S_n under the Kendall τ-metric is $B_K^n(t)|n!$. Since $B_K^n(t)$ has a prime factor $p > n$, we have $B_K^n(t) \nmid n!$. So, we prove the above result. □

4.2 The second sufficient condition of the nonexistence of perfect permutation codes

In this subsection, we give another sufficient condition of the nonexistence of a perfect t-error-correcting code under the Kendall τ-metric by using some properties of $B_K^n(r)$ and $S_K^n(r)$. For convenience, we define $S_{n,i} \triangleq \{\pi \in S_n | \pi(i) = 1\}$ for any $1 \leq i \leq n$. That is, π is an element of $S_{n,i}$ if 1 appears in the i-th position of π. Clearly, $|S_{n,i}| = (n-1)!$.

For $\pi \in S_{n,k}$, let $T_{n,(k,j)}^{(t)}(\pi) \triangleq B_K^n(\pi, t) \cap S_{n,j}$ for all $0 \leq t \leq \left\lfloor \frac{\binom{n}{2}-1}{2} \right\rfloor$. In the following, we prove that the size of $T_{n,(k,j)}^{(t)}(\pi)$ does not depend on π. For convenience, we denote the size of $T_{n,(k,j)}^{(t)}(\pi)$ as $T_{n,(k,j)}^{(t)}$.

Lemma 10 *For all* $1 \leq k, j \leq n$ *and any* $\pi \in S_{n,k}$*, the size of* $T_{n,(k,j)}^{(t)}(\pi)$ *does not depend on* π*.*

Proof For any two distinct permutations $\pi, \sigma \in S_{n,k}$, we prove $|T_{n,(k,j)}^{(t)}(\pi)| = |T_{n,(k,j)}^{(t)}(\sigma)|$. First, we have $T_{n,(k,j)}^{(t)}(\pi) = \{\beta \in S_n | d_K(\pi, \beta) \leq t, \beta(j) = 1\}$. By (2), we get

$$\begin{aligned} T_{n,(k,j)}^{(t)}(\pi) &= \{\beta \in S_n | d_K(\pi, \beta) \leq t, \beta(j) = 1\} \\ &= \{\beta \in S_n | d_K(\epsilon_n, \beta \circ \pi^{-1}) \leq t, \beta(j) = 1\} \\ &= \{\beta \in S_n | d_K(\sigma, \beta \circ \pi^{-1} \circ \sigma) \leq t, \beta(j) = 1\}. \end{aligned}$$

Hence, $|T_{n,(k,j)}^{(t)}(\pi)| = |\{\beta \in S_n | d_K(\sigma, \beta \circ \pi^{-1} \circ \sigma) \leq t, \beta(j) = 1\}| = |\{\beta \circ \pi^{-1} \circ \sigma \in S_n | d_K(\sigma, \beta \circ \pi^{-1} \circ \sigma) \leq t, \beta(j) = 1\}|$. Since $\beta \circ \pi^{-1} \circ \sigma(j) = \sigma(\pi^{-1}(\beta(j))) = 1$, we have $|\{\beta \circ \pi^{-1} \circ \sigma \in S_n | d_K(\sigma, \beta \circ \pi^{-1} \circ \sigma) \leq t, \beta(j) = 1\}| = |\{\beta \circ \pi^{-1} \circ \sigma \in S_n | d_K(\sigma, \beta \circ \pi^{-1} \circ \sigma) \leq t, \beta \circ \pi^{-1} \circ \sigma(j) = 1\}|$. Let $\gamma = \beta \circ \pi^{-1} \circ \sigma$. Then, we obtain that $|T_{n,(k,j)}^{(t)}(\pi)| = |\{\beta \circ \pi^{-1} \circ \sigma \in S_n | d_K(\sigma, \beta \circ \pi^{-1} \circ \sigma) \leq t, \beta \circ \pi^{-1} \circ \sigma(j) = 1\}| = |\{\gamma \in S_n | d_K(\sigma, \gamma) \leq t, \gamma(j) = 1\}| = |T_{n,(k,j)}^{(t)}(\sigma)|$. Thus, the size of $T_{n,(k,j)}^{(t)}(\pi)$ does not depend on π. □

Similarly, the exchange between k and j in $T_{n,(k,j)}^{(t)}$ does not change the size of $T_{n,(k,j)}^{(t)}$.

Lemma 11 *For all* $1 \leq k, j \leq n$*, we have*

$$T_{n,(k,j)}^{(t)} = T_{n,(j,k)}^{(t)}.$$

Proof By Lemma 10, we only prove $|T^{(t)}_{n,(k,j)}(\pi)| = |T^{(t)}_{n,(j,k)}(\sigma)|$ for any $\pi \in S_{n,k}$ and $\sigma \in S_{n,j}$. According to the definition of $T^{(t)}_{n,(k,j)}(\pi)$, we have

$$\begin{aligned} T^{(t)}_{n,(k,j)}(\pi) &= \{\beta \in S_n | d_K(\pi, \beta) \le t, \beta(j) = 1\} \\ &= \{\beta \in S_n | d_K(\pi \circ \beta^{-1}, \epsilon_n) \le t, \beta(j) = 1\} \\ &= \{\beta \in S_n | d_K(\pi \circ \beta^{-1} \circ \sigma, \sigma) \le t, \beta(j) = 1\}. \end{aligned}$$

Then, we get $\pi \circ \beta^{-1} \circ \sigma(k) = \sigma(\beta^{-1}(\pi(k))) = \sigma(j) = 1$. Given $\pi \in S_{n,k}$ and $\sigma \in S_{n,j}$, we obtain $|\{\beta \in S_n | d_K(\pi \circ \beta^{-1} \circ \sigma, \sigma) \le t, \beta(j) = 1\}| = |\{\pi \circ \beta^{-1} \circ \sigma \in S_n | d_K(\pi \circ \beta^{-1} \circ \sigma, \sigma) \le t, \pi \circ \beta^{-1} \circ \sigma(k) = 1\}|$, i.e., $|T^{(t)}_{n,(k,j)}(\pi)| = |T^{(t)}_{n,(j,k)}(\sigma)|$. □

Assume that there exists a perfect t-error-correcting code $C^{(t)} \subset S_n$. For any $1 \le i \le n$, we define $C^{(t)}_{n,i} \triangleq C^{(t)} \cap S_{n,i}$ and $x_i \triangleq |C^{(t)}_{n,i}|$. Since $C^{(t)}$ is a perfect t-error-correcting code, it follows that for any two distinct permutations $\pi, \sigma \in C^{(t)}$ we have $B^n_K(\pi, t) \cap B^n_K(\sigma, t) = \emptyset$. Moreover, we get $\bigcup_{\pi \in C^{(t)}} B^n_K(\pi, t) = S_n$. Clearly, we can obtain the following lemma.

