Designs, Codes and Cryptography (2022) 90:2563–2581
https://doi.org/10.1007/s10623-021-00922-3

Symmetric functions and spherical t-designs in $\mathbb{R}^2$

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Received: 16 December 2020 / Revised: 7 May 2021 / Accepted: 7 August 2021 /
Published online: 28 August 2021

Abstract
Given k points in S^1 satisfying certain conditions which are determined through their symmetric functions, we introduce a method for constructing spherical t-designs in $\mathbb{R}^2$ with $2t+k$ elements. This approach points toward a better understanding of the space of spherical t-designs as well as provides a systematic way to obtain examples.

Keywords t-Designs · Symmetric functions · Group-type t-designs · Polynomials with unit-norm roots

Mathematics Subject Classification 05B30

1 Introduction

A finite subset X of S^{n-1} in $\mathbb{R}^n$ is a *spherical t-design* if for any polynomial $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ of degree at most t, the value of the integral of $f(\mathbf{x})$ on S^{n-1} divided by the volume of S^{n-1} equals the average value of $f(\mathbf{x})$ on the finite set X. They were introduced by Delsarte et al. [5]. In the survey [1] a detailed explanation of the developments in the subject can be found, along with a discussion of the connections of spherical t-designs with other fields of mathematics such as group theory, number theory and orthogonal polynomials, among others.

The case of spherical t-designs in $\mathbb{R}^2$ was studied in [8]. Consider the elements of $\mathbb{R}^2$ as complex numbers, thus

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Denote σ_i, $i = 1, \ldots, k$ the elementary symmetric polynomials (the symmetric functions, for short) of a set of k elements $\{z_1, \ldots, z_k\}$. We usually write σ_i instead of $\sigma_i(z_1, \ldots, z_k)$. A spherical t-design X in $\mathbb{R}^2$ can be defined by means of the complex polynomial having

This is one of several papers published in *Designs, Codes and Cryptography* comprising the "Special Issue: On Coding Theory and Combinatorics: In Memory of Vera Pless".

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the elements of X as its roots (for this and other equivalent definitions of spherical t-designs, see [1]).

Definition 1 Let t, n be positive integers, and $n \geq t+1$. A set $X = \{z_1, \ldots, z_n\} \subset S^1$ is a spherical t-design, or simply a t-design, if

$$\sigma_i(z_1, \ldots, z_n) = 0 \text{ for } i = 1, \ldots, t. \tag{1}$$

Equivalently, the polynomial

$$f(z) = (z - z_1) \cdots (z - z_n) = z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n \tag{2}$$

is such that $c_i = 0$ for $i = 1, \ldots, t$, because $c_i = (-1)^i \sigma_i$.

As the elements of a given t-design X are unit-norm complex numbers, we can observe the following:

Remark 2 (cf. [8, Lemma 2])

(i) The condition $\sigma_i = 0$ for $i = 1, \ldots, t$ is equivalent to $\sigma_{n-i} = 0$ for $i = 1, \ldots, t$.
(ii) $\overline{\sigma_i}\sigma_n = \sigma_{n-i}$, for $i = 1 \ldots, n$ (σ_0 is taken as 1).

Notice that this definition of t-design allows repeated elements in the set X.

It is clear that the n-th roots of any unit-norm complex number (regular n-gons) form a t-design if $n > t$. Moreover, it is easy to see that, if $n_1, \ldots, n_s$ are positive integer numbers greater than t and $w_1, \ldots, w_s \in S^1$, then the roots of the polynomial

$$f(z) = \prod_{i=1}^{s} (z^{n_i} - w_i), \tag{3}$$

also form a t-design with $n = \sum_{i=1}^{s} n_i$ elements. In [8], spherical t-designs as such are called *group-type* t-designs, while a t-design that is not a group-type t-design is called a *non-group-type* t-design. In particular, for any positive integer t there exists always group-type t-designs with n elements, provided $n \geq t+1$. The natural question of deciding whether there are t-designs besides these easily obtained ones is answered by the following theorem.

Theorem 3 [8, Theorem A] *Let X be a spherical t-design in $\mathbb{R}^2$ with $|X| = n$. Then*

(i) for $t+1 \leq n \leq 2t+2$, X is always a group-type t-design; more precisely, when $t+1 \leq n \leq 2t+1$, X is a regular n-gon and when $n = 2t+2$, X is the union of two regular $(t+1)$-gons (this includes the $(2t+2)$-gon case);
(ii) for each $n \geq 2t+3$, besides group-type t-designs, there are as many as $\aleph_1$ non-group-type t-designs.

This paper goes deeper in the understanding of t-designs in $\mathbb{R}^2$ as started in [8], though our approach is different. We can say that, in some way, we parameterize the design with some of its points. The precise information to make this possible turn out to be in the symmetric functions of this points, more specifically, on algebraic and metric conditions upon such functions. Moreover, group-type t-designs can be obtained through easily described algebraic conditions.

In Sect. 4 our general method to construct a t-design X with $|X| = 2t+k$, given $a_1, \ldots, a_k \in S^1$ is given. The $a_i's$ will belong to X while the other $2t$ elements in the design turn out to be the roots of a polynomial whose coefficients are expressed in terms of the symmetric functions $\sigma_1, \ldots, \sigma_k$ of the $a_i's$. Such polynomial will be denoted by $G_{k,t}$. So, conditions must be imposed on $a_1, \ldots, a_k$ in order that: (1) the coefficients of the polynomial $G_{k,t}$ are well defined and, (2) the roots of $G_{k,t}$ are all unit-norm complex numbers. When these conditions are satisfied we will say that the $a_i's$ are in *good position*. Using this method, we construct a family of 3-designs in Example 34.

For clarity in the exposition, our method is illustrated first in Sect. 3 by means of the simplest case where a non-group spherical t-design can appear: a spherical 2-design of 7 elements. In Sect. 3.1 we show how conditions on three unit-norm complex numbers naturally appear when trying to construct a spherical 2-design of 7 elements containing them. Proposition 14 assures the existence and unicity of such a design under the obtained conditions and Example 18 gives a family of non-group-type 2-designs. In this case we are able to give a complete description of our object of study: Proposition 22 states that any 2-design with 7 elements, where at least five of them are distinct, can be obtained by this method. In Sect. 3.3 conditions for a 2-design with 7 elements to be of group-type are given and this result is generalized latter for any t and $k = t + 1$, in Proposition 37.

Another reason for giving a separate treatment to 2-designs is that they are an important kind of unit norm tight frames (see [2]), also known as *balanced unit norm tight frames*, which are studied in [7].

Our results indicate that it makes sense to consider the space of t-designs in the σ-space, that is, the space whose coordinates are the symmetric functions. This is explained in Sect. 5 and applied to the case of 2-designs. Further development of this approach could contribute to a new understanding of spherical t-designs.

2 Polynomials with roots on the unit circle

In this article, we are faced with the problem of deciding whether a given complex polynomial has all of its roots in S^1. This is not a trivial issue and it is particularly difficult to find necessary and sufficient conditions **on the coefficients** of the polynomials (see, for instance, [10] and [3]).

Definition 4 A polynomial $P(z) = \sum_{k=0}^{m} \alpha_k z^k \in \mathbb{C}[z]$ is self-inversive if $\alpha_{m-k} = w\overline{\alpha_k}$ for $k = 0, \ldots, m$, where $w \in \mathbb{C}$, $|w| = 1$.

It is easily seen that a polynomial with all of its roots in the unit circle is self-inversive, but the following classical result says much more:

Theorem 5 (Cohn [3]) *A complex polynomial P has all of its zeros in the unit circle if and only if the following two conditions holds:*

i) P is self-inversive,
ii) All zeros of P' lie in the unit disk.

It is also not an easy task to decide whether a given polynomial has its roots on the unit disk. However, for the case of degree 3, we can give the following criteria.

Proposition 6 *Let $g(z) = z^3 + az^2 + bz + c \in \mathbb{C}[z]$ such that $c - ba \neq 0$ and let $D = \{z \in \mathbb{C} : |z| \leq 1\}$. If $|a| < 1$ and $|c - ab| < 1 - |a|^2 - |\overline{a}c - b|$, then $g(z)$ has all of its roots in D.*

If $c = ba$ then the roots of $g(z)$ lie in D if and only if $a, b \in D$.

