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A new family of partial difference sets in 3-groups

John Polhill[1]

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Abstract
In this paper we construct several infinite families of partial difference sets of both the Latin and negative Latin square type. Among these constructions is a new family having parameters $(3^{2t}, r(3^t+1), -n+r^2+3r, r^2+r)$, where $r = 3^{t-1}+1$ (new for $t \geq 4$). For the cases where $r = 3^{t-1} - 1$ and 3^{t-1}, the constructions generalize previous results to a larger collection of abelian groups.

Keywords Partial difference set · Strongly regular graph · Two-weight code

Mathematics Subject Classification 05E30 · 05B10

1 Introduction

A partial difference set is a subset of a group which has a Cayley graph that is strongly regular and in the case of elementary abelian groups can be used to construct a projective two-weight code. We will construct several infinite families of partial difference sets, including one having new parameters. As a result, we obtain infinite families of strongly regular graphs as well as projective two-weight codes. In the case of strongly regular graphs, we will have some examples with new parameters.

Let G be a finite group of order v and let D be a subset of order k. Suppose further that the differences $d_1d_2^{-1}$ for $d_1, d_2 \in D, d_1 \neq d_2$ represent each of the nonidentity elements in D exactly λ times and each of the nonidentity elements of $G - D$ exactly μ times. Then we call D a (v, k, λ, μ)-*partial difference set* (*PDS*) in G. Though over twenty years old, the survey article of Ma still provides the best overview for these sets [13]. A partial difference set having parameters $(n^2, r(n-1), n+r^2-3r, r^2-r)$ is called a *Latin square type PDS*. Similarly, a partial difference set having parameters $(n^2, r(n+1), -n+r^2+3r, r^2+r)$ is called a *negative Latin square type PDS*. Originally, most constructions of both of these types of PDSs were in elementary abelian groups. Latin square type partial difference set constructions appear to be much more common than their negative Latin counterpart.

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✉ John Polhill
jpolhill@bloomu.edu

1 Bloomsburg University, Bloomsburg, PA, USA

A graph Γ with v vertices is said to be (v, k, λ, μ)-*strongly regular* (*SRG*) if each vertex has degree k and furthermore that two adjacent vertices have precisely λ common neighbors while two nonadjacent vertices have μ common neighbors. Brouwer [2], Van Lint and Wilson [20], and Godsil [10] provide an excellent background for these objects. Brouwer maintains a useful online table for SRGs [1]. Recently, Cohen and Pasechnik have used Sagemath to implement many of the graphs from Brouwer's database [4]. It is known that the Cayley graph of a partial difference set is strongly regular, and more specifically if the parameters of the PDS are (v, k, λ, μ) then the corresponding graph will have the same parameters.

In Sect. 2 of this paper we will form a relatively simple decomposition of a group into Latin square type partial difference sets and show how this can be used to construct negative Latin square type partial difference sets. In Sect. 3 we demonstrate that many abelian 3-groups have the necessary decomposition required for the constructions in the previous section, improving both on Sect. 2 as well as previous work [14]. We conclude with some final thoughts including possible avenues for further research.

Often PDSs are studied within the context of the group ring $\mathbf{Z[G]}$. For a subset D in G we can write $D = \sum_{d\in D} d$ and $D^{(-1)} = \sum_{d\in D} d^{-1}$. While this may be abuse of notation to have D represent two different but related objects, it is widely accepted and it should be clear from context whether D will represent the partial difference set D or the element $\sum_{d\in D} d$ in the group ring $\mathbf{Z[G]}$.

Another very powerful tool for simplifying calculations with partial difference sets is character theory. A *character* on an abelian group G is a homomorphism from the group to the set of complex numbers having modulus 1 under the operation of multiplication. The *principal character* sends all group elements to 1. The following theorem will be used extensively in subsequent sections of this work. See [18] for a proof of similar results.

