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On $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes and their Gray images

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Abstract
In this paper, we first generalize the polycyclic codes over finite fields to polycyclic codes over the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_4$, and we show that these codes can be identified as $\mathbb{Z}_4[x]$-submodules of $\mathcal{R}_{\alpha,\beta}$ with $\mathcal{R}_{\alpha,\beta} = \mathbb{Z}_2[x]/\langle t_1(x)\rangle \times \mathbb{Z}_4[x]/\langle t_2(x)\rangle$, where $t_1(x)$ and $t_2(x)$ are monic polynomials over $\mathbb{Z}_2$ and $\mathbb{Z}_4$, respectively. Then we provide the generator polynomials and minimal generating sets for this family of codes based on the strong Gröbner basis. In particular, under the proper defined inner product, we study the dual of $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes. Finally, we focus on the characterization of the $\mathbb{Z}_2\mathbb{Z}_4$-MDSS and MDSR codes, and as examples, we also present some (almost) optimal binary codes derived from the $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes.

Keywords $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes · Minimal generating sets · Duality · Optimal codes · Bounds

1 Introduction

Polycyclic codes over finite fields are a nature generalization of the concept of cyclic codes that are ideals modulo some other polynomials $t(x)$ [22]. It was shown that every polycyclic code over a finite field is a shortened cyclic code, and most of the best known random error-correcting codes are shortened cyclic codes [32]. Recently, there are many papers on the characterization of the polycyclic codes and their duality, see [2,7,29] and references therein.

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Extending the alphabets to the case of finite chain rings are considered later in [17,23]. Meanwhile, the description of the multivariable codes over finite chain rings, which contain the polycyclic codes as special cases with univariable, can be found in [25–27].

Mixed alphabet codes were first defined by Delsarte [14] in terms of the association schemes. Besides, Rifà and Pujol [34] introduced the translation invariant propelinear codes, and it was shown that such codes consistent with the additive codes in the binary Hamming scheme. Recently, Borges et al. [11] initially studied the generator and parity-check matrices of the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes systematically.

Since then, the research on this new topic has attracted the interest of many scholars. First, Aydogdu and Siap studied the algebraic structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes [3] and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes [4], and they also considered the duals of these additive codes. Borges and Fernández-Córdoba [9] investigated the $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes, and they showed that the class of $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes is exactly the class of $\mathbb{Z}_2$-linear codes with automorphism group of even order. Second, $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes were introduced in [1], and these codes can be used to construct binary codes with good parameters via Gray maps. On the other hand, the generator polynomials for their duality were exhibited in [12]. Lately, Aydogdu et al. [5] studied $\mathbb{Z}_2\mathbb{Z}_2[u]$-cyclic and constacyclic codes. In particular, some optimal binary linear codes were derived from this family of codes. Very recently, Qian and Cao [33] considered a more general case, i.e., $\mathbb{Z}_q\mathbb{Z}_q[u]$-additive cyclic codes, they obtained many MDSS (maximum distance separable w.r.t. the Singleton bound) codes and optimal q-ary codes using Gray map. Moreover, the construction of 1-perfect additive codes can be found in [10,36]. Generalizations of linear complementary dual (LCD for short) codes [28] over finite fields to additive complementary dual (ACD for short) codes over mixed alphabets can be found in [6] and [21]. In addition, the constructions of one-weight and two-weight additive codes were characterized in [15,37]. Another interesting topic on additive codes, presented in [16,20,40,41], is asymptotical property, and it has been shown that those codes are asymptotically good.

The main object of this manuscript is to study the additive polycyclic codes in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. More precisely, we show that these codes can be viewed as $\mathbb{Z}_4[x]$-submodules of $\mathcal{R}_{\alpha,\beta}$, where $\mathcal{R}_{\alpha,\beta} = \mathbb{Z}_2[x]/\langle t_1(x)\rangle \times \mathbb{Z}_4[x]/\langle t_2(x)\rangle$, $t_1(x)$ and $t_2(x)$ are monic polynomials of $\mathbb{Z}_2[x]$ and $\mathbb{Z}_4[x]$, respectively. The main tool used here is the strong Gröbner basis theory. Similarly, another natural topic is to study the duality. Recall that the standard inner product (see [11] for more details) between $(\mathbf{u}|\mathbf{v})$ and $(\mathbf{u}'|\mathbf{v}')$ in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is defined by:

$$\langle(\mathbf{u}|\mathbf{v}), (\mathbf{u}'|\mathbf{v}')\rangle = 2\sum_{i=0}^{\alpha-1} u_i u_i' + \sum_{j=0}^{\beta-1} v_j v_j' \in \mathbb{Z}_4,$$

where $(\mathbf{u}|\mathbf{v}) = (u_0, \dots, u_{\alpha-1}|v_0, \dots, v_{\beta-1})$, $(\mathbf{u}'|\mathbf{v}') = (u_0', \dots, u_{\alpha-1}'|v_0', \dots, v_{\beta-1}') \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then $\mathbb{Z}_2\mathbb{Z}_4$-dual of $\mathcal{C}$ is defined by

$$\mathcal{C}^{\perp_0} = \left\{(\mathbf{x}|\mathbf{y}) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta : \langle(\mathbf{x}|\mathbf{y}), (\mathbf{v}|\mathbf{w})\rangle = 0 \quad \text{for all } (\mathbf{v}|\mathbf{w}) \in \mathcal{C}\right\}.$$

It was shown in [1] that the $\mathbb{Z}_2\mathbb{Z}_4$-dual of a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code is still cyclic. However, in general this property is not valid for an additive polycyclic code. At this point, it might be one reason why there is little known on this kind of codes.