Proof Observe that z is a root of $g(z)$ if and only if

$$z^2 = \frac{-bz - c}{z + a}.$$

By hypothesis, the mapping $M : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $M(z) = \dfrac{-bz - c}{z + a}$ is a *Möbius transformation* (see, for instance, [4]) such that, if $|a| \neq 1$, $M(S^1)$ is a circle C with center $z_0 = -\dfrac{\overline{a}c - b}{|a|^2 - 1}$ and radius $\rho = \dfrac{|c - ab|}{||a|^2 - 1|}$.

If $|a| < 1$ the outside of D is mapped to the inside of C, while the second hypothesis is equivalent to $\rho < 1 - |z_0|$, which implies that C and its interior are contained in D. So, if z is a root of $g(z)$ and $|z| > 1$, then $M(z) = z^2 \in D$, a contradiction.

The last affirmation is obvious since $z^3 + az^2 + bz + ab = (z^2 + b)(z + a)$. □

The following result, due to Lakatos and Losonczi [9] (see also [10]) gives sufficient conditions for a self-inversive polynomial to have all of its roots on the unit circle.

Theorem 7 (Lakatos–Losonczi) *All zeros of a self-inversive polynomial* $P(z) = \sum_{k=0}^{m} \alpha_k z^k \in \mathbb{C}[z]$ *of degree* $m \geq 1$ *are on the unit circle if*

$$|\alpha_m| \geq \frac{1}{2} \sum_{k=1}^{m-1} |\alpha_k|. \tag{4}$$

For real polynomials with roots on the unit circle, the relation between them and their coefficients can be clarified: Let $b = (b_1, \dots, b_d) \in \mathbb{R}^d$ and denote g_b the polynomial

$$g_b = (x^{2d} + 1) + b_1(x^{2d-1} + x) + \cdots + b_{d-1}(x^{d+1} + x^{d-1}) + b_d x^d.$$

It is not difficult to see that if $g \in \mathbb{R}[z]$ is a monic polynomial of degree $2d$ such that all of its complex roots lie on S^1 and 1 and -1 are roots of g with even multiplicity (possibly 0), then $g = g_b$ for some $b \in \mathbb{R}^d$. Moreover, let V_d be the set of those $b \in \mathbb{R}^d$ such that all of the roots of g_b lie in S^1 and

$$I_d = \{(r_1, \dots, r_d) \in \mathbb{R}^d : -2 \leq r_1 \leq r_2 \leq \cdots \leq r_d \leq 2\}.$$

The following proposition is Lemma 2.1.1 in [6].

Proposition 8 *V_d is homeomorphic to I_d.*

Remark 9 The homemorphism Φ of Proposition 8 is given in the following way: for $r = (r_1, \dots, r_d) \in I_d$, define $\Phi(r)$ to be the unique $b \in \mathbb{R}^d$ such that

$$x^d \left(\sum_{i=0}^{d} (-1)^i \sigma_i(r_1, \dots, r_d) \left(x + \frac{1}{x} \right)^{d-i} \right) = g_b(x),$$

where σ_0 is taken as 1.

Example 10 The set V_1 is the closed interval $[-2, 2]$. For $d = 2$, we have that $(b_1, b_2) = \Phi(r_1, r_2) = (-r_1 - r_2, r_1 r_2 + 2)$ and, as Φ transforms the boundary of I_2 in the boundary of V_2, we have that

$$V_2 = \{(b_1, b_2) : 2|b_1| - 2 \leq b_2 \leq \frac{b_1^2}{4} + 2\}. \tag{5}$$

More generally, a description of the space of complex polynomials such that all of their roots lie in S^1 can be given.

Definition 11 If the complex polynomial $L(z) = z^k + c_1 z^{k-1} + \cdots + c_{k-1}z + c_k$ satisfies $c_k = 1$ and $\overline{c_i} = c_{k-i}$ for $i = 1 \ldots, k$, then it is called **conjugate reciprocal**.

In [11], the geometry, topology and Lebesgue measure of the space of conjugate reciprocal polynomials of fixed degree with all roots in S^1 is studied. To state some of their results, we introduce the matrix $X_k \in \mathbb{C}^{k-1\times k-1}$ by giving its l, j entry:

$$[X_k]_{l,j} = \begin{cases} \delta_{l,j} + \delta_{k-l,j} & \text{if } 1 \leq l < k/2 \\ \delta_{l,j} & \text{if } l = k/2 \\ i\delta_{k-l,j} - i\delta_{l,j} & \text{if } k/2 < l < k, \end{cases} \tag{6}$$

where $\delta_{l,j}$ is the Kronecker delta and i is the imaginary unit (in [11] this matrix is normalized, but this is unimportant for us). The key point here is that the polynomial

$$l(x) = (x^k + 1) + \sum_{n=1}^{k-1} c_n x^{k-n}$$

is conjugate reciprocal if and only if there exists $w \in \mathbb{R}^{k-1}$ such that $c = X_k w \in \mathbb{C}^{k-1}$, in which case it is denoted $w(x) = l(x)$. Define W_k as the set

$$W_k = \{w \in \mathbb{R}^{k-1} : w(x) \text{ has all roots in } S^1\}. \tag{7}$$

We have the following theorem which summarizes some of the results in [11]:

Theorem 12 *W_k is homeomorphic to the unit ball of $\mathbb{R}^{k-1}$ and its boundary is the set of those $w \in W_k$ such that $\Delta(w(x)) = 0$, where Δ denotes the discriminant of a given polynomial.*

Example 13 To describe the set W_3, recall that

$$\Delta(ax^3 + bx^2 + cx + d) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

Thus, we have

$$\begin{aligned} \Delta(w_1, w_2) &= \Delta(x^3 + (w_1 + iw_2)x^2 + (w_1 - iw_2)x + 1) \\ &= w_1^4 + 2w_1^2w_2^2 + w_2^4 - 8w_1^3 + 24w_1w_2^2 + 18w_1^2 + 18w_2^2 - 27 \end{aligned} \tag{8}$$

and the boundary of W_3 is the set $\{(w_1, w_2) \in \mathbb{R}^2 : \Delta(w_1, w_2) = 0\}$.

3 2-designs with 7 elements

3.1 Existence

Let $a_1, a_2, a_3 \in S^1$ and assume that there is a 2-design X, with $|X| = 7$ containing them. This is equivalent to the existence of a polynomial of degree 7

$$f(z) = z^7 + c_6 z^6 + \cdots + c_1 z + c_0 \tag{9}$$

such that

i) $c_6 = c_5 = 0$ and
ii) a_1, a_2, a_3 are roots of $f(z)$ and the other roots b_1, b_2, b_3, b_4 also belong to S^1.

As we saw, condition i) can be replaced by $c_2 = c_1 = 0$ and, moreover, these two conditions are equivalent if we assume ii) to be true. Taking this into account, we try to obtain the coefficients of such $f(z)$.

Let $\sigma_1, \sigma_2, \sigma_3$ be the symmetric functions of a_1, a_2, a_3 and write

$$f(z) = \prod_{i=1}^{3}(z-a_i)\prod_{i=1}^{4}(z-b_i) = (z^3-\sigma_1 z^2+\sigma_2 z-\sigma_3)(z^4-\tilde{\sigma_1}z^3+\tilde{\sigma_2}z^2-\tilde{\sigma_3}z+\tilde{\sigma_4}), \quad (10)$$

where $\tilde{\sigma_1}, \tilde{\sigma_2}, \tilde{\sigma_3}, \tilde{\sigma_4}$ are the symmetric functions of b_1, b_2, b_3, b_4. To obtain the $\tilde{\sigma_i}$'s we impose $c_6 = c_5 = c_2 = c_1 = 0$. By calculating the product of the two polynomials in the right-hand side of (10) and equating to zero the coefficients of z^6, z^5, z^2 and z, we come to

$$\tilde{\sigma_1} = -\sigma_1 \quad (11)$$

$$\tilde{\sigma_2} = -\tilde{\sigma_1}\sigma_1 - \sigma_2 = \sigma_1^2 - \sigma_2 \quad (12)$$

and to the system

$$\begin{cases} \sigma_2\tilde{\sigma_3} + \sigma_1\tilde{\sigma_4} = -\sigma_3(\sigma_1^2 - \sigma_2) \\ \sigma_3\tilde{\sigma_3} + \sigma_2\tilde{\sigma_4} = 0 \end{cases} \quad (13)$$

Assuming that $\sigma_2^2 - \sigma_3\sigma_1 \neq 0$, we obtain

$$\tilde{\sigma_3} = -\frac{\sigma_3\sigma_2(\sigma_1^2 - \sigma_2)}{\sigma_2^2 - \sigma_1\sigma_3}, \quad \tilde{\sigma_4} = \frac{\sigma_3^2(\sigma_1^2 - \sigma_2)}{\sigma_2^2 - \sigma_1\sigma_3}. \quad (14)$$

Thus we have obtained the following proposition, to be generalized in Sect. 4.