Theorem 1 *Let $\delta \in \{0, 1\}$. The subset D (with $1 \notin D$) of the abelian group G is a $(n^2, r(n+(-1)^\delta), (-1)^{\delta+1}n + r^2 + 3(-1)^\delta r, r^2 + (-1)^\delta r)$-PDS iff $\chi_0(D) = r(n + (-1)^\delta)$ for the principal character χ_0 and $\chi(D) = (-1)^\delta r$ or $(-1)^\delta(r-n)$ for every nonprincipal character χ. When $\delta = 0$, D is of the negative Latin square type and when $\delta = 1$, D is of the Latin square type.*

2 Main constructions of partial difference sets

In a previous paper [14], constructions of both Latin square type and negative Latin square type partial difference sets in 3-groups were given. If one could find the appropriate decomposition of a group G into Latin square type partial difference sets and another group G' into negative Latin square type partial difference sets, then a product theorem could be applied that would produce negative Latin square type partial difference sets in the product group. This paper will use a similar approach always using the following collection of partial difference sets in $G' = \mathbb{Z}_3^2 =< x, y >$ for the negative Latin square type portion of the product:

$$S_1 = \{x, x^2, y, y^2\}, S_2 = \{xy, x^2y^2, xy^2, x^2y\}.$$

For the other group G with identity 0_G, we require that $|G| = 3^{2t}$. G must also have a partition into the following sets:

$$\{0_G\}, H_1{}^*, H_2{}^*, D_0, D_1, D_2,$$

where H_1 and H_2 are subgroups of order 3^t, and the D_i are Latin square type partial difference sets having $r_i(3^t - 1)$ elements, where $r_0 = 3^{t-1} - 1$ and $r_1 = r_2 = 3^{t-1}$. Notice that this partition can be constructed by using partial difference sets of the well known partial congruence type (PCP) in the elementary abelian case, $\mathbb{Z}_3^{2t}$. A PCP construction in a group G of order n^2 consists of a union of trivially intersecting subgroups of order n with the identity removed. In general, the nonidentity elements of $\mathbb{Z}_3^{2t}$ can be partitioned into 3^t+1 subgroups of order 3^t. We let D_0 be a union of $3^{t-1}-1$ of these subgroups (removing the identity), while D_1 and D_2 will be a union of 3^{t-1} of the subgroups chosen so that the three PDSs are mutually disjoint. Then H_1 and H_2 are the remaining subgroups that have not been selected. For example, in $\mathbb{Z}_3^2$ we have the following:

$$H_1{}^* = \{x, x^2\},\ H_2{}^* = \{y, y^2\},\ D_0 = \{\},\ D_1 = \{xy, x^2y^2\},\ D_2 = \{xy^2, x^2y\}.$$

Before we state and prove our main result, we apply Theorem 1 to our specific partition.

Lemma 2 *Suppose the group G with identity 0_G has $|G| = 3^{2t}$. Suppose further that G has a partition into the following sets: $\{0_G\}$, $H_1{}^*$, $H_2{}^*$, D_0, D_1, D_2, where H_1 and H_2 are subgroups of order 3^t, and the D_i are Latin square type partial difference sets having $r_i(3^t-1)$ elements, where $r_0 = 3^{t-1}-1$ and $r_1 = r_2 = 3^{t-1}$. Then for any nonprincipal character χ on G, the character sums for $H_1{}^*$ and $H_2{}^*$ will be in $\{-1, 3^t-1\}$, the character sum for D_0 will be in $\{-3^{t-1}+1, 3^t-3^{t-1}+1\}$, and the character sums for D_1 and D_2 will be in $\{-3^{t-1}, 3^t-3^{t-1}\}$. Moreover, exactly one of the sets in $\{H_1, H_2, D_0, D_1, D_2\}$ will have positive character sum.*