Lemma 12 *For all $1 \le i \le n$, we have $\bigcup_{1 \le k \le n, \pi \in C^{(t)}_{n,k}} T^{(t)}_{n,(k,i)}(\pi) = S_{n,i}$.*

Proof For each permutation $\sigma \in S_{n,i}$, there must exist a codeword $\pi \in C^{(t)}$ such that $\sigma \in B^n_K(\pi, t)$, where $\pi(k) = 1$ for some $k \in [n]$. By the definition of $T^{(t)}_{n,(k,i)}(\pi)$, we have $\sigma \in T^{(t)}_{n,(k,i)}(\pi)$. Hence, $\bigcup_{1 \le k \le n, \pi \in C^{(t)}_{n,k}} T^{(t)}_{n,(k,i)}(\pi) = S_{n,i}$. □

For any two distinct permutations $\pi \in C^{(t)}_{n,k}$ and $\sigma \in C^{(t)}_{n,j}$, the relationship between $T^{(t)}_{n,(k,i)}(\pi)$ and $T^{(t)}_{n,(j,i)}(\sigma)$ is given as follows.

Lemma 13 *For any two distinct permutations $\pi \in C^{(t)}_{n,k}$ and $\sigma \in C^{(t)}_{n,j}$, we have $T^{(t)}_{n,(k,i)}(\pi) \bigcap T^{(t)}_{n,(j,i)}(\sigma) = \emptyset$ for all $1 \le i \le n$.*

Proof Since $\pi, \sigma \in C^{(t)}$ and $\pi \ne \sigma$, $B^n_K(\pi, t) \bigcap B^n_K(\sigma, t) = \emptyset$. Clearly, due to the definition of $T^{(t)}_{n,(k,i)}(\pi)$, we have $T^{(t)}_{n,(k,i)}(\pi) \bigcap T^{(t)}_{n,(j,i)}(\sigma) = \emptyset$ for all $1 \le i \le n$. □

By Lemmas 12 and 13, for all $1 \le i \le n$, we obtain that

$$\left| \bigcup_{1 \le k \le n, \pi \in C^{(t)}_{n,k}} T^{(t)}_{n,(k,i)}(\pi) \right| = \sum_{k=1}^{n} \sum_{\pi \in C^{(t)}_{n,k}} T^{(t)}_{n,(k,i)} = \sum_{k=1}^{n} T^{(t)}_{n,(k,i)} x_k = |S_{n,i}| = (n-1)!, \quad (21)$$

where $|C^{(t)}_{n,k}| = x_k$.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and let $\boldsymbol{I}$ be the all-ones column vector. By (21), we have a matrix form as

$$A\boldsymbol{x}^T = (n-1)! \cdot \boldsymbol{I}, \quad (22)$$

where $A = (a_{i,j})$ is an $n \times n$ matrix and $a_{i,j} = T^{(t)}_{n,(j,i)}$.

Furthermore, we discuss the sum of every row in A as follows.

Lemma 14 *For all $1 \le i \le n$, we have*

$$\sum_{k=1}^{n} T^{(t)}_{n,(k,i)} = B^n_K(t).$$

Proof For all $1 \le i \le n$ and $\pi \in S_{n,i}$, $B_K^n(\pi,t) = B_K^n(\pi,t) \bigcap S_n = B_K^n(\pi,t) \bigcap \left(\bigcup_{k=1}^n S_{n,k}\right) = \bigcup_{k=1}^n \left(B_K^n(\pi,t) \bigcap S_{n,k}\right)$. Since all the sets of $S_{n,k}$ are pairwise disjoint, we have

$$\begin{aligned} B_K^n(t) = |B_K^n(\pi,t)| &= \left|\bigcup_{k=1}^n \left(B_K^n(\pi,t) \bigcap S_{n,k}\right)\right| \\ &= \sum_{k=1}^n \left|B_K^n(\pi,t) \bigcap S_{n,k}\right| \\ &\overset{(a)}{=} \sum_{k=1}^n T_{n,(i,k)}^{(t)} \\ &\overset{(b)}{=} \sum_{k=1}^n T_{n,(k,i)}^{(t)}, \end{aligned}$$

where $\overset{(a)}{=}$ follows from the definition of $T_{n,(i,k)}^{(t)}$ and $\overset{(b)}{=}$ follows from Lemma 11. □

By Lemma 14, we obtain that the sum of every row in A is equal to $B_K^n(t)$. Then, it follows that the linear equation system defined in (22) has a solution $\boldsymbol{y} = \frac{(n-1)!}{B_K^n(t)} \cdot \boldsymbol{I}$. In order to discuss the solutions of (22), we need the following lemma that is an immediate conclusion of the well known Gerschgorin circle theorem [5].

Lemma 15 *Let $B = (b_{i,j})$ be an $n \times n$ matrix. If $|b_{i,i}| > \sum_{j \ne i} |b_{i,j}|$ for all i, $1 \le i \le n$, then B is nonsingular.*

Theorem 3 *For any $0 \le t \le \left\lfloor \frac{\binom{n}{2}-1}{2} \right\rfloor$, if $T_{n,(i,i)}^{(t)} \ge \frac{B_K^n(t)}{2} - 1$ for all $1 \le i \le n$ and $B_K^n(t)$ has a prime factor $p \ge n$, then there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric.*

Proof If $T_{n,(i,i)}^{(t)} \ge \frac{B_K^n(t)}{2} - 1$ for all $1 \le i \le n$, by Lemma 15, then A is a nonsingular matrix. Hence, $\boldsymbol{y}$ is the unique solution of (22). That is, we have $\boldsymbol{x} = \boldsymbol{y} = \frac{(n-1)!}{B_K^n(t)} \cdot \boldsymbol{I}$. If $B_K^n(t)$ has a prime factor $p \ge n$ and A is nonsingular, then $\frac{(n-1)!}{B_K^n(t)}$ is not an integer and perfect t-error-correcting codes do not exist. Hence, we prove the above result. □

4.3 The third sufficient condition of the nonexistence of perfect permutation codes

In this subsection, we give the third sufficient condition of the nonexistence of a perfect t-error-correcting code under the Kendall τ-metric by using some upper bounds on $A_K(n, 2t+1)$.