Proposition 14 *Let $a_1, a_2, a_3 \in S^1$ such that $\sigma_2^2 - \sigma_3\sigma_1 \neq 0$.*

i) If the roots b_1, b_2, b_3, b_4 of the polynomial

$$G(z) := z^4 + \sigma_1 z^3 + (\sigma_1^2 - \sigma_2)z^2 + \frac{\sigma_3\sigma_2(\sigma_1^2 - \sigma_2)}{\sigma_2^2 - \sigma_1\sigma_3}z + \frac{\sigma_3^2(\sigma_1^2 - \sigma_2)}{\sigma_2^2 - \sigma_1\sigma_3} \quad (15)$$

are unit-norm complex numbers then the set $X = \{a_1, a_2.a_3, b_1, b_2, b_3, b_4\}$ is the unique spherical 2-design with 7 elements that contains a_1, a_2, a_3.

ii) If some root of $G(z)$ does not belong to S^1 then there is not a spherical 2-design with 7 elements containing a_1, a_2, a_3.

Remark 15 Note that the condition on the polynomial $G(z)$ is actually a condition on a_1, a_2, a_3. We specify this in Definition 25 below.

Now, we can apply the results of Sect. 2 to establish when our 2-design actually exists. Surprisingly, **the polynomial $G(z)$ turns out to be self-inversive with no further conditions** on the numbers $a_1, a_2, a_3 \in S^1$ (this is easy to check using $\bar{\sigma_2}\sigma_3 = \sigma_1$ and $|\sigma_3| = 1$, see Proposition 31 below for the general case), so we apply Proposition 6 to $G'(z)$ in order to satisfy the second condition of Theorem 5.

Corollary 16 *Let $a_1, a_2, a_3 \in S^1$ such that $\sigma_2^2 - \sigma_3\sigma_1 \neq 0$ and $G(z)$ the polynomial defined in Eq.(15). In the following cases there exists a 2-design containing them.*

1. *If $2\sigma_3\sigma_2 \neq 3\sigma_1(\sigma_2^2 - \sigma_3\sigma_1)$, $0 < |\sigma_1| < \frac{4}{3}$ and $\left|3\sigma_1\sigma_2^2 - 3\sigma_1^2\sigma_3 - 2\sigma_2\sigma_3\right| < 8 - \frac{9}{2}|\sigma_1|^2 - \left|4\sigma_1\sigma_3 - \frac{5}{2}\sigma_2^2\right|$.*
2. *If $2\sigma_3\sigma_2 = 3\sigma_1(\sigma_2^2 - \sigma_3\sigma_1)$ and $\frac{3}{4}\sigma_1, \dfrac{\sigma_1^2 - \sigma_2}{2} \in D$.*

3. *If* $|\sigma_1| < \alpha$*, where* α *is the real root of the polynomial* $3x^3+10x^2+6x-8$ *(*$\alpha \simeq 0.60712$*).*

Proof The first two cases follow from applying Proposition 6 to

$$\frac{1}{4}G'(z) = z^3 + \frac{3}{4}\sigma_1 z^2 + \frac{\sigma_1^2-\sigma_2}{2}z + \frac{\sigma_3\sigma_2(\sigma_1^2-\sigma_2)}{4(\sigma_2^2-\sigma_1\sigma_3)},$$

using the fact that $|\sigma_1^2-\sigma_2| = |\sigma_2^2-\sigma_1\sigma_3|$, which follows from $\sigma_3^2\overline{(\sigma_1^2-\sigma_2)} = \sigma_2^2-\sigma_1\sigma_3$ (note also that the case $\sigma_1 = 0$ is excluded because, due to $\bar{\sigma_2}\sigma_3 = \sigma_1$, it would implies $\sigma_2 = 0$ and then $\sigma_2^2-\sigma_3\sigma_1 = 0$). The condition of the third case is stronger than the two previous ones. If $2\sigma_3\sigma_2 \neq 3\sigma_1(\sigma_2^2-\sigma_3\sigma_1)$, apply triangle inequality and, if $2\sigma_3\sigma_2 = 3\sigma_1(\sigma_2^2-\sigma_3\sigma_1)$, the two inequalities follows. □

Remark 17 For arbitrary t, we will define a general polynomial $G_{k,t}$ (see Definition 25 below) so that $G(z) = G_{3,2}(z)$ and obtain a general condition implying that all of its roots are in the unit circle (see Corollary 32). Nevertheless, in the case $t = 2$, the conditions given by Corollary 16 are more accurate.

We can use Example 10 to give a family of examples of 2-designs.

Example 18 Let $z_0 \in S^1$ and set

$$a_1 = 1,\ a_2 = z_0,\ a_3 = \overline{z_0}.$$

We want to find out when there is a 2-design containing $1, z_0, \overline{z_0}$. We have

$$\sigma_1 = \sigma_2 = 1 + 2\Re(z_0),\ \sigma_3 = 1,$$

then $\sigma_2^2-\sigma_1\sigma_3 \neq 0$ if and only if $z_0 \neq \pm i$ and $z_0 \neq -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, in order to be able to define the polynomial in (15), we discard these cases.

Thus, we need to check when the polynomial

$$G_0(z) = z^4 + (1+2\Re(z_0))z^3 + 2\Re(z_0)(1+2\Re(z_0))z^2 + (1+2\Re(z_0))z + 1 \tag{16}$$

has all of its roots in S^1. Note first that 1 is not a root of G_0, because in that case $\Re(z_0) \notin \mathbb{R}$, and if -1 is a root, has multiplicity 2. By (5), taking $b_1 = 1 + 2\Re(z_0)$ and $b_2 = 2\Re(z_0)(1 + 2\Re(z_0))$, we have that $G_0(z)$ has all of its roots in S^1 if and only if $\Re(z_0)$ belongs to the set

$$\left\{x \in \mathbb{R} : 2|2x+1| - 2 \leq 2x(2x+1) \leq \tfrac{(2x+1)^2}{4} + 2\right\} = \left[\tfrac{-1-\sqrt{28}}{6}, 0\right] \cup \left[\tfrac{1}{2}, \tfrac{-1+\sqrt{28}}{6}\right].$$

As $\frac{-1-\sqrt{28}}{6} \simeq -1,0485$ and $z_0 \in S^1$, we conclude that $G_0(z)$ has all of its roots in S^1 if and only if $\Re(z_0) \in \left[-1, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup \left[\frac{1}{2}, \frac{-1+\sqrt{28}}{6}\right]$. We depict, for some particular values of z_0, the corresponding 2-designs in Fig. 1. The grey points are the roots of the corresponding $G_0(z)$.

Remark 19 Notice that, in the previous example, the condition $\sigma_2^2-\sigma_1\sigma_3 = 0$ implies that our three input points are either the cubic roots of the unity or three of the four quartic roots of the unity. Therefore, in both cases there exists an infinite number of group-type 2-designs containing them. It is also worth noticing that the cases $z_0 = -1$ and $z_0 = \frac{1}{2}$ both correspond to the group-type 2-design given by the union of the quartic roots of 1 and the cubic roots of -1. Here -1 belongs to both sets of roots.

Remark 20 If we would have applied Corollary 16 to work this example, we would have obtained that for $\Re(z_0) \in \left(-1, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right)$ there exists a 2-design containing $1, z_0, \overline{z_0}$, that is, a less precise condition.

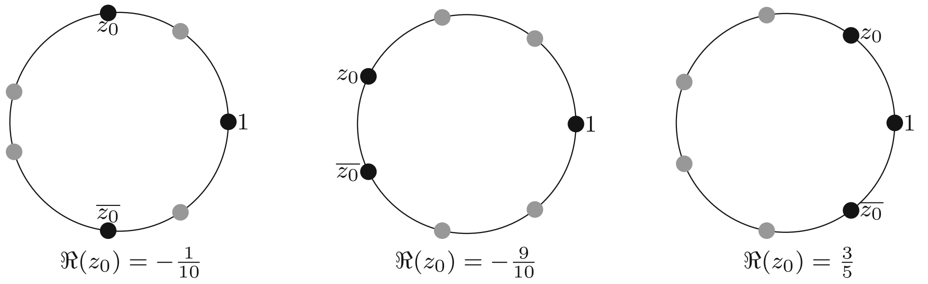

Fig. 1 Examples of 2-designs

3.2 Characterization of 2-designs

We would like to give a reciprocal statement to Proposition 14: Does any 2-design with 7 elements can be obtained in this way? To answer to this question we need the following lemmas.