Proof The possible character sums for the sets is an immediate consequence of Theorem 1. For any nonprincipal character χ on G we have:

$$\chi(H_1{}^*) = -1+\delta_1 3^t,\ \chi(H_2{}^*) = -1+\delta_2 3^t,\ \chi(D_0{}^*) = -3^{t-1}+1+\delta_3 3^t,$$
$$\chi(D_1{}^*) = -3^{t-1}+\delta_4 3^t,\ \chi(D_2{}^*) = -3^{t-1}+\delta_5 3^t,$$

where $\delta_i \in \{0, 1\}$ for $1 \le i \le 5$. We also have $-1 = \chi(G^*) = \chi(H_1{}^* \cup H_2{}^* \cup D_0 \cup D_1 \cup D_2) = \chi(H_1{}^*)+\chi(H_2{}^*)+\chi(D_0{}^*)+\chi(D_1{}^*)+\chi(D_2{}^*)$. It follows that $\delta_1+\delta_2+\delta_3+\delta_4+\delta_5 = 1$ and therefore there will be exactly one set in $\{H_1, H_2, D_0, D_1, D_2\}$ with a positive character sum. □

Now we are ready to state the main theorem. It includes construction of three different families of negative Latin square type partial difference sets, but we will only prove one of three since the other proofs are quite similar.

Theorem 3 (Main Theorem) *Suppose G is a group with identity 0_G and that $|G| = 3^{2t}$. Suppose that G has a partition into the following sets: $\{0_G\}$, $H_1{}^*$, $H_2{}^*$, D_0, D_1, D_2, where H_1 and H_2 are subgroups of order 3^t and the D_i are Latin square type partial difference sets having $r_i(3^t-1)$ elements, where $r_0 = 3^{t-1}-1$ and $r_1 = r_2 = 3^{t-1}$. Then the group $G \times \mathbb{Z}_3^2$ contains negative Latin square type partial difference set with parameters $(3^{2t+2}, r(3^{t+1}+1), -3^{t+1}+r^2+3r, r^2+r)$ where $r = 3^t-1, 3^t, 3^t+1$ respectively as follows:*

1. $D_a = (D_2 \cup H_2{}^*) \times (0, 0)) \cup ((D_0 \cup H_1{}^*) \times S_1) \cup (D_1 \times S_2)$;
2. $D_b = (D_1 \times (0, 0)) \cup ((D_2 \cup H_2) \times S_1) \cup ((D_0 \cup H_1{}^*) \times S_2)$;
3. *New family:* $D_c = (D_0 \times (0, 0)) \cup ((D_1 \cup H_1) \times S_1) \cup ((D_2 \cup H_2) \times S_2)$.

Proof $D_c = (D_0 \times (0, 0)) \cup ((D_1 \cup H_1) \times S_1) \cup ((D_2 \cup H_2) \times S_2)$. We will show that the set D_c is a partial difference set with parameters $(3^{2t+2}, r(3^{t+1}+1), -3^{t+1}+r^2+3r, r^2+r)$ for $r = 3^t+1$.

First observe that $|D_c| = |D_0| + 4(|D_1| + |H_1|) + 4(|D_2| + |H_2|) = (3^{t-1} - 1)(3^t - 1) + 8(3^{t-1}(3^t - 1) + 3^t) = (3^t + 1)(3^{t+1} + 1)$.

Let ϕ be a nonprincipal character on $G \times \mathbb{Z}_3^2$, so that $\phi = \chi \otimes \psi$ where χ is a character on G and ψ is a character on $\mathbb{Z}_3^2$. By Theorem 1, we will need to show that $\phi(D_c) = r$ or $r - n$ where $n = 3^{t+1}$.

Case 1: χ is principal, so that ψ is nonprincipal. Then we have $\phi(D_c) = |D_0| + (|D_1| + |H_1|)(\psi(S_1)) + (|D_2| + |H_2|)(\psi(S_2)) = |D_0| - |D_1| - |H_1| = -3^t - 3^t + 1 = (3^t + 1) - 3^{t+1} = r - n$.