Lemma 16 *[14, Theorem 23] If $A_K(n, 2t+1) \ge M$, then*

$$\binom{\lceil \frac{M}{2} \rceil}{2}(2t+2) + \binom{\lfloor \frac{M}{2} \rfloor}{2}(2t+2) + \left\lceil \frac{M}{2} \right\rceil \left\lfloor \frac{M}{2} \right\rfloor (2t+1) \le \binom{n}{2} \left\lceil \frac{M}{2} \right\rceil \left\lfloor \frac{M}{2} \right\rfloor.$$

Lemma 17 *[2, Theorem 10] If $\frac{2}{3}\binom{n}{2} < d \le \binom{n}{2}$, then*

$$A_K(n, d) = 2.$$

By Lemma 16, we present some sufficient conditions of the nonexistence of a perfect t-error-correcting code in S_n under the Kendall τ-metric as follows.

Theorem 4 *For any $\frac{1}{2}\binom{n}{2} \le 2t+1 \le \binom{n}{2}$, if $B_K^n(t) \cdot \frac{4t+4}{4t+3-\binom{n}{2}} < n!$, then there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric. Moreover, for $\frac{2}{3}\binom{n}{2} < 2t+1 \le \binom{n}{2}$, if $2B_K^n(t) < n!$, then there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric.*

Proof By Lemma 16, if $A_K(n, 2t+1)$ is even, then we have

$$A_K(n, 2t+1) \le \frac{4t+4}{4t+3-\binom{n}{2}} \tag{23}$$

for any $\frac{1}{2}\binom{n}{2} \le 2t+1 \le \binom{n}{2}$. If $A_K(n, 2t+1)$ is odd, then we obtain

$$A_K(n, 2t+1) \le \frac{\binom{n}{2}+1}{4t+3-\binom{n}{2}} \tag{24}$$

for any $\frac{1}{2}\binom{n}{2} \le 2t+1 \le \binom{n}{2}$. When $\frac{1}{2}\binom{n}{2} \le 2t+1 \le \binom{n}{2}$, then $4t+3 > \binom{n}{2}+1$. Hence, by (23) and (24), for any $\frac{1}{2}\binom{n}{2} \le 2t+1 \le \binom{n}{2}$, we have $A_K(n, 2t+1) \le \frac{4t+4}{4t+3-\binom{n}{2}}$. Moreover, the necessary condition of the existence of a perfect t-error-correcting code in S_n under the Kendall τ-metric is $B_K^n(t) \cdot A_K(n, t) = n!$. So, if $B_K^n(t)\frac{4t+4}{4t+3-\binom{n}{2}} < n!$, then there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric.

When $\frac{2}{3}\binom{n}{2} < 2t+1 \le \binom{n}{2}$, by Lemma 17, we have $A_K(n, 2t+1) = 2$. Thus, if $2B_K^n(t) < n!$, then there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric. So, we prove the above result. □

5 The nonexistence of a perfect *t*-error-correcting code in S_n under the Kendall τ-metric

In the section, we will discuss the nonexistence of a perfect t-error-correcting code in S_n for some n and t by using Theorems 2, 3 and 4.

5.1 The nonexistence of a perfect *t*-error-correcting code in S_n by using the first condition

In this subsection, we use Theorem 2 to prove the nonexistence of perfect t-error-correcting code in S_n for some n and $t = 2, 3, 4, 5$.

When $t = 2$, by (17), we have $B_K^n(2) = \frac{(n+2)(n-1)}{2}$. By Theorem 2, we can prove the nonexistence of a perfect two-error-correcting code in S_n, where $n+2 > 6$ is a prime.

When $t = 3$, by (18), we have $B_K^n(3) = \frac{(n+1)(n^2+2n-6)}{6}$. First, if $n+1 > 6$ is a prime, then $B_K^n(3)$ have a prime factor $n+1 > n$. Second, we compute n^2+2n-6 for $4 \le n \le 33$ and obtain that $(n+1)(n^2+2n-6)$ has a prime factor $p > n$ except for $n = 13$ and $n = 26$. If $n = 13$, $B_K^{13}(3) = 441 = 9 \times 7^2$. Thus, $441 \nmid 13!$. If $n = 26$, $B_K^{26}(3) = 3249 = 9 \times 19^2$. Hence, $3249 \nmid 26!$. So, by Theorem 2, we can prove the nonexistence of a perfect three-error-correcting code in S_n, where $n+1 > 6$ is a prime, or n^2+2n-6 has a prime factor $p > n$, or $4 \le n \le 33$.

When $t = 4$, by (19), we have $B_K^n(4) = \frac{(n+1)(n+2)(n^2+3n-12)}{24}$. First, if $n+1 > 6$ or $n+2 > 7$ is a prime, then $B_K^n(3)$ have a prime factor $p > n$. Second, we compute $n^2+3n-12$ for $5 \leq n \leq 19$ and obtain that $(n^2+3n-12)(n+1)(n+2)$ has a prime factor $p > n$ except for $n = 13$. If $n = 13$, $B_K^{13}(4) = 1715 = 5 \times 7^3$. Thus, $1715 \nmid 13!$. So, by Theorem 2, we can prove the nonexistence of a perfect four-error-correcting code in S_n, where $n+1 > 6$ or $n+2 > 7$ is a prime, or $n^2+3n-12$ has a prime factor $p > n$, or $5 \leq n \leq 19$.

When $t = 5$, by (20), $B_K^n(5) = \frac{(n+7)n(n^3+3n^2-6n-28)}{120}$. By Theorem 2, we can prove the nonexistence of a perfect five-error-correcting code in S_n, where $n+7 \geq 12$ is a prime or $n^3+3n^2-6n-28$ has a prime factor $p > n$.

By the above discussion, we have the following theorem.