Lemma 1 *Let $a_1, a_2, a_3 \in S^1$ such that $\sigma_2^2 - \sigma_3\sigma_1 = 0$. Then one of the following is true:*

i) a_1, a_2, a_3 are the cubic roots of a unit-norm complex number, or
ii) some pair of the a_i's are opposite.

Proof As the polynomial

$$\prod_{i=1}^{3}(z - a_i) = z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_3$$

is self inversive, we have $\overline{\sigma_1}\sigma_3 = \sigma_2$ and $|\sigma_3| = 1$. Then $|\sigma_1|^2 = |\sigma_2|^2 = |\sigma_1|$, that is $|\sigma_1| = 0$ or $|\sigma_1| = 1$. If $|\sigma_1| = 0$ we have $\sigma_1 = \sigma_2 = 0$ and then a_1, a_2, a_3 are the cubic roots of σ_3. On the other hand, if $|\sigma_1| = 1$, we have $\sigma_3 = \sigma_1\overline{\sigma_1}\sigma_3 = \sigma_1\sigma_2$ and then a_1, a_2, a_3 are the roots of

$$z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_1\sigma_2 = (z - \sigma_1)(z^2 + \sigma_2) \tag{17}$$

which implies *ii)*. □

Lemma 2 *Let X be a finite subset of S^1 such that $|X| \geq 3$ and for any three distinct points in X, $\sigma_2^2 - \sigma_3\sigma_1 = 0$, where $\sigma_1, \sigma_2, \sigma_3$ are the corresponding symmetric functions. Then one the following options is true:*

(i) $|X| = 3$ and the elements of X are the cubic roots of a unit-norm complex number, or
(ii) $|X| = 4$ and the elements of X are the quartic roots of a unit-norm complex number.

Proof If there exists a_1, a_2, a_3 distinct elements of X such that $\sigma_1 = 0$, then, by Lemma 1, they are the cubic roots of σ_3 and, moreover, there are not other (distinct) elements in X. In fact, let $a_4 \in X$ such that $a_4 \neq a_i$, $i = 1, 2, 3$. Applying Lemma 1 to a_1, a_2, a_4 we have that $a_4 = -a_1$ or $a_4 = -a_2$. If $a_4 = -a_1$ apply Lemma 1 to a_3, a_2, a_4 and get $a_1 = a_2$ or $a_1 = a_3$, a contradiction. Similarly if $a_4 = -a_2$. We conclude that a_1, a_2, a_3 are the only distinct elements of X.

So, we are left with the case of $|\sigma_1| = 1$: take three distinct points, they are the roots of a polynomial as in (17). As we have $\sigma_2 = \sigma_1^2$, any other distinct point in X must be equal to $-\sigma_1$ so the elements of X are the roots of $z^4 - \sigma_1^4$. □

Corollary 21 *Let X be a spherical 2-design with 7 elements, and at least five of them are distinct. Then there exists three elements of X such that $\sigma_2^2 - \sigma_3\sigma_1 \neq 0$, where $\sigma_1, \sigma_2, \sigma_3$ are the corresponding symmetric functions.*

Now we can apply Proposition 14 to obtain our characterization result.

Proposition 22 *Let X a 2-design with 7 elements, and at least five of them distinct. Then there exist $a_1, a_2, a_3 \in X$ such that $X = \{a_1, a_2.a_3, b_1, b_2, b_3, b_4\}$ where b_1, b_2, b_3, b_4 are the roots of the polynomial $G(z)$ defined in (15) and $\sigma_1, \sigma_2, \sigma_3$ are the symmetric functions of a_1, a_2, a_3.*

3.3 Group-type 2-designs

What can be said about group-type 2-designs under this point of view? To answer that question, first note the following fact, whose proof is evident.

Proposition 23 *A spherical 2-design of 7 elements is of group-type if and only if it is given by the roots of a polynomial of the shape*

$$f(z) = z^7 + \alpha z^4 + \beta z^3 + \gamma \tag{18}$$

where one of the following options holds:

i) $\alpha, \beta \in S^1$ *and* $\gamma = \alpha\beta$.
ii) $\alpha = \beta = 0$ *and* $\gamma \in S^1$.

We would like to describe group-type t-designs in terms of symmetric functions. The following proposition shows that this is possible.

Proposition 24 *Let $a_1, a_2, a_3 \in S^1$ such that $\sigma_2^2 - \sigma_3\sigma_1 \neq 0$. If $\sigma_3 = \sigma_2\sigma_1$ then there is a spherical 2-design X containing a_1, a_2, a_3. Moreover, X is a group-type 2-design.*

Proof We know by Proposition 14 that if there is a 2-design with 7 elements containing a_1, a_2, a_3, their other four elements must be the roots of the polynomial $G(z)$ in equation (15), provided that all of them are in S^1. Instead of finding this out, we make the product

$$f(z) = (z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_3)\left(z^4 + \sigma_1 z^3 + (\sigma_1^2 - \sigma_2)z^2 + \frac{\sigma_3\sigma_2(\sigma_1^2 - \sigma_2)}{\sigma_2^2 - \sigma_1\sigma_3} z + \frac{\sigma_3^2(\sigma_1^2 - \sigma_2)}{\sigma_2^2 - \sigma_1\sigma_3}\right) \tag{19}$$

and show that $f(z)$ satisfies the first condition on Proposition 23.

First, observe that, due to $a_1, a_2, a_3 \in S^1$, we have $\overline{\sigma_1}\sigma_3 = \sigma_2$, which, together with our assumption, gives

$$|\sigma_1|^2\sigma_2 = \sigma_2.$$

If $\sigma_2 = 0$ then $\sigma_1 = 0$ (because $|\sigma_3| = 1$), which would imply $\sigma_2^2 - \sigma_3\sigma_1 = 0$, a contradiction. So $|\sigma_1| = |\sigma_2| = 1$. We also have,

$$\sigma_2^2 - \sigma_1\sigma_3 = \sigma_2(\sigma_2 - \sigma_1^2)$$

and then

$$\begin{aligned} f(z) &= (z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_1\sigma_2)(z^4 + \sigma_1 z^3 + (\sigma_1^2 - \sigma_2)z^2 - \sigma_1\sigma_2 z - \sigma_1^2\sigma_2) = \\ &= z^7 - \sigma_1^3 z^4 - \sigma_2^2 z^3 + \sigma_1^3\sigma_2^2. \end{aligned}$$

□

Alternatively, we can combine Propositions 23, 14 and 22 to obtain necessary and sufficient conditions for 2-design with 7 elements to be of group-type . In fact, both two cases in Proposition 23 applied to the polynomial

$$\left(z^3-\sigma_1 z^2+\sigma_2 z-\sigma_3\right)\left(z^4+\sigma_1 z^3+(\sigma_1^2-\sigma_2)z^2+\frac{\sigma_3\sigma_2(\sigma_1^2-\sigma_2)}{\sigma_2^2-\sigma_1\sigma_3}z+\frac{\sigma_3^2(\sigma_1^2-\sigma_2)}{\sigma_2^2-\sigma_1\sigma_3}\right) \tag{20}$$

turn out in algebraic and norm conditions on the σ_i's, though these conditions will be rather cumbersome. In Sect. 5 further tools to understand the space of t-designs in the σ-space will be given.

4 Spherical *t*-designs in $\mathbb{R}^2$

4.1 Existence

In this section we generalize the principal results obtained for the case $t = 2$. To state our main theorem we need the following definition.