Case 2: ψ is principal, so that χ is nonprincipal. Exactly one of the sets $H_1{}^*$, $H_2{}^*$, D_0, D_1, D_2, χ will have positive character sum. If $\chi(D_0) > 0$ we have the following $\phi(D_c) = 1(\chi(D_0)) + 4(\chi(H_1) + \chi(D_1)) + 4(\chi(H_2) + \chi(D_2)) = (3^t - 3^{t-1} + 1) + 4(-3^{t-1}) + 4(-3^{t-1}) = (3^t + 1) - 3^{t+1} = r - n$. Now suppose that $\chi(D_1) > 0$. Then we have: $\phi(D_c) = 1(\chi(D_0)) + 4(\chi(H_1) + \chi(D_1)) + 4(\chi(H_2) + \chi(D_2)) = (-3^{t-1} + 1) + 4(3^t - 3^{t-1}) + 4(-3^{t-1}) = (3^t + 1) = r$. The other subcases are nearly identical to this last case.

Case 3: Both χ and ψ are nonprincipal. Note that $\{\psi(S_1), \psi(S_2)\} = \{-2, 1\}$, and the cases are similar so suppose that $\psi(S_1) = -2$. If $\chi(D_0) > 0$, we will have: $\phi(D_c) = 1(\chi(D_0)) + -2(\chi(H_1) + \chi(D_1)) + (1)(\chi(H_2) + \chi(D_2)) = (3^t - 3^{t-1} + 1) + -2(-3^{t-1}) + 1(-3^{t-1}) = (3^t + 1) = r$. If $\chi(D_1) > 0$ or $\chi(H_1) > 0$, we have: $\phi(D_c) = 1(\chi(D_0)) + -2(\chi(H_1) + \chi(D_1)) + (1)(\chi(H_2) + \chi(D_2)) = (-3^{t-1} + 1) + -2(3^t - 3^{t-1}) + 1(-3^{t-1}) = (3^t + 1) - 3^{t+1} = r - n$. If $\chi(D_2) > 0$ or $\chi(H_2) > 0$, we have: $\phi(D_c) = 1(\chi(D_0)) + -2(\chi(H_1) + \chi(D_1)) + (1)(\chi(H_2) + \chi(D_2)) = (-3^{t-1} + 1) + -2(-3^{t-1}) + 1(3^t - 3^{t-1}) = (3^t + 1) = r$. □

While D_c is an infinite family with new parameters, the remaining two parameter sets have similar constructions in [14]. We now have constructed negative Latin square type partial difference sets in groups of the form $G \times \mathbb{Z}_3^2$, where the order of $|G| = 3^{2t}$ provided that G has a partition as above. We conclude this section with a recursive theorem that will be useful when we introduce groups other than elementary abelian to this construction in Sect. 3.

Theorem 4 *Suppose G and G' are groups with identities 0_G and $0_{G'}$ respectively with $|G| = 3^{2t}$ and $|G'| = 3^{2s}$. Suppose that G has a partition into the following sets: $\{0_G\}$, $H_1{}^*$, $H_2{}^*$, D_0, D_1, D_2, where H_1 and H_2 are subgroups of order 3^t, and the D_i are Latin square type partial difference sets having $x_i(3^t - 1)$ elements, where $x_0 = 3^{t-1} - 1$ and $x_1 = x_2 = 3^{t-1}$. Suppose that G' has the same type of partition: $\{0_{G'}\}$, $H_1'^*$, $H_2'^*$, D_0', D_1', D_2', where H_1' and H_2' are subgroups of order 3^s, and the D_i' are Latin square type partial difference sets having $y_i(3^s - 1)$ elements, where $y_0 = 3^{s-1} - 1$ and $y_1 = y_2 = 3^{s-1}$. Then the product group $G \times G'$ has a partition into the following sets: $\{0_{G\times G'}\}$, $K_1{}^*$, $K_2{}^*$, P_0, P_1, P_2, where K_1 and K_2 are subgroups of order 3^{t+s}, and the D_i are Latin square type partial difference sets having $z_i(3^{t+s} - 1)$ elements, where $z_0 = 3^{t+s-1} - 1$ and $z_1 = z_2 = 3^{t+s-1}$.*