Theorem 5 *When $t = 2$, there does not exist a perfect two-error-correcting code in S_n, where $n+2 > 6$ is a prime. When $t = 3$, there does not exist a perfect three-error-correcting code in S_n, where $n+1 > 6$ is a prime, or n^2+2n-6 has a prime factor $p > n$, or $4 \leq n \leq 33$. When $t = 4$, there does not exist a perfect four-error-correcting code in S_n, where $n+1 > 6$ or $n+2 > 7$ is a prime, or $n^2+3n-12$ has a prime factor $p > n$, or $5 \leq n \leq 19$. When $t = 5$, there does not exist a perfect five-error-correcting code in S_n, where $n+7 \geq 12$ is a prime or $n^3+3n^2-6n-28$ has a prime factor $p > n$.*

5.2 The nonexistence of a perfect t-error-correcting code in S_n by using the second condition

In this subsection, we use Theorem 3 to prove the nonexistence of perfect t-error-correcting code in S_n for some n, t. When $t = 1$ or 5, $B_K^n(t)$ has a factor n. Buzaglo and Etzion [2] proved that A is nonsingular for $n > 4$ and therefore, there is no perfect single-error-correcting codes in S_n if $n \geq 4$ is a prime. In the following, we discuss the condition of $T_{n,(i,i)}^{(5)} \geq \frac{B_K^n(5)}{2} - 1$ for all $1 \leq i \leq n$ which makes A nonsingular.

First, due to $B_K^n(5) = \frac{n(n+7)(n^3+3n^2-6n-28)}{120}$, we obtain that $B_K^n(5)$ has a prime factor $p > n$ for all $5 \leq n \leq 16$ except for $B_K^7(5) = 7^3$ and $B_K^{11}(5) = 2^4 \times 3 \times 5 \times 11$. By Theorem 2, we obtain that there is no perfect five-error-correcting codes in S_n for $5 \leq n \leq 10$ or $12 \leq n \leq 16$. Second, we only consider the condition of $T_{n,(i,i)}^{(5)} \geq \frac{B_K^n(5)}{2} - 1$ for all $1 \leq i \leq n$, where $n \geq 17$.

When $i = 1$ and $\pi \in S_{n,1}$, we obtain all the elements of $T_{n,(1,1)}^{(5)}(\pi)$ by only moving the right $n-1$ elements at most 5 adjacent transpositions. Hence, we have

$$T_{n,(1,1)}^{(5)} = B_K^{n-1}(5).$$

In order to compute $T_{n,(i,i)}^{(5)}$ for $2 \leq i \leq n$, we need the following lemmas.

Lemma 18 *For all $1 \leq i \leq n$, we have*

$$T_{n,(i,i)}^{(5)} = T_{n,(n+1-i,n+1-i)}^{(5)}.$$

Proof Choose $\pi \in S_{n,i}$ and $\pi^r \in S_{n,n+1-i}$. By Lemma 10, we obtain that

$$T_{n,(i,i)}^{(5)} = |T_{n,(i,i)}^{(5)}(\pi)| = |\{\beta \in S_n | d_K(\pi, \beta) \leq 5, \beta(i) = 1\}|,$$

and

$$T^{(5)}_{n,(n+1-i,n+1-i)} = |T^{(5)}_{n,(n+1-i,n+1-i)}(\pi^r)| = |\{\beta \in S_n | d_K(\pi^r, \beta) \le 5, \beta(n+1-i) = 1\}|.$$

We just need to prove that $|T^{(5)}_{n,(i,i)}(\pi)| = |T^{(5)}_{n,(n+1-i,n+1-i)}(\pi^r)|$. First we define a function $f : T^{(5)}_{n,(i,i)}(\sigma) \to T^{(5)}_{n,(n+1-i,n+1-i)}(\sigma^r)$, where $f(\sigma) = \sigma^r$ for any $\sigma \in T^{(5)}_{n,(i,i)}(\pi)$.

If $\beta \in T^{(5)}_{n,(i,i)}(\pi)$, then $d_K(\beta, \pi) \le 5$ and $\beta(i) = 1$. Hence, $d_K(\pi^r, \beta^r) \le 5$ and $\beta^r(n+1-i) = 1$. Then, $\beta^r \in T^{(5)}_{n,(n+1-i,n+1-i)}(\pi^r)$. Moreover, we can easily prove that the function f is reasonable and bijection. Thus, $T^{(5)}_{n,(i,i)} = T^{(5)}_{n,(n+1-i,n+1-i)}$ for all $1 \le i \le n$. □

Lemma 19 *For $\pi, \sigma \in S_{n,i}$ and $2 \le i \le n-1$, let $\pi = [a_1, \ldots a_{i-1}, 1, a_{i+1}, \ldots, a_n]$ and $\sigma = [b_1, \ldots b_{i-1}, 1, b_{i+1}, \ldots, b_n]$. If the number of different elements between $\{a_1, \ldots, a_{i-1}\}$ and $\{b_1, \ldots, b_{i-1}\}$ is t, then $d_K(\pi, \sigma) \ge t^2 + t$.*

Proof Assume that the former t elements between $[a_1, \ldots a_{i-1}]$ and $[b_1, \ldots b_{i-1}]$ are different. Then, for all $1 \le j, k \le t$, we have that $\{a_j, b_k\}$, $\{1, a_j\}$ and $\{1, b_j\}$ in π and σ are different ordered pairs. Thus, we have $d_K(\pi, \sigma) \ge t^2 + t$. □

By Lemma 18, we only consider $T^{(5)}_{n,(i,i)}$ for $2 \le i \le n-2$. By Lemma 19, for $\pi = [a_1, \ldots a_{i-1}, 1, a_{i+1}, \ldots, a_n] \in S_{n,i}$, we can divide $T^{(5)}_{n,(i,i)}(\pi)$ into two disjoint subsets as follows. For convenience, we denote by $Fr_i(\pi) = \cup_{j=1}^{i-1}\{\pi(j)\}$ a set of the former $i-1$ elements of π, i.e., $Fr_i(\pi) = \{a_1, \ldots a_{i-1}\}$. Moreover, we denote by $L_i(\pi) = \{\beta | \beta(i) = 1, Fr_i(\beta) = Fr_i(\pi)$, and $d_K(\beta, \pi) \le 5\}$ and $R_i(\pi) = \{\beta |$the number of different elements between $Fr_i(\pi)$ and $Fr_i(\beta)$ is1, $\beta(i) = 1$, and $d_K(\beta, \pi) \le 5\}$.