The paper is organized as follows. Section 2 collects the basic definition of $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes. In Sect. 3, we investigate the generator polynomials and the standard generating sets for $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes. Section 4 is devoted to studying the duality of $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes based on a proper defined inner product. In Sect. 5, we characterize the $\mathbb{Z}_2\mathbb{Z}_4$-MDSS and MDSR codes, and present some (almost) optimal binary codes derived from $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes. Finally, we conclude this paper in Sect. 6.

2 Preliminaries

2.1 $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

Let $\mathbb{Z}_4$ be the ring of integers modulo 4, then a code over $\mathbb{Z}_4$ of length n is said to be linear if it is an additive submodule of $\mathbb{Z}_4^n$. In general, $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are defined as subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ (see [11] for more details), they are generalizations of the usual binary and quaternary linear codes. For any $r = r_0 + 2r_1 \in \mathbb{Z}_4$ with $r_0, r_1 \in \mathbb{Z}_2$, we then define a scalar multiplication on $\mathbf{c} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ as

$$r\mathbf{c} = r(\mathbf{v}|\mathbf{w}) = \big(r_0v_0, r_0v_1, \ldots, r_0v_{\alpha-1}|rw_0, rw_1, \ldots, rw_{\beta-1}\big),$$

where $\mathbf{c} = (\mathbf{v}|\mathbf{w}) = (v_0, v_1, \ldots, v_{\alpha-1}|w_0, w_1, \ldots, w_{\beta-1}) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Thus, any $\mathbb{Z}_2\mathbb{Z}_4$-additive code can always be identified as a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ under this multiplication.

Let X (resp. Y) be the set of $\mathbb{Z}_2$ (resp. $\mathbb{Z}_4$) coordinate positions. Denote $\mathcal{C}_X$ (resp. $\mathcal{C}_Y$) the punctured code of $\mathcal{C}$ by deleting the coordinates Y (resp. X). Let $\mathcal{C}_b$ be the subcode of $\mathcal{C}$ which contains all order two codewords, and κ be the dimension of the linear code $(\mathcal{C}_b)_X$. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, i.e., a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, then the code $\mathcal{C}$ is isomorphic to $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$, for some positive integers γ and δ, and $\mathcal{C}$ is called a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta, \kappa)$. Further, we have $|\mathcal{C}_b| = 2^{\gamma+\delta}$. A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is called separable if and only if $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$.

2.2 Gray map

The classical Gray map [19] ϕ from the ring $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ is given by

$$\phi(0) = (0, 0), \quad \phi(1) = (0, 1), \quad \phi(2) = (1, 1), \quad \phi(3) = (1, 0).$$

This mapping is widely used to construct binary codes from linear quaternary codes. The main property of ϕ from this point of view is that it is an isometry between $\mathbb{Z}_4$ with the Lee metric and $\mathbb{Z}_2$ with the Hamming metric. In fact, this important property holds when we generalize it to the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_4$.

For this purpose, let $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{Z}_4^n$, then the coordinate-wise extension ψ can be expressed as $\psi(\mathbf{x}) = (\phi(x_0), \phi(x_1), \ldots, \phi(x_{n-1})) \in \mathbb{Z}_2^{2n}$. Then, the Gray-like map Φ for element $(\mathbf{v}|\mathbf{w}) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is defined as $\Phi((\mathbf{v}|\mathbf{w})) = (\mathbf{v}|\psi(\mathbf{w}))$. Obviously Φ is a weight-preserving map from $(\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, Lee weight) to $(\mathbb{Z}_2^{\alpha+2\beta}$, Hamming weight), that is,

$$\mathrm{wt}((\mathbf{v}|\mathbf{w})) = \mathrm{wt}_H(\Phi(\mathbf{v}|\mathbf{w})) = \mathrm{wt}_H(\mathbf{v}) + \mathrm{wt}_L(\mathbf{w}),$$

where $\mathrm{wt}_H(\mathbf{v})$ is the Hamming weight of $\mathbf{v}$ and $\mathrm{wt}_L(\mathbf{w})$ is the Lee weight of $\mathbf{w}$. Throughout this paper we use the calligraphic $\mathcal{C}$ to denote the codes over the mixed $\mathbb{Z}_2\mathbb{Z}_4$ alphabet, and we use the standard C to denote the codes over $\mathbb{Z}_2$ or $\mathbb{Z}_4$.

2.3 Polycyclic codes

Now, we begin with a brief review of polycyclic codes over the residue class ring $\mathbb{Z}_4$. We called a linear code C of length n over $\mathbb{Z}_4$ is polycyclic if there exists a vector $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{Z}_4^n$ such that for every codeword $(a_0, a_1, \ldots, a_{n-1}) \in C$ yields

$$(0, a_0, \ldots, a_{n-2}) + a_{n-1}(c_0, c_1, \ldots, c_{n-1}) \in C.$$

We refer to $\mathbf{c}$ as an associate vector of C. Note that such a vector is not necessarily unique. On the other hand, let $c(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in \mathbb{Z}_4[x]$ under the usual correspondence between vectors and polynomials, and denote $t(x) = x^n - c(x)$. Then, a polycyclic code C over $\mathbb{Z}_4$ can also be viewed as an ideal of $\mathbb{Z}_4[x]/\langle t(x)\rangle$. Now, we are ready to introduce the definition of the $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes which are a nature generalization of the classical polycyclic codes over $\mathbb{Z}_2$ (or $\mathbb{Z}_4$) and the $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes.