Proof The two subgroups in the partition are given by $K_1 = (H_1 H_1')$ and $K_2 = (H_2 H_2')$. The two larger PDSs are given by:

$$P_1 = ((D_0 \cup H_1 \cup H_2) \times D_2') \cup (D_1 \times (H_1' \cup H_2' \cup D_0')) \cup (D_2 \times D_1'),$$
$$P_2 = ((D_0 \cup H_1 \cup H_2) \times D_1') \cup (D_1 \times D_2')) \cup (D_2 \times (H_1' \cup H_2' \cup D_0'))),$$

Then the remaining PDS is:

$$\begin{aligned} P_0 &= (G \times G') - (K_1 \cup K_2 \cup P_1 \cup P_2) \\ &= (\{0_G\} \times D_0') \cup ((D_0) \times (H_1' \cup H_2' \cup D_0')) \cup (D_1 \times D_1') \end{aligned}$$

$$\cup(D_2 \times D_2') \cup (H_1{}^* \times (H_2'^* \cup D_0')) \cup (H_2{}^* \times (H_1'^* \cup D_0')).$$

The fact that K_1 and K_2 are subgroups is obvious, and it will follow that P_0 is a partial difference set provided that both P_1 and P_2 are both PDSs since they are Latin square type PDSs of the appropriate size. In the context of SRGs and association schemes, Van Dam proved that the union of Latin square PDSs [19] gives that $K_1{}^* \cup K_2{}^* \cup P_1 \cup P_2$ is a Latin square type partial difference set since it is the fusion of Latin square type partial difference sets, and then P_0 is simply the complement of this union.

To see that P_1 and P_2 are $(3^{2s+2t}, r(3^{s+t}-1), 3^{s+t}+r^2-3r, r^2-r)$-PDSs for $r = 3^{s+t-1}$, we observe that $(D_0 \cup H_1{}^* \cup H_2{}^*)$, D_1, and D_2 are Latin square type PDSs in G with cardinality $(3^{t-1}+1)(3^t-1)$, $3^{t-1}(3^t-1)$, and $3^{t-1}(3^t-1)$ respectively while $(D_0' \cup H_1'^* \cup H_2'^*)$, D_1', and D_2' are Latin square type PDSs in G' with cardinality $(3^{s-1}+1)(3^s-1)$, $3^{s-1}(3^s-1)$, and $3^{s-1}(3^s-1)$. These are the necessary criteria to apply Theorem 2.2 of [14] and it follows that P_1 and P_2 are both partial difference sets in $G \times G'$. □

We summarize our results as follows.

Corollary 5 *There exist negative Latin square type partial difference sets with parameters* $(3^{2t}, r(3^t+1), -3^t+r^2+3r, r^2+r)$ *for every positive integer* $t \geq 2$ *and for* $r = 3^t - 1, 3^t, 3^t+1$.

Proof Either by using a PCP-type construction or by repeatedly applying Theorem 4 to $\mathbb{Z}_3^2$ and the initial partition $H_1{}^* = \{x, x^2\}$, $H_2{}^* = \{y, y^2\}$, $D_0 = \{\}$, $D_1 = \{xy, x^2y^2\}$, $D_2 = \{xy^2, x^2y\}$, we can construct in the group $G = \mathbb{Z}_3{}^{2s}$ the necessary partition : $\{0_G\}$, $H_1'^*$, $H_2'^*$, D_0', D_1', D_2' to use with Theorem 3. □

We have three infinite families of partial difference sets, and it immediately follows that the Cayley graphs of these sets correspond to strongly regular graphs with the same parameters, namely $(3^{2t}, r(3^t+1), -3^t+r^2+3r, r^2+r)$ for every positive integer $t \geq 2$ and for $r = 3^t-1, 3^t, 3^t+1$. In Brouwer's table [1], one finds that previous constructions have been given for $t = 2$ where the degrees is $k = 20, 30$, and 40 and also for $t = 3$, where the degrees are $k = 234, 252$, and 280. The cases $k = 234, 252$ can be constructed using quadratic forms [13] or by the methods of [14] and are known to be part of an infinite family. De Resmini constructed the graph with $k = 280$ in [6]. This paper gives the first construction to include this parameter set in an infinite family. Note the graph with degree $k = 40$ has the well known Paley parameters.