Lemma 20 *Let $2 \le i \le n-2$. For any $\pi \in S_{n,i}$, we obtain that $T^{(5)}_{n,(i,i)}(\pi) = L_i(\pi) \cup R_i(\pi)$ and $L_i(\pi) \cap R_i(\pi) = \emptyset$.*

Proof By the definitions of $L_i(\pi)$ and $R_i(\pi)$, we clearly have $L_i(\pi) \cap R_i(\pi) = \emptyset$. Choose $\beta \in T^{(5)}_{n,(i,i)}(\pi)$ and let t be the number of different elements between $Fr_i(\pi)$ and $Fr_i(\beta)$. By Lemma 19, we obtain that $d_K(\beta, \pi) \ge t^2 + t$. Since $\beta \in T^{(5)}_{n,(i,i)}(\pi)$, we have $d_K(\beta, \pi) \le 5$ and $t \le 1$. Hence, if $t = 1$, then $\beta \in R_i(\pi)$. If $t = 0$, then $\beta \in L_i(\pi)$. So, we obtain that $T^{(5)}_{n,(i,i)}(\pi) = L_i(\pi) \cup R_i(\pi)$ and $L_i(\pi) \cap R_i(\pi) = \emptyset$. □

Example 5 Let $n = 11$ and $\pi = [3, 2, 1, 4, 5, 6, 7, 8, 9, 10, 11]$. Consider $T^{(5)}_{11,(3,3)}(\pi)$, we obtain the two kinds of permutations in $T^{(5)}_{11,(3,3)}(\pi)$. By using an adjacent transpositions on the former 2 elements of π, we obtain the first kind of permutation $\sigma = [2, 3, 1, 4, 5, 6, 7, 8, 9, 10, 11]$. By using three adjacent transpositions on the elements $\{3, 1, 4\}$ of σ, we obtain the second kind of permutation $\sigma' = [2, 4, 1, 3, 5, 6, 7, 8, 9, 10, 11]$ (i.e., $[2, 3, 1, 4, 5, 6, 7, 8, 9, 10, 11] \to [2, 3, 4, 1, 5, 6, 7, 8, 9, 10, 11] \to [2, 4, 3, 1, 5, 6, 7, 8, 9, 10, 11] \to [2, 4, 1, 3, 5, 6, 7, 8, 9, 10, 11]$).

By Lemma 10, the size of $L_i(\pi)$ and $R_i(\pi)$ does not depend on $\pi \in S_{n,i}$. Then, we denote by L_i and R_i the size of $L_i(\pi)$ and $R_i(\pi)$, respectively. The values of L_i and R_i is given in the following lemma and the proof of the next lemma is given in Appendix A.

Lemma 21 *For $2 \le i \le n-1$, we obtain that*

$$L_i = \sum_{t=0}^{5} S_K^{i-1}(t) B_K^{n-i}(5-t)$$

and

$$R_i = \begin{cases} B_K^{n-2}(2) + B_K^{n-2}(1) + B_K^{n-2}(0) & \text{if } i = 2 \text{ or } n-1, \\ B_K^{n-3}(2) + 3B_K^{n-3}(1) + 3 & \text{if } i = 3 \text{ or } n-2, \\ \sum_{t=0}^{2} S_K^{i-1}(t) B_K^{n-i}(2-t) + 2\sum_{t=0}^{1} S_K^{i-1}(t) B_K^{n-i}(1-t) + 2 & \text{if } 4 \le i \le n-3, \end{cases}$$

where $S_K^{i-1}(t) = 0$ for all $t \ge \binom{i-1}{\lfloor \frac{i-1}{2} \rfloor}$.

By Lemma 21, when $i = 3$, 4 and 5, we have

$$T_{n,(2,2)}^{(5)} = B_K^{n-2}(5) + B_K^{n-2}(2) + B_K^{n-2}(1) + 1, \tag{25}$$

$$T_{n,(3,3)}^{(5)} = B_K^{n-3}(5) + B_K^{n-3}(4) + B_K^{n-3}(2) + 3B_K^{n-3}(1) + 3, \tag{26}$$

$$T_{n,(i,i)}^{(5)} = \sum_{t=0}^{5} S_K^{i-1}(t) B_K^{n-i}(5-t) + \sum_{t=0}^{2} S_K^{i-1}(t) B_K^{n-i}(2-t) + 2\sum_{t=0}^{1} S_K^{i-1}(t) B_K^{n-i}(1-t) + 2, \tag{27}$$

for all $4 \le i \le n-3$, where $S_K^{i-1}(t) = 0$ for all $t \ge \binom{i-1}{\lfloor \frac{i-1}{2} \rfloor}$. By (17)–(20) and (25)–(27), we compute the size of $T_{n,(i,i)}^{(5)}$ as follows. For all $1 \le i \le n$, we have

$$T_{n,(i,i)}^{(5)} = \begin{cases} \frac{1}{120}(n^5 + 5n^4 - 15n^3 - 65n^2 - 46n + 120) & \text{if } i = 1 \text{ or } n, \\ \frac{1}{120}(n^5 - 25n^3 + 60n^2 - 36n) & \text{if } i = 2 \text{ or } n-1, \\ \frac{1}{120}(n^5 - 45n^3 + 120n^2 + 164n - 480) & \text{if } i = 3 \text{ or } n-2, \\ \frac{1}{120}(n^5 - 45n^3 + 60n^2 + 344n - 240) & \text{if } i = 4 \text{ or } n-3, \\ \frac{1}{120}(n^5 - 45n^3 + 60n^2 + 224n) & \text{if } i = 5 \text{ or } n-4, \\ \frac{1}{120}(n^5 - 45n^3 + 60n^2 + 224n - 120) & \text{if } 6 \le i \le n-5, \end{cases} \tag{28}$$

where $16 \le n$.