Definition 1 A subset $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is called a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code if

(i) $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, and
(ii) there exist two vectors $(c_0, c_1, \ldots, c_{\alpha-1}) \in \mathbb{Z}_2^\alpha$, $(c'_0, c'_1, \ldots, c'_{\beta-1}) \in \mathbb{Z}_4^\beta$ such that for any codeword $(\mathbf{v}|\mathbf{w}) = (v_0, v_1, \ldots, v_{\alpha-1}|w_0, w_1, \ldots, w_{\beta-1}) \in \mathcal{C}$, its poly-shift

$$\begin{aligned}\mathrm{T}(\mathbf{v}|\mathbf{w}) = \big(&(0, v_0, \ldots, v_{\alpha-2}) + v_{\alpha-1}(c_0, c_1, \ldots, c_{\alpha-1})|\\ &(0, w_0, \ldots, w_{\beta-2}) + w_{\beta-1}(c'_0, c'_1, \ldots, c'_{\beta-1})\big) \in \mathcal{C}.\end{aligned}$$

From Definition 1, we see that there exists a one-to-one correspondence between the elements in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and $\mathcal{R}_{\alpha,\beta}$, where $\mathcal{R}_{\alpha,\beta} = \mathbb{Z}_2[x]/\langle t_1(x)\rangle \times \mathbb{Z}_4[x]/\langle t_2(x)\rangle$, $t_1(x) = x^\alpha - (c_0 + c_1 x + \cdots + c_{\alpha-1}x^{\alpha-1}) \in \mathbb{Z}_2[x]$ and $t_2(x) = x^\beta - (c'_0 + c'_1 x + \cdots + c'_{\beta-1}x^{\beta-1}) \in \mathbb{Z}_4[x]$. From now on, let $f(x) \in \mathbb{Z}_4[x]$ and $(v(x)|w(x)) \in \mathcal{R}_{\alpha,\beta}$, define the multiplication $*$ as follows:

$$f(x) * (v(x)|w(x)) = (\overline{f(x)}v(x)|f(x)w(x)),$$

where $f(x) \equiv \overline{f(x)}$ (mod 2). This property induces the following characterization for $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes with the module-theoretic language.

Theorem 2 *A subset $\mathcal{C}$ is called a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code if and only if $\mathcal{C}$ is a $\mathbb{Z}_4[x]$-submodule of $\mathcal{R}_{\alpha,\beta}$.*

Proof Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code, then its polynomial representation is closed under scalar multiplication by elements of $\mathbb{Z}_4$. Let $(v_0, v_1, \ldots, v_{\alpha-1}|w_0, w_1, \ldots, w_{\beta-1}) \in \mathcal{C}$ correspond an element $(v(x)|w(x)) \in \mathcal{R}_{\alpha,\beta}$. Then the word

$$\begin{aligned}x * (v(x)|w(x)) = &(v_{\alpha-1}c_0 + (v_0 + v_{\alpha-1}c_1)\,x + \cdots + (v_{\alpha-2} + v_{\alpha-1}c_{\alpha-1})\,x^{\alpha-1}|\\ &w_{\beta-1}c'_0 + \big(w_0 + w_{\beta-1}c'_1\big)\,x + \cdots + \Big(w_{\beta-2} + w_{\beta-1}c'_{\beta-1})x^{\beta-1}\Big)\end{aligned}$$

corresponds the word

$$\Big(v_{\alpha-1}c_0, \ldots, v_{\alpha-2} + v_{\alpha-1}c_{\alpha-1})|w_{\beta-1}c'_0, \ldots, w_{\beta-2} + w_{\beta-1}c'_{\beta-1}\Big) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta,$$

which is a codeword of $\mathcal{C}$ from Definition 1, and this implies that $\mathcal{C}$ is closed under multiplication by x, and hence it is a $\mathbb{Z}_4[x]$-submodule of $\mathcal{R}_{\alpha,\beta}$. Conversely, it is easy to check that a $\mathbb{Z}_4[x]$-submodule of $\mathcal{R}_{\alpha,\beta}$ is exactly a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code. □

Remark 3 Let $\mathcal{C}$ be a $\mathbb{Z}_4[x]$-submodule of $\mathcal{R}_{\alpha,\beta}$ with $t_1(x) = x^\alpha - 1$, $t_2(x) = x^\beta - \lambda$, $\lambda \in \{1, 3\}$ and β odd, which is a special case of a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code. Then it is easy to check that the $\mathbb{Z}_2\mathbb{Z}_4$-dual $\mathcal{C}^{\perp_0}$ is also a $\mathbb{Z}_4[x]$-submodule of $\mathcal{R}_{\alpha,\beta}$. However, in general this property is no longer true for other polynomials with $t_1(x) \in \mathbb{Z}_2[x]$ and $t_2(x) \in \mathbb{Z}_4[x]$ under the usual inner product, we will focus on the duality of $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes in Sect. 4.

3 The structure of $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes

A polynomial over $\mathbb{Z}_2$ is called square-free if it has no multiple irreducible factors in its decomposition, and the square-free part of a polynomial over $\mathbb{Z}_2$ is the product of all its distinct irreducible factors. Recall that a polynomial $t(x) \in \mathbb{Z}_4[x]$ is called basic irreducible if $\overline{t(x)}$ is irreducible over $\mathbb{Z}_2$. From [35, Lemma 3.1], it is clear that $\mathbb{Z}_4[x]/\langle t(x)\rangle$ is a principal ideal ring if $\overline{t(x)}$ is square-free. Then $t(x)$ factors uniquely into monic and coprime basic irreducibles, an assumption we make throughout this paper.

Before preceding with the structure of $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes, we need the following two elementary lemmas for polycyclic codes over $\mathbb{Z}_2$ and $\mathbb{Z}_4$, respectively, and they can be found in [23,32].