3 PDSs for groups with exponent 3^n, any $n \geq 1$

In Sect. 2, we constructed 3 infinite families of partial difference sets with parameters $(3^{2+2n}, r(3^{n+1}+1), -3^{n+1}+r^2+3r, r^2+r)$ and $r = 3^n-1, 3^n, 3^n+1$. So far, all constructions have been in the group $\mathbb{Z}_3{}^{2+2n}$. In this section, we show that the new family having $r = 3^n+1$ will include many abelian 3-groups and not just exponent 3. For the other two parameter sets, [14] gave constructions in all groups of the form $\mathbb{Z}_3^{2t_1} \times \mathbb{Z}_9^{2t_2} \times \mathbb{Z}_{81}^{2t_4} \times \cdots \times \mathbb{Z}_{3^{2s}}^{2t_{2s}}$ ($t_1 > 0$, $t_k \geq 0$ for $k > 1$). In this section, we improve on this result by removing the stipulation that the component groups beyond $\mathbb{Z}_3^{2t_1}$ necessarily have exponent an even power of 3. For example, the previous paper did not include $\mathbb{Z}_3^2 \times \mathbb{Z}_{27}^2$ and this paper shows that this group contains PDSs with these parameters.

Lemma 6 *The nonidentity elements of the group $\mathbb{Z}_9^2$ can be partitioned into 2 subgroups H_1 and H_2 of order 9 and 3 Latin square type partial difference sets D_0, D_1, D_2 so that $|D_0| = 16$ and $|D_1| = |D_2| = 24$ as needed for Theorem 4.*

Proof Let $\mathbb{Z}_9^2 =< a, b >$. The appropriate partition can be found using the methods of [5] or [12]. We give it explicitly: $D_1 = (< a > \cup < ab > \cup < ab^2 > \cup < b >) \cap (G - 3G)$, $D_2 = (< ab^3 > \cup < ab^4 > \cup < ab^5 > \cup < a^3b >) \cap (G - 3G)$, $H_1 =< ab^6 >$, $H_2 =< ab^7 >$, and $D_0 = G^* - D_1 - D_2 - H_1^* - H_2^*$. □

Theorem 7 *For all positive integers r there exists a partition of the nonidentity elements in the group $G = \mathbb{Z}_{3^r}^2$ into 2 subgroups H_1 and H_2 of order 3^r and three Latin square type partial difference sets D_0, D_1, D_2 so that $|D_0| = (3^{r-1}-1)(3^r-1)$ and $|D_1| = |D_2| = 3^{r-1}(3^r-1)$.*

Proof We have now seen that both $\mathbb{Z}_3^2$ and $\mathbb{Z}_9^2$ have this partition. We now use induction on r.

Suppose that the group $\mathbb{Z}_{3^{(r-2)}}^2$ contains the partition into two subgroups K_1 and K_2 of order 3^{r-2} and the PDSs P_0, P_1, and P_2. Let $\Pi : \mathbb{Z}_{3^{r-1}}^2 \to \mathbb{Z}_{3^{r-2}}^2$ be the natural projection map given by $\Pi(x) = 3x$. Note that $3\mathbb{Z}_{3^r}^2 \cong \mathbb{Z}_{3^{r-1}}^2$ and $9\mathbb{Z}_{3^r}^2 \cong \mathbb{Z}_{3^{r-2}}^2$ so we can identify the sets P_i and K_j in $9G$.