By (28), then we have

$$T_{n,(i,i)}^{(5)} - \frac{B_K^n(5)}{2} - 1 = \begin{cases} \frac{1}{240}n^5 - \frac{1}{16}n^3 - \frac{1}{4}n^2 + \frac{13}{30}n & \text{if } i = 1 \text{ or } n, \\ \frac{1}{240}n^5 - \frac{1}{24}n^4 - \frac{13}{48}n^3 + \frac{19}{24}n^2 + \frac{31}{60}n - 1 & \text{if } i = 2 \text{ or } n-1, \\ \frac{1}{240}n^5 - \frac{1}{24}n^4 - \frac{7}{16}n^3 + \frac{31}{24}n^2 + \frac{131}{60}n - 5 & \text{if } i = 3 \text{ or } n-2, \\ \frac{1}{240}n^5 - \frac{1}{24}n^4 - \frac{7}{16}n^3 + \frac{19}{24}n^2 + \frac{221}{60}n - 3 & \text{if } i = 4 \text{ or } n-3, \\ \frac{1}{240}n^5 - \frac{1}{24}n^4 - \frac{7}{16}n^3 + \frac{19}{24}n^2 + \frac{161}{60}n - 1 & \text{if } i = 5 \text{ or } n-4, \\ \frac{1}{240}n^5 - \frac{1}{24}n^4 - \frac{7}{16}n^3 + \frac{19}{24}n^2 + \frac{161}{60}n - 2 & \text{if } 6 \le i \le n-5. \end{cases} \tag{29}$$

By simple computations, for all $n \ge 16$, we obtain that

$$T_{n,(i,i)}^{(5)} - \frac{B_K^n(5)}{2} - 1 > 0, \tag{30}$$

for all $1 \le i \le n$.

By (30) and Theorem 3, if $n \geq 16$ and $t = 5$, we have A is nonsingular. Thus there does not exist a perfect five-error-correcting code in S_n if $n \geq 16$ is a prime. By the above discussion and Theorem 5, we have the following theorem.

Theorem 6 *There does not exist a perfect five-error-correcting code in S_n, where $n \geq 16$ is a prime or $n + 7 \geq 12$ is a prime or $n^3 + 3n^2 - 6n - 28$ has a prime factor $p > n$.*

5.3 The nonexistence of a perfect *t*-error-correcting code in S_n by using the third condition

In this subsection, we use Theorem 4 to prove the nonexistence of perfect t-error-correcting code in S_n for some n and $\frac{5}{8}\binom{n}{2} < 2t + 1 < \binom{n}{2}$.

Assume that $\frac{2}{3}\binom{n}{2} < 2t + 1 \leq \binom{n}{2}$. If $2t + 1 = \binom{n}{2}$, by Lemma 2 or Corollary 8 [2], we have $2B_K^n(t) = n!$ and there exists a perfect t-error-correcting code in S_n under the Kendall τ-metric. Otherwise, by Lemma 17, we obtain that $2B_K^n(t) < n!$ and there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric for all $\frac{2}{3}\binom{n}{2} < 2t + 1 \leq \binom{n}{2}$ except for $2t + 1 = \binom{n}{2}$.

Let $2t + 1 = \alpha\binom{n}{2}$ for $\frac{5}{8} \leq \alpha \leq \frac{2}{3}$. If $A_K(n, 2t + 1)$ is even, by (23), we have

$$A_K(n, 2t + 1) < 1 + \frac{1}{2\alpha - 1} \leq 5.$$

If $A_K(n, 2t + 1)$ is odd, by (24), we obtain

$$A_K(n, 2t + 1) < \frac{1}{2\alpha - 1} \leq 4.$$

Hence, we have that

$$A_K(n, 2t + 1) \leq 4 \tag{31}$$

for all $\frac{5}{8}\binom{n}{2} < 2t + 1 \leq \frac{2}{3}\binom{n}{2}$.

In order to prove the nonexistence of perfect t-error-correcting code in S_n for all $\frac{5}{8}\binom{n}{2} < 2t + 1 \leq \frac{2}{3}\binom{n}{2}$, we need some lemmas in the following. The proof of the next lemma is given in Appendix B.

Lemma 22 *Let p, q be two integers such that $1 \leq p < q$ and $p + q \leq \frac{1}{2}\binom{n}{2}$. Then, we have*

$$S_K^n(p) + S_K^n(q) + p - 2 < S_K^n(p + q)$$

for all $n \geq 6$.

By Lemma 22, we can obtain the following lemma.

Lemma 23 *For any $n \geq 5$, we obtain that*

$$4B_K^n\left(\left\lfloor \frac{1}{3}\binom{n}{2} - \frac{1}{2} \right\rfloor\right) < n!. \tag{32}$$

Proof By Lemma 2, we have $B_K^n\left(\left\lfloor \frac{1}{2}(\binom{n}{2} - 1) \right\rfloor\right) \leq \frac{1}{2}n!$. In order to obtain the result in (32), we only need to prove that

$$\sum_{i=0}^{\left\lfloor \frac{1}{3}\binom{n}{2} - \frac{1}{2} \right\rfloor} S_K^n(i) < \sum_{i=\left\lfloor \frac{1}{3}\binom{n}{2} - \frac{1}{2} \right\rfloor + 1}^{\left\lfloor \frac{1}{2}(\binom{n}{2} - 1) \right\rfloor} S_K^n(i) \tag{33}$$

for all $n \geq 6$.

Assume that $\binom{n}{2} = 6m$ for some $m \geq 3$. Then $\lfloor \frac{1}{3}\binom{n}{2} - \frac{1}{2} \rfloor = 2m - 1$ and $\lfloor \frac{1}{2}(\binom{n}{2} - 1) \rfloor = 3m - 1$. Hence, we have

$$\sum_{i=0}^{\lfloor \frac{1}{3}\binom{n}{2} - \frac{1}{2} \rfloor} S_K^n(i) = \sum_{i=0}^{2m-1} S_K^n(i)$$

and

$$\sum_{i=\lfloor \frac{1}{3}\binom{n}{2} - \frac{1}{2} \rfloor + 1}^{\lfloor \frac{1}{2}(\binom{n}{2} - 1) \rfloor} S_K^n(i) = \sum_{i=2m}^{3m-1} S_K^n(i).$$

By Lemma 22, we have $S^n(m+t) + S^n(m-t) + m - t - 2 < S^n(2m)$ for all $1 \leq t \leq m-1$. Hence, we obtain $S^n(m+t) + S^n(m-t) \leq S^n(2m)$ for all $1 \leq t \leq m-1$. Since $S^n(t)$ is a strictly increasing sequence for $1 \leq t \leq 3m$, then $S^n(0) + S^n(m) \leq S^n(2m)$. So, we have

$$\left(S^n(0) + S^n(m)\right) + \sum_{t=1}^{m-1} \left(S^n(m+t) + S^n(m-t)\right) \leq mS^n(2m) < \sum_{i=2m}^{3m-1} S_K^n(i). \quad (34)$$

Therefore, when $\binom{n}{2} = 6m$ and $n \geq 6$, by (34), we can obtain this result of (33).