Lemma 4 *Let $C \subset \mathbb{Z}_2[x]/\langle t_1(x)\rangle$ be a polycyclic code over $\mathbb{Z}_2$ with $t_1(x) \in \mathbb{Z}_2[x]$ and* $\deg(t_1(x)) = \alpha \geq 1$. *Then, there exists a unique monic polynomial $f(x) \in \mathbb{Z}_2[x]$ of minimal degree such that*

- (i) *$C = \langle f(x)\rangle$, and $f(x)|t_1(x)$;*
- (ii) *If $\deg(f(x)) = k$, then the ideal $\langle f(x)\rangle$ in the algebra polynomial modulo $t_1(x)$ has dimension $\alpha - k$, that is, $|C| = 2^{\alpha-k}$.*

Lemma 5 *Let $C \subset \mathbb{Z}_4[x]/\langle t_2(x)\rangle$ be a polycyclic code over $\mathbb{Z}_4$ with $t_2(x) \in \mathbb{Z}_4[x]$ and* $\deg(t_2(x)) = \beta \geq 1$. *Then C admits a set of generator polynomials $\{g_0(x), 2g_1(x)\}$ such that*

- (i) *$g_0(x)$ and $g_1(x)$ are monic polynomials over $\mathbb{Z}_4$;*
- (ii) *$\deg(g_0(x)) > \deg(g_1(x))$, and $g_1(x)|g_0(x)|t_2(x)$;*
- (iii) *The linear code C has $4^{\beta-\deg(g_0(x))}2^{\deg(g_0(x))-\deg(g_1(x))}$ codewords.*

In particular, $C = \langle g_0(x), 2g_1(x)\rangle = \langle g_0(x) + 2g_1(x)\rangle$.

In fact, the set $\{g_0(x), 2g_1(x)\}$ for the linear code C described in Lemma 5 is called the strong Gröbner basis for the principal ideal ring $\mathbb{Z}_4[x]/\langle t_2(x)\rangle$, the reader may refer to [31, Sect. 4] for more details. In addition, it should be noticed that a strong Gröbner basis of C is not necessarily unique [30, Theorem 7.5]. However, the degrees of $g_0(x)$ and $g_1(x)$ are unique. After these preparations, we are now in a position to state and prove the main result in this section.

Theorem 6 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code in $\mathcal{R}_{\alpha,\beta}$ defined above. Then $\mathcal{C}$ can be identified as*

$$\mathcal{C} = \langle (f(x)|0), (e(x)|g_0(x) + 2g_1(x))\rangle, \tag{1}$$

where $f(x)|t_1(x) \pmod 2$, $g_1(x)|g_0(x)|t_2(x) \pmod 4$, and $e(x)$ is a binary polynomial satisfying $\deg(e(x)) < \deg(f(x))$, and $f(x)|\frac{t_2(x)}{g_1(x)}e(x) \pmod 2$.

Proof Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code, then the projection of $\mathcal{C}$ on $\mathbb{Z}_4[x]/\langle t_2(x)\rangle$ according to its quaternary part is the mapping

$$\varphi : \mathcal{C} \longrightarrow \mathbb{Z}_4[x]/\langle t_2(x)\rangle \text{ by } (f_1(x), f_2(x)) \longmapsto f_2(x).$$

It is clear that φ induces a $\mathbb{Z}_4$-module homomorphism, and $\mathrm{Im}(\varphi)$ is an ideal of $\mathbb{Z}_4[x]/\langle t_2(x)\rangle$, that is, $\mathrm{Im}(\varphi) = \langle g_0(x) + 2g_1(x))\rangle$ with $g_1(x)|g_0(x)|t_2(x)$ from Lemma 5.

On the other hand, according to Lemma 4, we know that $\ker(\varphi)$ is a submodule of $\mathcal{C}$ with the generator polynomial formed by $(f(x)|0)$, where $f(x)|t_1(x) \pmod 2$. Then, the remainder of the proof follows as in the proof of [1], we omit the details here. □

Corollary 7 *Let $\mathcal{C}$ be a code defined in (1). If $\gcd(f(x), \frac{t_2(x)}{g_1(x)}) \equiv 1 \pmod 2$, then we have $e(x) = 0$, i.e., the additive code $\mathcal{C}$ is separable.*

The next theorem gives a minimal generating set for a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code viewed as a $\mathbb{Z}_4$-module.

Theorem 8 *Suppose that $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code defined in (1). Let $\deg(f(x)) = a_0$, $\deg(g_0(x)) = b_0$, $\deg(g_1(x)) = b_1$ and $g_0(x)h_0(x) = t_2(x) \pmod 4$. Let*

$$S_1 = \{x^i * (f(x)|0) : 0 \le i \le \alpha - a_0 - 1\},$$
$$S_2 = \{x^i * (e(x)|g_0(x) + 2g_1(x)) : 0 \le i \le \beta - b_0 - 1\},$$
$$S_3 = \{x^i * (\overline{h_0(x)}e(x)|2h_0(x)g_1(x)) : 0 \le i \le b_0 - b_1 - 1\}.$$

Then the union $S_1 \cup S_2 \cup S_3$ forms a minimal generating set for the code $\mathcal{C}$. In addition, the code $\mathcal{C}$ has $2^{\alpha - a_0 + b_0 - b_1}4^{\beta - b_0}$ codewords.