Let $\mathbb{Z}_{3^r}^2 =< a, b >$. Define the two partial difference sets D_1 and D_2 as:

$$\begin{aligned} D_1 &= \Pi^{-1}(P_1) \cup [(G - 3G) \cap \\ &\quad ((< a > \cup < ab > \cup \cdots \cup < ab^{3^{r-1}-1} > \cup < b > \cup < a^3b > \cup \cdots \cup < a^{3(p^{r-2}-1)}b)] \\ D_2 &= \Pi^{-1}(P_2) \cup [(G - 3G) \cap \\ &\quad (< ab^{3^{r-1}} > \cup < ab^{3^{r-1}+1} > \cup \cdots \cup < ab^{2(3^{r-1})-1} > \cup < a^{3p^{r-2}}b > \cup \cdots \cup < a^{3(2p^{r-2}-1)}b)] \end{aligned}$$

Let $H_i < \mathbb{Z}_{3^r}^2$ be a subgroup of order 3^r such that $\Pi(3H_i) = K_i$ for $i = 1, 2$. We also choose the H_i so that they are each disjoint from the D_j. It follows that D_1, D_2, H_1^*, and H_2^* are mutually disjoint. Define $D_0 = \mathbb{Z}_{3^r}^2 - H_1 - H_2^* - D_1 - D_2$. We will prove that D_1 is a partial difference set with the appropriate parameters. The proof of D_2 is identical, and the fact that D_0 will be a partial difference set with the appropriate parameters follows immediately.

Let χ be a character on G. If χ is principal, $\chi(D_1) = |D_1|$. D_1 contains exactly $\frac{1}{3}$ of the elements of $G - 3G$, and also $|\Pi^{-1}(P_1)| = 9|P_1|$ since $|\ker(\Pi)| = 9$. So $\chi(D_1) = \frac{1}{3}(3^{2r} - 3^{2r-2}) + 9(|P_1|) = \frac{1}{3}(3^{2r} - 3^{2r-2}) + 9(3^{r-3}(3^{r-2} - 1) = 3^{r-1}(3^r) - 3^{2r-3} + 3^{2r-3} - 3^{r-1} = 3^{r-1}(3^r - 1)$.

If χ has order 3, then $\chi(\Pi^{-1}(P_1)) = 9|P_1| = 3^{2r-3} - 3^{r-1}$. Meanwhile χ will be principal on exactly one of $< a >, < ab >, < ab^2 >$, and $< b >$ and order 3 on the other 3. It follows that $\chi(< a >)+\chi(< ab >)+\chi(< ab^2 >)+\chi(< b >) = 3^r - 3^{r-1} - 3(3^{r-1}) = -3^{r-1}$.Then we have: $\chi[(G - 3G) \cap ((< a > \cup < ab > \cup \cdots \cup < ab^{3^{r-1}-1} > \cup < b > \cup < a^3b > \cup \cdots \cup < a^{3(3^{r-2}-1)}b)] = 3^{r-2}(-3^{r-1}) = -3^{2r-3}$ since there are 3^{r-2} 4-tuples of subgroups that will have the same character values as $< a >, < ab >, < ab^2 >$, and $< b >$. Putting it all together gives gives $\chi(D_1) = 3^{2r-3} - 3^{r-1} - 3^{2r-3} = -3^{r-1}$.

If χ has order 3^k, $1 < k < r$, then $\chi[(G - 3G) \cap ((< a > \cup < ab > \cup \cdots \cup < ab^{3^{r-1}-1} > \cup < b > \cup < a^3b > \cup \cdots \cup < a^{3(3^{r-2}-1)}b)] = 0$. This results from the fact that for every $< x >$ in the collection on which χ is principal will be exactly 2 subgroups in the collection on which χ will be order 3. These triples yield a character sum of $3^r - 3^{r-1} - 2(3^{r-1}) = 0$. χ has order greater than 3 on all other subgroups and yields character sum 0 on these. So $\chi(D_1) = \chi(\Pi^{-1}(P_1)) = 9\chi(P_1) = 9(\delta 3^{r-2} - 3^{r-3}) = \delta 3^r - 3^{r-1}$, where $\delta = 0, 1$.