Definition 25 Let t, k be positive integer numbers. Let $a_1, \ldots, a_k \in S^1$ and $\sigma_1, \ldots, \sigma_k$ their corresponding symmetric functions. We say that $a_1, \ldots, a_k$ are in ***t*-good position** if the following two conditions hold:

i) The determinant $\Delta_{k,t} = \Delta_{k,t}(\sigma_1, \ldots, \sigma_k)$ of the matrix

$$T_{k,t} = \begin{bmatrix} \sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \ldots & \sigma_{k-t+1} & \sigma_{k-t} \\ \sigma_k & \sigma_{k-1} & \sigma_{k-2} & \ldots & \sigma_{k-t+2} & \sigma_{k-t+1} \\ 0 & \sigma_k & \sigma_{k-1} & \ldots & \sigma_{k-t+3} & \sigma_{k-t+2} \\ 0 & 0 & \sigma_k & \ldots & \sigma_{k-t+4} & \sigma_{k-t+3} \\ \vdots & \vdots & \ldots & \ldots\ldots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 & \sigma_k & \sigma_{k-1} \end{bmatrix} \in \mathbb{C}^{t\times t} \tag{21}$$

is non-zero. Here σ_i is taken as zero if $i \le 0$.

ii) All of the roots of the polynomial

$$G_{k,t}(z) = z^{2t} - \tilde{\sigma}_1 z^{2t-1} + \tilde{\sigma}_2 z^{2t-2} - \cdots - \tilde{\sigma}_{2t-1} z + \tilde{\sigma}_{2t} \tag{22}$$

lie in S^1, where the coefficients $\tilde{\sigma}_i$ are defined as follows: for $i = 1, \ldots, t$ they satisfy the recursion formula

$$\tilde{\sigma}_1 = -\sigma_1 \text{ and } \tilde{\sigma}_i = -\sigma_i - \textstyle\sum_{j=1}^{i-1} \tilde{\sigma}_j \sigma_{i-j} \text{ for } i = 2, \ldots, t \tag{23}$$

while, for $i = t+1, \ldots, 2t$, they are obtained as the unique solutions of the system

$$T_{k,t} \cdot \begin{bmatrix} \tilde{\sigma}_{t+1} \\ \tilde{\sigma}_{t+2} \\ \vdots \\ \tilde{\sigma}_{2t} \end{bmatrix} = \begin{bmatrix} -\sigma_k \tilde{\sigma}_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{24}$$

Example 26 For any t the matrix $T_{1,t}$ is singular, so one single point is never in t-good position.

Example 27 Equations (11),(12) and (14) give the expressions for the $\tilde{\sigma}_i$'s if $t = 2$, $k = 3$ and $a_1, a_2, a_3 \in S^1$ are such that $\Delta_{3,2} = \sigma_2^2 - \sigma_1\sigma_3 \neq 0$. The polynomial $G_{3,2}(z)$ is equal to the polynomial $G(z)$ in (15).

Example 28 Set $t = 3$ and $k = 4$. The expressions for $\tilde{\sigma}_i$ are, for $i = 1, 2, 3$:

$$\tilde{\sigma}_1 = -\sigma_1,\ \tilde{\sigma}_2 = -\sigma_2 + \sigma_1^2,\ \tilde{\sigma}_3 = -\sigma_3 + 2\sigma_1\sigma_2 - \sigma_1^3.$$

Assuming $\Delta_{4,3} = \sigma_3^3 - 2\sigma_2\sigma_3\sigma_4 + \sigma_1\sigma_4^2 \neq 0$, we can solve the system

$$\begin{bmatrix} \sigma_3 & \sigma_2 & \sigma_1 \\ \sigma_4 & \sigma_3 & \sigma_2 \\ 0 & \sigma_4 & \sigma_3 \end{bmatrix} \cdot \begin{bmatrix} \tilde{\sigma}_4 \\ \tilde{\sigma}_5 \\ \tilde{\sigma}_6 \end{bmatrix} = \begin{bmatrix} -\sigma_4\tilde{\sigma}_3 \\ 0 \\ 0 \end{bmatrix}$$

to obtain

$$\tilde{\sigma}_4 = \frac{\sigma_4(\sigma_3 - 2\sigma_1\sigma_2 + \sigma_1^3)(\sigma_3^2 - \sigma_4\sigma_2)}{\Delta_{4,3}},$$
$$\tilde{\sigma}_5 = \frac{-\sigma_4^2\sigma_3(\sigma_3 - 2\sigma_1\sigma_2 + \sigma_1^3)}{\Delta_{4,3}},$$
$$\tilde{\sigma}_6 = \frac{\sigma_4^3(\sigma_3 - 2\sigma_1\sigma_2 + \sigma_1^3)}{\Delta_{4,3}}.$$

Our main theorem below, allows to find a t-design with $2t + k$ elements, given k points in t-good position.

Theorem 29 *For k complex numbers in S^1 in t-good position, there is an unique t-design X containing them with $|X| = k + 2t$.*

Proof Assume that $a_1, \ldots, a_k \in S^1$ are in t-good position and let

$$L(z) = \prod_{i=1}^{k}(z - a_i) = z^k + \sum_{i=1}^{k}(-1)^i \sigma_i z^{k-i}. \tag{25}$$

By hypothesis, the set X of the roots of the polynomial

$$L(z)G_{k,t}(z) = z^n + \sum_{i=1}^{n} c_{n-i} z^{n-i},$$

where $n = k + 2t$, is a subset of S^1. Thus, to show that X is a spherical t-design it only suffices to check that $c_{n-1} = c_{n-2} = \cdots = c_{n-t} = 0$ and this is clear for the construction of the coefficients of $G_{k,t}(z)$.

The unicity also follows from the construction of $G_{k,t}(z)$ because, if $a_1, \ldots, a_k$ belongs to a spherical t-design X with $n = k + 2t$ elements, and $\{b_1, \ldots, b_{n-k}\} = X \setminus \{a_1, \ldots, a_k\}$ then the b_i's are the roots of a polynomial

$$H(z) = z^{2t} + \sum_{i=1}^{2t}(-1)^i \phi_i z^{2t-i},$$

where the ϕ_i's are the symmetric functions of $b_1, \ldots, b_{n-k}$. Being X a spherical t-design, the polynomial

$$L(z)H(z) = z^n + \sum_{i=1}^{n} d_{n-i} z^{n-i},$$

must satisfy $d_{n-i} = d_i = 0$, $i = 1, \ldots, t$ and this implies that $\phi_i = \tilde{\sigma_i}$ for $i = 1, \ldots, 2t$, where the $\tilde{\sigma_i}$'s are as in Definition 25. Thus, $b_1, \ldots, b_{n-k}$ are the roots of $G_{k,t}(z) = H(z)$. □

Remark 30 Assume $t \geq 2$. From Theorems 29 and 3 we conclude that if 2 points in S^1 are in t-good position then there exists a unique group-type t-design (the union of two $t+1$-gons) containing them.

In order to construct t-designs, we need to decide, for k complex numbers in S^1 such that $\Delta_{k,t} \neq 0$, whether the polynomial $G_{k,t}(z)$ have all of its roots in S^1. This can be done using Theorem 5. Unexpectedly, $G_{k,t}(z)$ satisfies the first one of the conditions of the theorem if $k \geq t+1$.

Proposition 31 *Let t, k be positive integers such that $k \geq t+1$. If $a_1, \ldots, a_k \in S^1$ satisfy that $\Delta_{k,t} \neq 0$, then the polynomial $G_{k,t}(z)$ is self-inversive.*

Proof As $G_{k,t}(z)$ is monic, we need to show that the constant term $\tilde{\sigma}_{2t}$ is a unit-norm complex number and that $\tilde{\sigma}_{2t}\overline{\tilde{\sigma}_i} = \tilde{\sigma}_{2t-i}$, for $i = 1, \ldots, t$. Note that, by Cramer's rule,

$$\tilde{\sigma}_{2t} = \frac{(-1)^t \sigma_k^t \tilde{\sigma}_t}{\Delta_{k,t}}. \tag{26}$$

As $|\sigma_k| = 1$, it is enough to prove that $|\tilde{\sigma_t}| = |\Delta_{k,t}|$. We will show that, for any t,

$$\sigma_k^t \overline{\Delta_{k,t}} = (-1)^t \tilde{\sigma}_t, \tag{27}$$

which implies the previous assertion. Observe that, if $t = 1$, $\sigma_k \overline{\Delta_{k,1}} = \sigma_k \overline{\sigma_{k-1}} = \sigma_1 = -\tilde{\sigma}_1$ and that if $t = 2$, $\sigma_k^2 \overline{\Delta_{k,2}} = \sigma_k^2 \overline{(\sigma_{k-1}^2 - \sigma_k \sigma_{k-2})} = \sigma_1^2 - \sigma_2 = \tilde{\sigma}_2$. So, suppose that $t \geq 3$ and that (27) is true for lower values of t. We can express $\Delta_{k,t}$ as

$$\Delta_{k,t} = \sum_{i=1}^{t-2} (-1)^{i-1} \sigma_k^{i-1} \sigma_{k-i} \Delta_{k,t-i} + (-1)^{t-2} \sigma_k^{t-2} (\sigma_{k-t+1}\sigma_{k-1} - \sigma_{k-t}\sigma_k). \tag{28}$$