Similarly, when $n \geq 6$ and $\binom{n}{2} = 6m + s$ for any $1 \leq s \leq 5$, we also have the result in (33). Thus, when $n \geq 6$, we obtain $4B_K^n\left(\lfloor \frac{1}{3}\binom{n}{2} - \frac{1}{2} \rfloor\right) < n!$. Moreover, when $n = 5$, we have $4B_K^5\left(\left\lfloor \frac{1}{3}\binom{5}{2} - \frac{1}{2} \right\rfloor\right) = 4B_K^5(2) = 64 < 5!$. So, when $n \geq 5$, we prove the above lemma. □

By (31) and Lemma 23, we can prove the following theorem.

Theorem 7 *Let $n \geq 5$. For any $\frac{5}{8}\binom{n}{2} < 2t + 1 < \binom{n}{2}$, there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric. For $2t + 1 = \binom{n}{2}$, there exists a perfect t-error-correcting code in S_n under the Kendall τ-metric.*

Proof When $\frac{2}{3}\binom{n}{2} < 2t+1 < \binom{n}{2}$, by the above discussion, we have that there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric except for $2t+1 = \binom{n}{2}$. When $\frac{5}{8}\binom{n}{2} < 2t+1 \leq \frac{2}{3}\binom{n}{2}$, by (31), we have $A_K(n, 2t+1) \leq 4$. Furthermore, by Lemma 23, we obtain $A_K(n, 2t+1) \cdot B_K^n(t) < n!$ for $\frac{5}{8}\binom{n}{2} < 2t+1 \leq \frac{2}{3}\binom{n}{2}$. Thus, there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric for all $\frac{5}{8}\binom{n}{2} < 2t+1 \leq \frac{2}{3}\binom{n}{2}$. So, when $\frac{5}{8}\binom{n}{2} < 2t+1 < \binom{n}{2}$, there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric. For $2t+1 = \binom{n}{2}$, by Lemmas 2 and 17, we have $A_K(n, d) = 2$ and $2B_K^n(t) = n!$. Therefore, there exists a perfect t-error-correcting code in S_n under the Kendall τ-metric for $2t+1 = \binom{n}{2}$. □

6 Conclusions

Permutation codes under the Kendall τ-metric have attracted lots of research interest due to their applications in flash memories. In this paper, we considered the nonexistence of perfect codes under the Kendalls τ-metric. We gave the polynomial representations of the size of a ball or a sphere with radius r when $t = 2, 3, 4,$ or 5. Moreover, we presented three sufficient conditions of the nonexistence of perfect permutation codes under the Kendall τ-metric. Finally, we used these sufficient conditions to prove that there does not exist a perfect t-error-correcting code in S_n under the Kendall τ-metric for some n and $t = 2, 3, 4, 5$, or $\frac{5}{8}\binom{n}{2} < 2t + 1 \leq \binom{n}{2}$. Specifically, we proved that there does not exist a perfect two-error-correcting code in S_n, where $n + 2 > 6$ is a prime. We also proved that there does not exist a perfect three-error-correcting code in S_n, where $n + 1 > 6$ is a prime or $n^2 + 2n - 6$ has a prime factor $p > n$ or $4 \leq n \leq 33$. We further proved that there does not exist a perfect four-error-correcting code in S_n, where $n + 1 > 6$ or $n + 2 > 7$ is a prime or $n^2 + 3n - 12$ has a prime factor $p > n$ or $5 \leq n \leq 19$. We proved that there does not exist a perfect five-error-correcting code in S_n, where $n \geq 16$ or $n + 7 \geq 12$ is a prime or $n^3 + 3n^2 - 6n - 28$ has a prime factor $p > n$. For $\frac{5}{8}\binom{n}{2} < 2t + 1 \leq \binom{n}{2}$ and $n \geq 5$, we proved that there does not exist a perfect t-error-correcting code in S_n except for $2t + 1 = \binom{n}{2}$.

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Appendix A

The purpose of this appendix is to prove Lemma 21 given in Section 5.2.

Proof We choose any permutation $\beta \in L_i(\pi)$. Then β is obtained by applying some t adjacent transpositions and at most $5-t$ adjacent transpositions on the former $i-1$ elements of π and the latter $n-i$ elements of π, respectively, where $0 \leq t \leq 5$. Thus, these operations can produce $S_K^{i-1}(t)B_K^{n-i}(5-t)$ permutations. If $t \geq \binom{i-1}{\lfloor \frac{i-1}{2} \rfloor}$, we have $S_K^{i-1}(t) = 0$. So, the size of the first kind of permutations is $\sum_{t=0}^{5} S_K^{i-1}(t)B_K^{n-i}(5-t)$.

Next, we choose $\pi = [2, 1, 3, \ldots, n] \in S_{n,2}$ such that $\pi(i) = i$ for all $i \geq 3$. By Lemma 10, we obtain $R_2 = |R_2(\pi)|$. If $\sigma \in R_2(\pi)$, then we have $\sigma(1) = 3, 4,$ or 5. When $\sigma(1) = 3$, we can obtain that elements 2 and 3 are exchanged and this operation needs at least 3 adjacent transpositions. Then the number of this kind of permutations in $R_2(\pi)$ is $B_K^{n-2}(2)$. When $\sigma(1) = 4$, we have that elements 2 and 4 are exchanged and this operation needs at least 4 adjacent transpositions. Hence, the number of this kind of permutations in $R_2(\pi)$ is $B_K^{n-2}(1)$. When $\sigma(1) = 5$, we obtain that elements 2 and 5 are exchanged and this operation needs at least 5 adjacent transpositions. Hence, the number of this kind of permutations in $R_2(\pi)$ is $B_K^{n-2}(0)$. So when $i = 2$, we obtain that $R_2 = B_K^{n-2}(2) + B_K^{n-2}(1) + B_K^{n-2}(0)$. By Lemma 18, when $i = n - 1$, we also get $R_{n-1} = B_K^{n-2}(2) + B_K^{n-2}(1) + B_K^{n-2}(0)$.