Proof Without loss of generality we may assume that $c(x) \in \mathcal{C}$. Then there exist two polynomials $f_1(x), f_2(x) \in \mathbb{Z}_4[x]$ such that

$$c(x) = f_1(x) * (f(x)|0) + f_2(x) * (e(x)|g_0(x) + 2g_1(x)).$$

It is easy to check that $f_1(x) * (f(x)|0) \in \mathrm{Span}(S_1)$. Suppose that $\deg(f_2(x)) \le \beta - b_0 - 1$, then $f_2(x) * (e(x)|g_0(x) + 2g_1(x)) \in \mathrm{Span}(S_2)$. Otherwise, we have $f_2(x) = h_0(x) f_3(x) + f_4(x)$, where $f_3(x), f_4(x) \in \mathbb{Z}_4[x]$ and $\deg(f_4(x)) \le \beta - b_0 - 1$, which yields

$$f_2(x) * (e(x)|g_0(x) + 2g_1(x)) = f_3(x) * (\overline{h_0(x)}e(x)|2h_0(x)g_1(x)) + \triangle,$$

where $\triangle \in \mathrm{Span}(S_2)$, and we only need to show that $f_3(x) * (\overline{h_0(x)}e(x)|2h_0(x)g_1(x)) \in \mathrm{Span}(S_3)$. In fact, it is true from a similar argument above. In addition, the elements in the union $S_1 \cup S_2 \cup S_3$ are linear independent over $\mathbb{Z}_4$, and this completes the proof. □

Proposition 9 *Let $\mathcal{C}$ be the code defined in (1), where $\deg(f(x)) = a_0$, $\deg(g_0(x)) = b_0$, $\deg(g_1(x)) = b_1$ and $g_0(x)h_0(x) = t_2(x) \pmod 4$. Then $\mathcal{C}$ is of type*

$$(\alpha, \beta; \alpha - a_0 + b_0 - b_1, \beta - b_0, \alpha - a_1),$$

where $a_1 = \deg(\gcd(f(x), \overline{h_0(x)}e(x)))$.

Proof By Theorem 8 it suffices to show $\kappa = \alpha - a_1$. Since $\mathcal{C}_b$ is the subcode of $\mathcal{C}$ which contains all codewords of order 2, we have

$$\mathcal{C}_b = \langle (f(x)|0), (0|2g_0(x)), (\overline{h_0(x)}e(x)|2h_0(x)g_1(x))\rangle.$$

Hence, it is clear that $(\mathcal{C}_b)_X = \langle \gcd(f(x), \overline{h_0(x)}e(x))\rangle$. This completes the proof. □

Example 10 Let $t_1(x) = x^4 + x^3 + x^2 + 1$, $t_2(x) = x^7 + 2x^5 + 2x^4 + x^3 + 2x^2 + x + 3$, $f(x) = x + 1$, $e(x) = 1$, $g_0(x) = x^5 + 3x^4 + 2x^3 + x^2 + 2x + 3$ and $g_1(x) = x + 3$. Then the code $\mathcal{C}$ is of type $(4, 7; 7, 2, 4)$ and $|\mathcal{C}| = 2^{12}$.

In the following, a polynomial $g(x) \in \mathbb{Z}_2[x]$ or $\mathbb{Z}_4[x]$ will be denoted simply by g without ambiguity.

4 Duality of the additive polycyclic codes in $\mathcal{R}_{\alpha,\beta}$

In this section, we always assume that $t_1 = t_2$ in $\mathcal{R}_{\alpha,\beta}$, where $t_1 \in \mathbb{Z}_2[x]$ and $t_2 \in \mathbb{Z}_4[x]$. Following the previous section, we define an inner product on $\mathcal{R}_{\alpha,\beta}$ by the rule:

$$\langle (u_0|u_1), (v_0|v_1)\rangle_{\alpha,\beta} = 2u_0v_0 + u_1v_1 \in \mathbb{Z}_4[x]/\langle t_2\rangle,$$

where $(u_0|u_1), (v_0|v_1) \in \mathcal{R}_{\alpha,\beta}$. In particular, if $\alpha = 0$ (resp. $\beta = 0$), that is, when $\mathcal{C}$ is a quaternary code (resp. $\mathcal{C}$ is a binary code), then for $u_1, v_1 \in \mathbb{Z}_4[x]$ (resp. $u_0, v_0 \in \mathbb{Z}_2[x]$), the inner product is defined by $\langle u_1, v_1\rangle_\beta = u_1v_1 \in \mathbb{Z}_4[x]/\langle t_2(x)\rangle$ (resp. $\langle u_0, v_0\rangle_\alpha = u_0v_0 \in \mathbb{Z}_2[x]/\langle t_1(x)\rangle$). Next, on the basis of the inner product defined above, the dual $\mathcal{C}^\perp$ of a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code can be described as follows.

Definition 11 Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code in $\mathcal{R}_{\alpha,\beta}$. Then the dual $\mathcal{C}^\perp$ of $\mathcal{C}$ is defined by the formula:

$$\mathcal{C}^\perp = \{(u_0|u_1) \in \mathcal{R}_{\alpha,\beta} : \langle (u_0|u_1), (v_0|v_1)\rangle_{\alpha,\beta} = 0 \text{ for all } (v_0|v_1) \in \mathcal{C}\}.$$

Lemma 12 *Keep the notion above. The form $\langle ., .\rangle_{\alpha,\beta}$ is a symmetric $\mathbb{Z}_4[x]$-bilinear form on $\mathcal{R}_{\alpha,\beta}$ with $t_1 = t_2$. Further, this bilinear form is also non-degenerate.*

Proof Let $a, b \in \mathbb{Z}_4[x]$ and $(u_0|u_1), (u_0'|u_1'), (v_0|v_1) \in \mathcal{R}_{\alpha,\beta}$. An easy argument shows that

$$\langle a * (u_0|u_1) + b * (u_0'|u_1'), (v_0|v_1)\rangle_{\alpha,\beta} = a\langle (u_0|u_1), (v_0|v_1)\rangle_{\alpha,\beta} + b\langle (u_0'|u_1'), (v_0|v_1)\rangle_{\alpha,\beta}.$$

In particular, for a fixed $(u_0|u_1) \in \mathcal{R}_{\alpha,\beta}$, if $\langle (u_0|u_1), (v_0|v_1)\rangle_{\alpha,\beta} = 0$ for all $(v_0|v_1) \in \mathcal{R}_{\alpha,\beta}$, then we have $u_0 = u_1 = 0$, i.e., the bilinear form $\langle ., .\rangle_{\alpha,\beta}$ is non-degenerate. □

First we remind a lemma of bilinear form over Frobenius ring of [24,38,39].