Then,

$$\begin{aligned}
\sigma_k^t \overline{\Delta_{k,t}} =& \sum_{i=1}^{t-2} (-1)^{i-1} \sigma_i \sigma_k^{t-i} \overline{\Delta_{k,t-i}} + (-1)^{t-2} \sigma_k^2 \overline{(\sigma_{k-t+1}\sigma_{k-1} - \sigma_{k-t}\sigma_k)} = \\
=& \sum_{i=1}^{t-2} (-1)^{i-1} \sigma_i \tilde{\sigma}_{t-i} (-1)^{t-i} + (-1)^{t-2} (\sigma_{t-1}\sigma_1 - \sigma_t) = \\
=& (-1)^t \left(-\sum_{i=1}^{t-2} \sigma_i \tilde{\sigma}_{t-i} - \sigma_{t-1}\tilde{\sigma}_1 - \sigma_t \right) = (-1)^t \tilde{\sigma}_t.
\end{aligned}$$

We now prove that

$$\tilde{\sigma}_{2t} \overline{\tilde{\sigma}_i} = \tilde{\sigma}_{2t-i}, \tag{29}$$

for $i = 1, \ldots, t-1$. If $i = 1$,

$$\tilde{\sigma}_{2t} \overline{\tilde{\sigma}_1} = -\tilde{\sigma}_{2t} \overline{\sigma_1} = \frac{(-1)^{t-1} \sigma_{k-1} \sigma_k^{t-1} \tilde{\sigma}_t}{\Delta_{k,t}}$$

and this is easily seen to be equal to $\tilde{\sigma}_{2t-1}$ using, again, Cramer's rule. Now, take $1 < i \leq t-1$ and assume that $\tilde{\sigma}_{2t}\overline{\tilde{\sigma}_j} = \tilde{\sigma}_{2t-j}$ holds for $j < i$. By (24) we have that

$$\sum_{j=0}^{i} \sigma_{k-j}\tilde{\sigma}_{2t-(i-j)} = 0.$$

Thus,

$$\sigma_k\tilde{\sigma}_{2t-i} = -\sum_{j=1}^{i} \sigma_{k-j}\tilde{\sigma}_{2t-(i-j)} = -\sum_{j=1}^{i} \sigma_{k-j}\tilde{\sigma}_{2t}\overline{\tilde{\sigma}_{i-j}},$$

and we finally obtain

$$\tilde{\sigma}_{2t-i} = -\tilde{\sigma}_{2t}\sum_{j=1}^{i} \overline{\sigma_k}\sigma_{k-j}\overline{\tilde{\sigma}_{i-j}} = -\tilde{\sigma}_{2t}\overline{\sum_{j=1}^{i} \sigma_j\tilde{\sigma}_{i-j}} = \tilde{\sigma}_{2t}\overline{\tilde{\sigma}_i}.$$

The only thing left to prove to conclude is $\tilde{\sigma}_{2t}\overline{\tilde{\sigma}_t} = \tilde{\sigma}_t$, but this is straightforward, because we have $\tilde{\sigma}_{2t}\Delta_{k,t} = (-1)^t\sigma_k^t\tilde{\sigma}_t$ and from (27) it follows easily that

$$\Delta_{k,t} = (-1)^t\sigma_k^t\overline{\tilde{\sigma}_t} \tag{30}$$

which finishes the proof. □

Finding under what conditions on $a_1, \ldots, a_k$ the polynomial $G_{k,t}(z)$ satisfies the second condition of Cohn' Theorem could be a difficult task and we do not have at our disposal a result as Corollary 16 for $t > 2$. Instead of that, we can apply Theorem 7 to arrive to sufficient conditions for the existence of t-designs.

Corollary 32 *Let t, k be positive integers such that $k \geq t+1$. If $a_1, \ldots, a_k \in S^1$ satisfy that $\Delta_{k,t} \neq 0$, and that*

$$\sum_{i=1}^{t-1} 2|\tilde{\sigma}_i| + |\tilde{\sigma}_t| \leq 2 \tag{31}$$

then $a_1, \ldots, a_k$ are in t-good position. Consequently, Theorem 29 guarantees the existence and unicity of a t-design X, with $|X| = k + 2t$, containing them.

Proof As we saw, $|\tilde{\sigma}_{2t}| = 1$ and $\tilde{\sigma}_{2t}\overline{\tilde{\sigma}_i} = \tilde{\sigma}_{2t-i}$ for $i = 1, \ldots, t-1$, which implies that $|\tilde{\sigma}_i| = |\tilde{\sigma}_{2t-i}|$ for $i = 1, \ldots, t-1$. Then, Theorem 7 applied to the monic polynomial $G_{k,t}$ gives the result. □

Example 33 Applying Corollary 32 to the case $t = 2, k = 3$, we obtain that the condition

$$2|\sigma_1| + |\sigma_1^2 - \sigma_2| \leq 2 \tag{32}$$

assures the existence and unicity of a 2-design containing $a_1, a_2, a_3 \in S^1$ such that $\sigma_2^2 - \sigma_1\sigma_3 \neq 0$. If we set $a_1 = z_0$, $a_2 = \overline{z_0}$ and $a_3 = 1$ with $z_0 \in S^1$ we obtain the existence of 2-designs for $\frac{1-\sqrt{17}}{4} \simeq -0,78077 \leq \Re(z_0) < -\frac{1}{2}$. This comes from solving the equation

$$2|1 + 2\Re z_0| + |(1 + 2\Re z_0)^2 - (1 + 2\Re z_0)| \leq 2.$$

Compare this with Example 18 and with Remark 20.

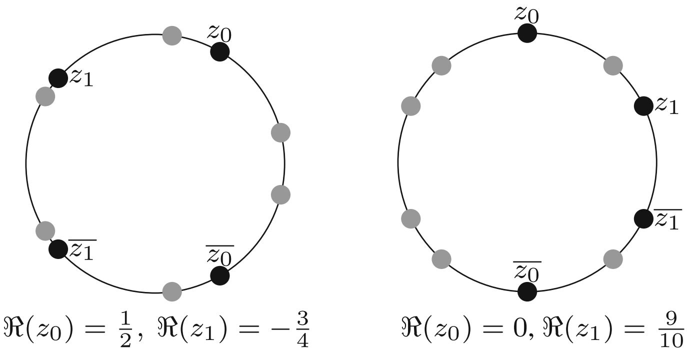

Fig. 2 Examples of 3-designs

Example 34 In this example, we construct a family of 3-designs in a similar fashion than we did in Example 18. In order to apply Corollary 32, we must take 4 points in S^1 such that $\Delta_{4,3} \neq 0$.

Let $z_0, z_1 \in S^1$ and set

$$a_1 = z_0,\ a_2 = \overline{z_0},\ a_3 = z_1,\ a_4 = \overline{z_1}.$$

We have

$$\sigma_1 = \sigma_3 = 2(\Re(z_0) + \Re(z_1)),\ \sigma_2 = 2 + 2(\Re(z_0 z_1) + \Re(z_0\overline{z_1})),\ \sigma_4 = 1.$$

Based on Example 28, we obtain the expressions for the $\tilde{\sigma_i}'s$. We put $z_0 = e^{i\theta_0}$ and $z_1 = e^{i\theta_1}$ so that $\Re(z_0) = \cos\theta_0$ and $\Re(z_1) = \cos\theta_1$ and we obtain:

$$\tilde{\sigma_1} = \tilde{\sigma_5} = -2(\cos\theta_0 + \cos\theta_1),\ \tilde{\sigma_2} = \tilde{\sigma_4} = 4\cos^2\theta_0 + 4\cos^2\theta_1 + 4\cos\theta_0\cos\theta_1 - 2,\ \tilde{\sigma_6} = 1.$$

whereas

$$\Delta_{4,3} = -\tilde{\sigma_3} = 2(\cos\theta_0 + \cos\theta_1)(4\cos^2\theta_0 + 4\cos^2\theta_1 - 3)$$

Notice that we could obtain, through Proposition 8 conditions for $G_{4,3}(z) = \sum_{i=0}^{6}(-1)^i\tilde{\sigma_i}z^{6-i} \in \mathbb{R}[z]$ to have all of its roots lying in S^1, however this will lead us to too many calculations. Moreover, even an application of Corollary 32 in the general case would carry too much work to obtain a condition on θ_0 and θ_1. So, instead of doing that, we put $\theta_0 = \frac{\pi}{3}$ and look for those θ_1 such that condition (31) holds. This will give us a family of 3-designs. Writing θ instead of θ_1, condition (31) is equivalent to

$$|1 + 2\cos\theta| + |4\cos^2\theta + 2\cos\theta - 1| + |1 + 2\cos\theta||2\cos^2\theta - 1| \leq 1. \tag{33}$$

which holds for $-0.82948... \leq \cos\theta \leq -0.61803...$. Then, for the corresponding values of θ, we can assure the existence of 3-designs. The corresponding design for $\cos\theta_1 = -\frac{3}{4}$ is depicted in Fig. 2 on the left. As in Example 33, conditions for existence are not sharp: on the right of Fig. 2 there is a 3-design whose existence is not provided by the calculations in this example, but obtained in the same fashion.