Similarly, when $i = 3$ or $n-2$, we can obtain that $R_i = B_K^{n-3}(2) + 3B_K^{n-3}(1) + 3$. When $4 \leq i \leq n-3$, we can prove that $R_i = \sum_{t=0}^{2} S_K^{i-1}(t) \cdot B_K^{n-i}(2-t) + 2\sum_{t=0}^{1} S_K^{i-1}(t)B_K^{n-i}(1-t) + 2$. □

Appendix B

The purpose of this appendix is to prove Lemma 22 given in Sect. 5.3.

Proof In order to obtain this result, we first prove that when $1 \le p < q$ and $p + q \le \frac{1}{2}\binom{n}{2}$, then

$$S_K^n(p) + S_K^n(q-1) < S_K^n(q) + S_K^n(p-1) \tag{35}$$

by induction for all $n \ge 6$.

When $n = 6$, we obtain $\binom{6}{2} = 15$. By Lemma 4, we compute $S_K^6(t)$ to obtain that $S_K^6(0) = 1$, $S_K^6(1) = 5$, $S_K^6(2) = 14$, $S_K^6(3) = 29$, $S_K^6(4) = 49$, $S_K^6(5) = 71$, $S_K^6(6) = 90$, $S_K^6(7) = 101$. Hence, when $n = 6$, we clearly have that $S_K^6(p) + S_K^6(q-1) < S_K^6(q) + S_K^6(p-1)$ for $1 \le p < q \le 7$ and $p + q \le 7$. So when $n = 6$, $S_K^6(t)$ satisfies the condition in (35).

Now we assume that $S_K^m(t)$ satisfies the condition in (35) for some integers $m \ge 6$, that is, if $1 \le p < q$ and $p + q \le \frac{1}{2}\binom{m}{2}$, then

$$S_K^m(p) + S_K^m(q-1) < S_K^m(q) + S_K^m(p-1). \tag{36}$$

When $n = m + 1$, by Lemma 4, we obtain

$$S_K^{m+1}(q) = S_K^{m+1}(q-1) + S_K^m(q) - S_K^m(q-m-1)$$
$$S_K^{m+1}(p) = S_K^{m+1}(p-1) + S_K^m(p) - S_K^m(p-m-1)$$

for $1 \le p < q$ and $p + q \le \frac{1}{2}\binom{m+1}{2}$. Hence, we have

$$\begin{aligned} S_K^{m+1}(q) + S_K^{m+1}(p-1) + S_K^m(p) + S_K^m(q-m-1) &= S_K^{m+1}(q-1) \\ +S_K^{m+1}(p) + S_K^m(q) + S_K^m(p-m-1). \end{aligned} \tag{37}$$

Using the induction hypothesis on $S_K^m(q)$, we can obtain some results as follows. When $p \ge m + 1$, then $1 \le p - m < q$ and $p + q - m - 1 < \frac{1}{2}\binom{m}{2} - \frac{m}{2} - 1$. Thus, by (36), we have

$$S_K^m(p-m) + S_K^m(q-1) < S_K^m(q) + S_K^m(p-m-1). \tag{38}$$

Since $q - 1 \ge \max\{p, q-m-1\}$, by (36), then $S_K^m(p) + S_K^m(q-m-1) < S_K^m(p-m) + S_K^m(q-1)$. Hence, by (38), we obtain $S_K^m(p) + S_K^m(q-m-1) < S_K^m(q) + S_K^m(p-m-1)$.

When $p < m+1$, then $S_K^m(p-m-1) = 0$. If $q < m+1$, we also have $S_K^m(q-m-1) = 0$. Since $p < q$, then $S_K^m(p) < S_K^m(q)$. If $q \ge m + 1$, then $1 \le p, q - m$ and $p + q - m < \frac{1}{2}\binom{m}{2} - \frac{m}{2}$. Hence, by (36), we obtain $S_K^m(p) + S_K^m(q-m) \le S_K^m(p+q-m)$. Assume $q \le \frac{1}{2}\binom{m}{2}$, then $p+q-m \le q$. Since $S_K^m(t)$ is an increasing sequence for all $0 \le t \le \frac{n}{2}\binom{m}{2}$, then $S_K^m(p+q-m) \le S_K^m(q)$. Assume $\frac{1}{2}\binom{m}{2} < q$ and $p+q \le \frac{1}{2}\binom{m+1}{2}$, then $\binom{m}{2} - q < \frac{1}{2}\binom{m}{2}$. By Lemma 2, $S_K^m(q) = S_K^m(\binom{m}{2} - q)$. Since $p + q - m < \binom{m}{2} - q < \binom{m}{2}$, we also have $S_K^m(p) + S_K^m(q-m-1) < S_K^m(p) + S_K^m(q-m) \le S_K^m(p+q-m) < S_K^m(\binom{m}{2} - q) = S_K^m(q)$. By the above discussion, we always have $S_K^m(p) + S_K^m(q-m-1) < S_K^m(q) + S_K^m(p-m-1)$ for $0 \le p < q$ and $p + q \le \frac{1}{2}\binom{m+1}{2}$. By (37), we obtain that

$$S_K^{m+1}(q-1) + S_K^{m+1}(p) < S_K^{m+1}(q) + S_K^{m+1}(p-1),$$

for $1 \le p < q$ and $p + q \le \frac{1}{2}\binom{m+1}{2}$. So, by induction, if $1 \le p < q$ and $p + q \le \frac{1}{2}\binom{n}{2}$, then

$$S_K^n(p) + S_K^n(q-1) < S_K^n(q) + S_K^n(p-1). \tag{39}$$

By (39), we obtain

$$S_K^n(p+q-t) + S_K^n(t) < S_K^n(p+q-t+1) + S_K^n(t-1)$$

for all $1 \leq t \leq p < q$ and $p + q \leq \frac{1}{2}\binom{n}{2}$. Hence, we can get

$$S_K^n(p) + S_K^n(q) + p - 1 < S_K^n(p+q) + S_K^n(0)$$

for $1 \leq p < q$ and $p + q \leq \frac{1}{2}\binom{n}{2}$. Since $S_K^n(0) = 1$, then we have

$$S_K^n(p) + S_K^n(q) + p - 2 < S_K^n(p+q)$$

for $1 \leq p < q$ and $p + q \leq \frac{1}{2}\binom{n}{2}$. □

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