Lemma 13 *Let R be a Frobenius ring. Let M be an R-module. Suppose $\epsilon : M \times M \to R$ is a non-degenerate bilinear form. Let $C \subseteq M^n$ be an R-submodule, $C^{\perp_\epsilon}$ is the dual of C with respect to ϵ. Then $|C| \cdot |C^{\perp_\epsilon}| = |M|^n$.*

Now we return to the polynomial ring over $\mathbb{Z}_4$.

Proposition 14 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code in $\mathcal{R}_{\alpha,\beta}$. Then*

(i) *$\mathcal{C}^\perp$ is a $\mathbb{Z}_4[x]$-submodule of $\mathcal{R}_{\alpha,\beta}$;*
(ii) $|\mathcal{C}| \cdot |\mathcal{C}^\perp| = |\mathcal{R}_{\alpha,\beta}|$.

Proof The proof of assertion (i) is trivial. For (ii), according to Lemma 12, we know the form $\langle ., .\rangle_{\alpha,\beta}$ is a non-degenerate bilinear form over $\mathbb{Z}_4[x]$. Since $\mathcal{R}_{\alpha,\beta}$ is a $\mathbb{Z}_4[x]$-module, and $\mathcal{C} \subseteq \mathcal{R}_{\alpha,\beta}$ is a $\mathbb{Z}_4[x]$-submodule, then the result follows from Lemma 13 directly. □

As just observed, the $\mathbb{Z}_2\mathbb{Z}_4$-dual of a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code does not coincide with the dual in this section. However, Proposition 14 allows us to give a analogous result to Theorem 8 that depends on $\mathcal{C}$. In the following, we first make some notation.

Notation Let $d_1 = \gcd(f, \frac{et_2}{g_0})$, and $f = d_1 d_2$. Let $d = \gcd(f, e)$, then there exist $e_1, e_2 \in \mathbb{Z}_2[x]$ such that $e_1 f + e_2 e = d$. Set $d_1 = dd'$ and $g_0 = g_1 h_1$. Moreover, we define $e' = e_2(\frac{d' t_1}{f} + \frac{d_2 t_1}{f})$, $g_0' = \frac{d_1 t_2}{g_1 f}$ and $g_1' = \frac{d t_2}{d_1 g_0}$.

Remark 15 A similar argument in [5] shows that $d_2 | h_1$, and this implies that $g_1' | g_0' | t_2$.

Theorem 16 *Keep the notion above. Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code defined in (1). Then the dual $\mathcal{C}^\perp$ of the code $\mathcal{C}$ is given by*

$$\mathcal{C}^\perp = \langle (t_1/\gcd(f, e)|0), \big(e'|g_0' + 2g_1'\big)\rangle. \tag{2}$$

Proof Write $\mathcal{C}' = \langle (t_1/\gcd(f, e)|0), (e'|g_0' + 2g_1')\rangle$, and we first show that $\mathcal{C}' \subseteq \mathcal{C}^\perp$. It is easy to verify that the following three equations:

$$\langle (t_1/\gcd(f, e)|0), (f|0)\rangle_{\alpha,\beta} = 0,$$
$$\langle (t_1/\gcd(f, e)|0), (e|g_0 + 2g_1)\rangle_{\alpha,\beta} = 0,$$
$$\langle (e'|g_0' + 2g_1'), (f|0)\rangle_{\alpha,\beta} = 0.$$

For the rest case, we have

$$\langle (e'|g_0' + 2g_1'), (e|g_0 + 2g_1)\rangle_{\alpha,\beta} = 2ee' + g_0 g_0' + 2g_1 g_0' + 2g_0 g_1'.$$

Since $g_0 g_0' = \frac{d_1 t_2}{g_1 f} g_0 \equiv 0 \pmod{t_2}$ and

$$2ee' = 2ee_2\left(\frac{d' t_1}{f} + \frac{d_2 t_1}{f}\right) \equiv 2\left(\frac{t_1}{d_2} + \frac{t_1}{d_1}\right) = 2g_1 g_0' + 2g_0 g_1' \pmod{t_2},$$

where the last equality comes from the assumption $t_1 = t_2$.

On the other hand, it remains to show that $|\mathcal{C}'| = |\mathcal{C}^\perp|$. As in Theorem 8 and Proposition 14, $\mathcal{C}'$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code of size $2^{\deg(g_1)+\deg(g_0)+\deg(f)}$, which coincides with the size of $\mathcal{C}^\perp$. So we have $\mathcal{C}' = \mathcal{C}^\perp$. □

Proposition 17 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code defined in (1) with $t_1 = t_2$, where* $\deg(f) = a_0$, $\deg(g_0) = b_0$, $\deg(g_1) = b_1$, $g_0 h_0 = t_2 \pmod 4$ *and* $\deg(\gcd(f, \bar{h}_0 e)) = a_1$. *Then $\mathcal{C}^\perp$ is of type*

$$(\alpha, \alpha; 2a_1 + b_0 - a_0 - b_1, a_0 + b_1 - a_1, a_1).$$

Proof According to Proposition 9 and Theorem 16, the result is straightforward. □

Example 18 Let $\mathcal{C} = \langle (x^2 + x + 1|2(x^2 + 3x + 1))\rangle$, and type $(3, 3; 1, 0, 1)$ with $t_1 = t_2 = f = g_0 = x^3 + 1$. Then we have $\mathcal{C}^\perp = \langle (x + 1|0), (x|3)\rangle$ with type $(3, 3; 2, 3, 2)$. In fact, the dual $\mathcal{C}^\perp$ is equal to the $\mathbb{Z}_2\mathbb{Z}_4$-dual $\mathcal{C}^{\perp_0}$.