It is clear that a rotation applied to all elements of a t-design gives another t-design. So, we expect our method for constructing t-designs from a subset of S^1 will produce a consistent result if we rotate that subset.

Proposition 35 *Let $a_1, \dots, a_k$ be in t-good position and X the t-design containing them. If $\theta \in \mathbb{R}$ then $e^{i\theta}a_1, \dots, e^{i\theta}a_k$ are in t-good position and the corresponding t-design X_θ is obtained from X by a rotation in the angle θ.*

Proof We claim that the coefficient of z^j of the polynomial $G_{k,t}(z)$ is a homogeneous function of degree $2t - j$ when considered as a function on $a_1, \dots, a_k$. Thus, denoting $G_\theta(z)$ the corresponding polynomial for $e^{i\theta}a_1, \dots, e^{i\theta}a_k$, we have that $G_\theta(z) = e^{i2t\theta}G_{k,t}(e^{-i\theta}z)$. Then, the roots of G_θ are of the form $e^{i\theta}b$ where b is a root of $G_{k,t}$.

We now prove the claim. For $j = 1, \dots, t$, it is clear by definition (cf.(23)) that the symmetric functions $\tilde{\sigma_i}$ are homogeneous of degree j. For $j = t+1, \dots, 2t$, note first that $\Delta_{k,t}$ is homogeneous of degree $tk - t$ (considered as a function on $a_1, \dots, a_k$). In fact, from (27), we easily obtain $\Delta_{k,t} = (-1)^t \sigma_k^t \overline{\tilde{\sigma_t}}$ (note that this also implies that $\Delta_{k,t} \neq 0$ when applied to $e^{i\theta}a_1, \dots, e^{i\theta}a_k$).

Now, from (26), $\tilde{\sigma}_{2t}$ is homogeneous of degree $tk + t - (tk - t) = 2t$ and finally, (29) concludes the proof. □

It would be nice to have a result like Proposition 22 in the general case. Even in the case $t = 3, k = 4$ it is not clear what implies for 4 points in S^1 to satisfy $\Delta_{4,3} = \sigma_3^3 - 2\sigma_2\sigma_3\sigma_4 + \sigma_1\sigma_4^2 = 0$. In particular, any 4 of the quintic roots of a unit-norm complex number fulfill this condition and it is clear that they belong to an infinite number of 3-designs with 10 elements. Nevertheless, we believe that the following statement is true:

Conjecture 36 *Every t-design of $2t + k$ elements can be obtained from k points in t-good position using the methods developed here.*

4.2 Group-type *t*-designs

In the general case, it is also possible to describe group-type spherical t-designs in the σ-space. We focus on the case $k = t + 1$. The following proposition generalizes Proposition 24.

Proposition 37 *Let $k = t + 1$ and $a_1, \dots, a_k \in S^1$ such that $\Delta_{k,t} \neq 0$. If $\sigma_k = \sigma_1\sigma_{k-1}$ and $\sigma_i = 0$ for $i \neq 1, k-1, k$ then $a_1, \dots, a_k$ are in t-good position and the t-design containing them is of group-type.*

Proof As before, $\sigma_k = \sigma_1\sigma_{k-1}$ is equivalent to $|\sigma_1| = |\sigma_{k-1}| = 1$. Then, it is enough to show that

$$L(z)G_{k,t}(z) = (z^{2t} - \sigma_{k-1}^2)(z^k - \sigma_1^k),$$

where $L(z)$ is as in (25). In this case

$$L(z) = z^k - \sigma_1 z^{k-1} + (-1)^{k-1}\sigma_{k-1}z + (-1)^k\sigma_1\sigma_{k-1}.$$

In order to calculate $G_{k,t}$, we use (23) to obtain

$$\tilde{\sigma_i} = (-1)^i\sigma_1^i, \tag{34}$$

for $i = 1, \dots, t-1$ and

$$\tilde{\sigma_t} = (-1)^t\sigma_1^t - \sigma_{k-1}. \tag{35}$$

From (30) and (35) we obtain that in this case

$$\Delta_{k,t} = (-1)^{t+1}\sigma_1^t\sigma_{k-1}^{t-1} + \sigma_{k-1}^t,$$

thus $\frac{\tilde{\sigma_t}}{\Delta_{k,t}} = \frac{-1}{\sigma_{k-1}^{t-1}}$ and we have that

$$\tilde{\sigma}_{2t-i} = \tilde{\sigma}_{2t}\overline{\tilde{\sigma_i}} = \frac{(-1)^t\sigma_k^t\tilde{\sigma}_t}{\Delta_{k,t}}\overline{\tilde{\sigma_i}} = (-1)^{t-1+i}\sigma_1^{t-i}\sigma_{k-1} \tag{36}$$

for $i = 0, \ldots, t-1$. Finally, we have

$$G_{k,t} = \sum_{i=0}^{t-1}\sigma_1^i z^{2t-i} + (\sigma_1^t + (-1)^{t+1}\sigma_{k-1})z^t + (-1)^{t-1}\sum_{i=t+1}^{2t}\sigma_1^i\sigma_{k-1}z^{2t-i}$$

and performing the product we arrive at the desired result. □

Moreover, we can obtain others group-type t-designs in a similar fashion. Observe that due to $k = t+1$, the obtained t-design X is such that $|X| = 3t+1$ then X is the union of, at most, two n-gons. In the case of Proposition 37, we get the union of a $2t$-gon and a $t+1$-gon, but we get other combinations if we ask additional algebraic conditions on the σ's, as the following examples shows.

Example 38 If we take $k = 5$ and $t = 4$ and we ask the conditions $\sigma_1 = \sigma_4 = 0$ and $\sigma_5 = \sigma_2\sigma_3$ we get

$$L(z)G_{5,4}(z) = (z^7 + \sigma_2^2\sigma_3)(z^6 - \sigma_3^2),$$

that is, the 4-design obtained is the union of a hexagon and a heptagon.

Example 39 If we take $k = 4$ and $t = 3$ and we ask the conditions $\sigma_4 = \sigma_1\sigma_3$, $\sigma_1^2 = \sigma_2$ and $\sigma_3 = \sigma_1\sigma_2$ we get

$$L(z)G_{4,3}(z) = (z^5 + \sigma_1^3\sigma_2)^2,$$

that is, the 3-design obtained is a pentagon with all points repeated twice.

It would be interesting to have a result describing the general pattern for obtaining group-type t-designs for general k and t.

5 The space of *t*-designs

To finish this article, we would like to discuss how t-designs in $\mathbb{R}^2$ could be understood as elements of a subset lying in a bigger space. Our results show that when conditions are imposed on the symmetric functions of complex numbers in S^1 instead of on the numbers themselves, the existence (or the non-existence) of a t-design and the condition to be of group-type becomes clearer than it would be if we simply analyzed the given finite subset of S^1.

It is known (see, for instance [12, Appendix V]) that, if we denote $< a_1, \ldots, a_k >$ a k-uple in $\mathbb{C}^k$ without regarding the ordering of the elements, the k-th symmetric power of $\mathbb{C}$

$$\mathbb{C}^k_{sym} = \{< a_1, \ldots, a_k >: a_1, \ldots, a_k \in \mathbb{C}\},$$

is in bijective correspondence with $\mathbb{C}^k$ by means of the mapping $\Sigma : \mathbb{C}^k_{sym} \to \mathbb{C}^k$ given by

$$\Sigma(< a_1, \ldots, a_k >) = (\sigma_1(a_1, \ldots, a_k), \ldots, \sigma_k(a_1, \ldots, a_k)).$$

Moreover, the space $\mathbb{C}^k_{sym}$ can be realized as a analytic variety and the mapping Σ is holomorphic. This shows that one can work indistinctly in the set of unordered k-uples of complex numbers or in the $\sigma-$space. Nevertheless, it is not that clear what the rank of the mapping Σ is if instead we consider unordered k-uples of elements in S^1.