Example 19 Consider $\mathcal{C} = \langle (x + 1|0), (1|2(x^3 + 2x^2 + 3x + 3))\rangle$, of type $(4, 4; 4, 0, 4)$, with $t_1 = t_2 = g_0 = x^4 + x^3 + x^2 + 1$. Then we have $\mathcal{C}^\perp = \langle (x^3 + x + 1|3)\rangle$ of type $(4, 4; 0, 4, 0)$. However, for $\mathbf{u} = (1, 1, 0, 1|1, 0, 0, 0) \in \mathcal{C}^\perp$ and $\mathbf{v} = (0, 1, 1, 0|0, 0, 0, 0) \in \mathcal{C}$, it is obvious that $\langle \mathbf{u}, \mathbf{v}\rangle = 2 \neq 0$. This means that although the additive codes $\mathcal{C}^\perp$ and $\mathcal{C}^{\perp_0}$ have the same type (see Proposition 17 and [11, Theorem 2]), they might not be equivalent.

5 Examples of $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes with good parameters

Recall that if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, of type $(\alpha, \beta; \gamma, \delta, \kappa)$, with minimum distance d, then its parameters satisfy one of the following two inequalities (see [1,8]):

$$\frac{d-1}{2} \le \frac{\alpha}{2} + \beta - \frac{\gamma}{2} - \delta, \tag{3}$$

$$\left\lfloor \frac{d-1}{2} \right\rfloor \le \alpha + \beta - \gamma - \delta. \tag{4}$$

More precisely, the code $\mathcal{C}$ is called MDSS (maximum distance separable w.r.t. the Singleton Bound) if it satisfies (3) with equality, and it is called MDSR (maximum distance separable w.r.t. the rank bound) if it satisfies (4) with equality.

Theorem 20 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code defined in* (1) *with $e = g_0 = g_1 = 1$, $f = x+1$, $(x+1)|t_1$ and $(x+1)|\overline{t_2}$. Then $\mathcal{C}$ is MDSS.*

Proof In fact, we have $\overline{h_0} = \overline{t_2}$ and $a_1 = 1$ in Proposition 9. Therefore, it is easy to check that $\mathcal{C}$ is of type $(\alpha, \beta; \alpha-1, \beta, \alpha-1)$ with minimum distance 2. □

Remark 21 Notice that if $t_1 = x^\alpha - 1$ and $t_2 = x^\beta - 1$ with β an odd integer, then [1, Theorem 18] is just the case of Theorem 20. However, we do not require β to be an odd number in other cases. That is, we have no limitation on the values of β here.

We now turn our attention to the case of MDSR codes. Suppose that $(f|0) \in \mathcal{C} \subseteq \mathcal{R}_{\alpha,\beta}$ with $\deg(f) > 0$, then the code $\mathcal{C}$ could never be MDSR. Otherwise, the minimum distance d of $\mathcal{C}$ is $\deg(f)+1$ or less and according to Proposition 9, we have

$$\alpha + \beta - \gamma - \delta = a_0 + b_1 = \left\lfloor \frac{d-1}{2} \right\rfloor \le \left\lfloor \frac{a_0}{2} \right\rfloor,$$

where $a_0 = \deg(f) > 0$ and $b_1 \ge 0$, which is a contradiction.

If $\deg(f) = 0$, i.e., $(1|0) \in \mathcal{C} \subseteq \mathcal{R}_{\alpha,\beta}$ and $\mathcal{C}$ is MDSR. Then the minimum distance of $\mathcal{C}$ is 1, and with a similar discussion above, $\mathcal{C}$ must be of the form:

$$\mathcal{C} = \langle (1|0), (0|g_0 + 2) \rangle,$$

where $g_0 \in \mathbb{Z}_4[x]$ and $g_0|t_2$. Actually, $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code of type $(\alpha, \beta; \alpha + \deg(g_0), \beta - \deg(g_0), \alpha)$, and it is also separable. We summarize these results.

Theorem 22 *Let $\mathcal{C}$ be any $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code of the form* (1). *If $\mathcal{C}$ is MDSR, then $\mathcal{C}$ is either:*

(i) $\mathcal{C} = \langle (e|g_0 + 2g_1) \rangle$, *where* $t_1 | \frac{t_2}{g_1} e \pmod 2$, $g_1 | g_0 | t_2 \pmod 4$ *and* $\alpha + \deg(g_1) = \left\lfloor \frac{d-1}{2} \right\rfloor$ *with* $f = t_1$, *or*
(ii) $\mathcal{C} = \langle (1|0), (0|g_0 + 2) \rangle$, *where* $g_0 | t_2 \pmod 4$.