We can obtain a more suitable insight applying Theorem 12, considering each t-design being parametrized by an element of $\mathbb{C}^k_{sym}$ or, equivalently, by the k-uple $(\sigma_1, \ldots, \sigma_k)$.

Definition 40 Let $\sigma = (\sigma_1, \ldots, \sigma_{k-1}) \in \mathbb{C}^{k-1}$ and X_k, W_k as in equations (6), (7), respectively. We say that σ is t-compatible if

1. $\Delta_{k,t}(\sigma_1, \ldots, \sigma_{k-1}, (-1)^k) \neq 0$
2. $(-1)^i \sigma_i = X_k w,\ w \in W_k$.

We call the set of all t-compatible σ, the t-compatible space.

Observe that the t-compatible space can be described by means of the set W_k.

Proposition 41 *For every set $\{a_1, \ldots, a_k\}$ of points in S^1 in t-good position, there exists a σ in the t-compatible space.*

Proof Let $a_1, \ldots, a_k$ in t-good position. By Proposition 35, we can rotate them in order that the resulting set is also in good position and satisfies $\sigma_k = (-1)^k$, where we denote also by σ_i the simmetric functions of the rotated points. Thus, the polynomial $z^k - \sigma_1 z^{k-1} + \cdots + (-1)^{k-1}\sigma_{k-1} z + 1$ is conjugate reciprocal and all of its roots are in the unit cicle, that is, $(-1)^i \sigma_i = X_k w$ for some $w \in W_k$. □

Using Proposition 22 we obtain a nice description of the space of 2-designs with 7 elements.

Corollary 42 *For every 2-design with 7 elements such that at least five of them are distinct, we have a $\sigma = (\sigma_1, \bar{\sigma_1})$ in the 2-compatible space.*

To give a description of the space of 2-designs with 7 elements we first need to find all $\sigma_1 \in \mathbb{C}$ such that $\sigma = (\sigma_1, \bar{\sigma_1})$ is 2-compatible, that is, to find all $(w_1, w_2) \in W_3$, so we consider the polynomial

$$z^3 - \sigma_1 z^2 + \bar{\sigma_1} z + 1 = z^3 + (w_1 + i w_2) z^2 + (w_1 - i w_2) z + 1.$$

Since $\sigma_1 = -w_1 - i w_2$, we consider $\Delta(-w_1, -w_2) = 0$ (cf. eq.(8)) as the boundary of the set of (w_1, w_2) such that the corresponding σ are compatible.

- The condition $\Delta_{3,2} = 0$ is $\overline{\sigma_1}^2 + \sigma_1 = 0$, which happens if and only if $\sigma_1 \in \{0, 1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}\}$, so we exclude these points.
- By Proposition 24, the equation $\sigma_3 = \sigma_1 \sigma_2$ corresponds to group-type 2-designs, in this case this is equivalent to $|\sigma_1| = 1$.
- Finally, in Corollary 16, item 3, we saw that for every σ with $|\sigma_1| \leq \alpha$, $\alpha \simeq 0.60712$, we obtain a 2-design.

These considerations are reflected in Fig. 3, draw in the w-space. The interior of the curve corresponds to the set of compatible σ, and the shaded regions correspond to actual 2-designs. Note that there may be other 2-designs besides these ones (see Corollary 16, items 1 and 2).

We also note that different points in this picture could give raise to the same 2-design. To elucidate this appropriately, we have to decide for what values of σ_1 the polynomial

$$\begin{aligned}&(z^3 - \sigma_1 z^2 - \overline{\sigma_1} z + 1)\left(z^4 + \sigma_1 z^3 + (\sigma_1^2 + \overline{\sigma_1}) z^2 + \frac{\overline{\sigma_1}^2 + |\sigma_1|^2 \sigma_1}{\overline{\sigma_1}^2 + \sigma_1} z + \frac{\sigma_1^2 + \overline{\sigma_1}}{\overline{\sigma_1}^2 + \sigma_1}\right)\\ &= z^7 + c_3 z^4 + c_4 z^3 + c_0\end{aligned}$$

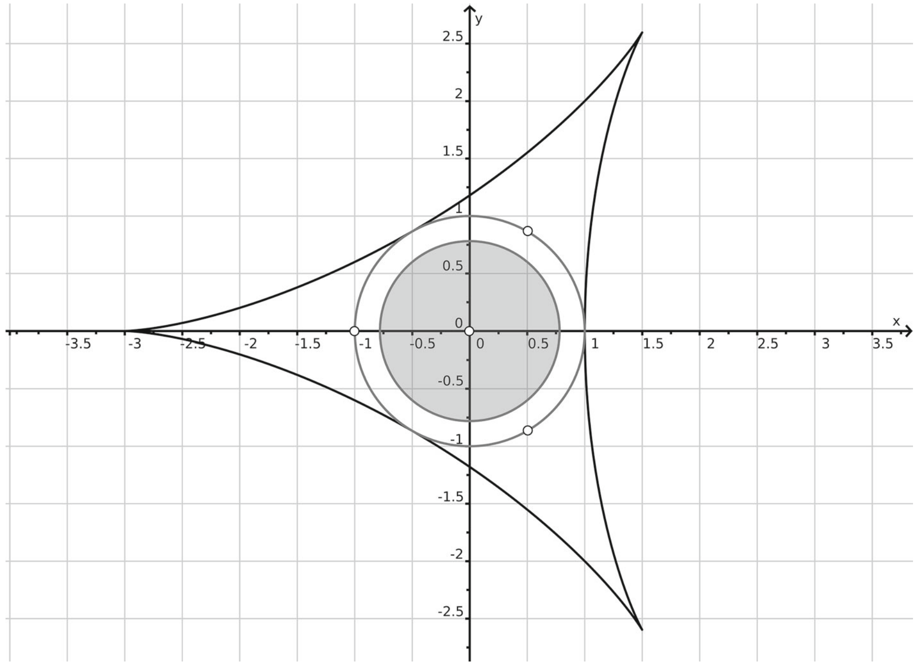

Fig. 3 The space of 2-designs of 7 elements

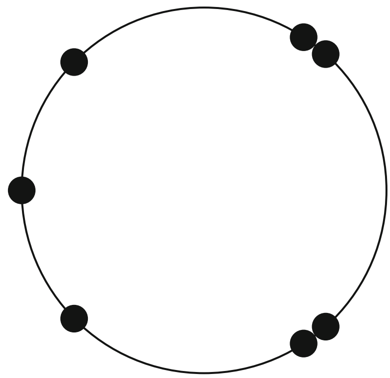

Fig. 4 The 2-design whose elements are the zeroes of $z^7 + \frac{29}{27}z^4 + \frac{29}{27}z^3 + 1$

is the same one. As $c_4 = c_0\overline{c_3}$, it suffices to see for what values of compatible σ_1

$$\frac{\sigma_1^2 + \overline{\sigma_1}}{\overline{\sigma_1}^2 + \sigma_1} = c_0, \ \frac{\overline{\sigma_1}^2 + |\sigma_1|^2\sigma_1}{\overline{\sigma_1}^2 + \sigma_1} - \sigma_1^3 - 2|\sigma_1|^2 + 1 = c_3$$

remain equal. For instance, for the values $\frac{1}{3}$, $\frac{-21+\sqrt{513}}{18} \simeq 0.09164$ and $\frac{-21-\sqrt{513}}{18} \simeq -2.425$ of σ_1 we obtain the same 2-design, that is depicted in Fig. 4.

This discussion also helps to clarify the question raised in Sect. 3.3: which are all the group-type 2-designs with 7 elements? Solving $c_3c_4 = c_0$ we obtain $|\sigma_1| = 1$ plus a finite number of points and solving $c_3 = c_4 = 0$ also gives a finite number of points. This illustrates Theorem 3ii) in this case.

Acknowledgements The author would like to thank Professor Patricia Morillas (Departamento de Matemática, Universidad Nacional de San Luis-CONICET) for suggesting the present research subject and for many useful discussions that helped him to clarify the ideas and concepts developed in this article. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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