Example 23 (a) Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic code $\mathcal{C} = \langle (x+1|0), (1|3) \rangle$ in $\mathbb{Z}_2[x]/\langle x^3 + x^2 \rangle \times \mathbb{Z}_4[x]/\langle (x+1)(x^2+x+1) \rangle$. Then $\mathcal{C}$ is an MDSS code of type $(3, 3; 2, 3, 2)$ with minimum distance 2 using computer system Magma [13].
(b) Consider $\mathcal{C} = \langle (x+1|0), (1|3) \rangle$ in $\mathbb{Z}_2[x]/\langle (x+1)(x^3+x+1) \rangle \times \mathbb{Z}_4[x]/\langle (x+1)(x^6 + x^3 + 1) \rangle$. Then, the code $\mathcal{C}$ is MDSS and of type $(4, 7; 3, 7, 3)$ with minimum distance 2.
(c) Consider $\mathcal{C} = \langle (1|(x+3)(x^3+2x^2+x+3)+2) \rangle$ in $\mathbb{Z}_2[x]/\langle (x+1) \rangle \times \mathbb{Z}_4[x]/\langle x(x+3)(x^3+2x^2+x+3) \rangle$. Then, the code $\mathcal{C}$ is MDSR and of type $(1, 5; 4, 1, 1)$ with minimum distance 3.

Table 1 The list of some binary optimal codes from $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes $\mathcal{C}$ with $\mathcal{C} = \langle (e|g_0+2g_1) \rangle$

t_1	t_2	e	g_0	g_1	$\mathbb{Z}_2\mathbb{Z}_4$-type	$\Phi(\mathcal{C})$
101	3131	11	101	1	[2, 3; 1, 1, 0]	$[8, 3, 4]^\diamond$
10^21	12^21	01	1^3	1	[3, 3; 2, 1, 2]	$[9, 4, 4]^\diamond$
1^2	01231^2	1	1231^2	3121	[1, 5; 1, 1, 1]	$[11, 3, 5]^*$
101^3	1231^2	1^2	0	31	[4, 4; 3, 0, 3]	$[12, 3, 6]^\diamond$
101^3	01231^2	1^2	1231^2	31	[4, 5; 3, 1, 3]	$[14, 5, 6]^\diamond$
1^20101	0312321	1^2	312321	31	[5, 6; 4, 1, 4]	$[17, 6, 6]^*$
10^31^3	320^2131	1^301	3^21321	1	[6, 6; 5, 1, 5]	$[18, 7, 6]^*$
$1^20^31^3$	31020131	1^201	0	3121	[7, 7; 4, 0, 4]	$[21, 4, 9]^*$

We tried to find more MDSR codes from $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes as Example 23 (c) according to Theorem 22 (i), i.e., the nontrivial case. However, it is not easy to find such codes when α (i.e., $\deg(t_1)$) becomes bigger, and these experiments are limited to codes of small length, we believe that there exist more MDSR codes for larger length. Fortunately, in this process, we find some binary optimal codes (w.r.t. the online database [18]) from $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes via the Gray map, and the codes are listed in Table 1 by Magma [13].

In general, we call a linear binary code optimal (or distance-optimal) if it has the highest minimum distance for given length and dimension. For convenience, we write coefficients of generator polynomials in increasing order. For instance, we write 10^21^332 to present the polynomial $1 + x^3 + x^4 + x^5 + 3x^6 + 2x^7 \in \mathbb{Z}_4[x]$.

In Table 1, the binary codes with asterisk "$*''$ are almost optimal, and we call a binary $[n, k, d]$ linear code almost optimal if its minimum distance is at most one less than the largest possible value, that is, the code with parameters $[n, k, d+1]$ is optimal.

Further, the diamond "$\diamond$" indicates that the corresponding binary codes are both optimal and new according to the current database [18], and we confirm that these codes are inequivalent to the currently known codes even though they have the same parameters. For instance, let $t_1 = f = (x+1)(x^3+x+1)$, $t_2 = x(x+3)(x^3+2x^2+x+3)$, $e = x+1$, $g_0 = (x+3)(x^3+2x^2+x+3)$ and $g_1 = x+3$ in (1). Then the image $\Phi(\mathcal{C})$, a linear code with parameters [14, 5, 6], has generator matrix G given by

$$G = \begin{pmatrix} 1&0&0&1&0&0&0&0&1&1&0&0&1&1\\ 0&1&0&1&0&0&1&1&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&0&1&1&1&1&0&0\\ 0&0&0&0&1&0&1&1&0&1&1&0&1&0\\ 0&0&0&0&0&1&1&1&1&0&0&1&0&1 \end{pmatrix}.$$

It is easy to check that $\Phi(\mathcal{C})$ and the best known linear code from the Magma BKLC($\mathbb{F}_2$, 14, 5) or online database [18], that is, shortening the extended QR code of length 17 at positions {15, 16, 17, 18}, are not permutation equivalent by Magma [13].

6 Conclusion

In this paper, we have studied $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes, and these codes can be viewed as $\mathbb{Z}_4[x]$-submodules of $\mathcal{R}_{\alpha,\beta}$. We determine the generator polynomials and minimal generating sets of this family of codes. Moreover, we study the dual of the $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes under a proper defined inner product. It is worth noting that the dual code considered here is still additive polycyclic. And finally, we give some concrete examples of $\mathbb{Z}_2\mathbb{Z}_4$-additive polycyclic codes which lead to optimal binary codes via the Gray map.

In fact, let R be a finite chain ring with residue field F. If $f \in R[x]$ and $\bar{f}$ square-free, then $R/\langle f\rangle$ is a principal ideal ring, and every polycyclic code in $R/\langle f\rangle$ admits a strong Gröbner basis like Lemma 5 (more details refer to [30,31]). Thus, the main results of this paper can be generalized to FR-additive polycyclic codes, and it may derive more linear codes with good parameters over a finite field F.

In future studies, it would be interesting to consider quasi-polycyclic codes and ACD polycyclic codes over the mixed alphabets.

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