If χ has order 3^r, $\chi(\Pi^{-1}(P_1)) = 0$ since χ is no longer trivial on the kernel of Π. Then $\chi(D_1) = \chi[(G-3G) \cap ((<a> \cup <ab> \cup \cdots \cup <ab^{3^{r-1}-1}> \cup <b> \cup <a^3b> \cup \cdots \cup <a^{3(3^{r-2}-1)}b)] = \delta 3^r - 3^{r-1}$ where again $\delta = 0, 1$. In this case, there is exactly one subgroup in the entire collection on which χ will be order 3 (so $\delta = 0$) or principal ($\delta = 1$). This completes the proof. □

As an example let $G = \mathbb{Z}_{27}^2 =< a, b >$ we use:

$$
\begin{aligned}
S_1 &= ((< a > \cup < ab > \cup \cdots \cup < ab^8 > \cup < b > \cup < a^3b > \cup < a^6b >) \cap (G - 3G)) \\
&\quad \cup ((< a^3 > \cup < a^3b^9 > \cup < a^3b^{18} >) \cap (3G - 9G) \\
S_2 &= ((< ab^9 > \cup < ab^{10} > \cup \cdots \cup < ab^{17} > \cup < a^9b > \cup < a^{12}b > \cup < a^{15}b >) \cap (G - 3G)) \\
&\quad \cup ((< b^3 > \cup < a^9b^3 > \cup < a^{18}b^3 >) \cap (3G - 9G) \\
H_1 &= < ab^{19} >, H_2 =< ab^{20} >, \\
D_0 &= G^* - D_1 - D_2 - H_1 - H_2.
\end{aligned}
$$

4 Final remarks and remaining questions

We have constructed a new family of negative Latin square type partial difference sets. It would appear that, while less easily constructed than their Latin square type partial difference set counterparts, negative Latin square type partial difference sets are not so uncommon in p-groups. On the other hand, partial difference sets with parameters not of Latin or negative Latin square type do seem rather rare as do constructions of PDSs of any kind in groups not having order the power of a prime. De Winter and his co-authors have several recent nonexistence results along these lines [7–9]. They have proved nonexistence for each of the open cases in Ma's table [13] for PDSs having cardinality $k \leq 100$. We conclude by posing a few possible directions for future work.

1. In this paper, constructions of infinite families of partial difference sets/strongly regular graphs that generalize the examples with $v = 729$ and with $k = 224, 252,$ and 280. Brouwer's table [1] indicates that the case of $k = 140$ is still completely undetermined, while some other negative Latin square type parameters have constructions. Constructing the graph with $k = 140$ would be quite interesting, but it would also be of interest to find new families that include some of the other cases for negative Latin square type PDSs. We note that $k = 364$ has the well known Paley parameters, a family with many known examples such as in [5,15,16].
2. Partial difference sets also can be used to construct projective 2-weight codes. It might be possible to use results similar to those in this paper to find interesting codes. Equally plausible is the possibility at finding some interesting results by looking at some of the known projective two-weight codes to get new families of partial difference sets. For example, Gulliver constructs the SRGs with $k = 168, 196$ in [11] using a coding approach.
3. This paper and [14] give constructions of negative Latin square type partial difference sets in both 2-groups and 3-groups. In the case of 3-groups, the constructions depend on the partition of $\mathbb{Z}_3^2$ into the identity and two $(9, 4, 1, 2)$-PDSs. In the case of 2-groups, the constructions depend on the fact that both $\mathbb{Z}_2^4$ and $\mathbb{Z}_4^2$ have partitions into the identity and three $(16, 5, 0, 2)$-PDSs. For primes p larger than 3, a few examples of negative Latin square type partial difference sets were constructed in [3]. However, the group $\mathbb{Z}_p^2$

does not have a partition into the identity and $p-1$ negative Latin square type partial difference sets so the methods in this paper do not immediately apply. It is possible with some adjustments new constructions could be found for larger primes.

4. Find new Latin or negative Latin square type partial difference sets in non p-groups, or find new PDSs that are of neither the Latin nor negative Latin square type. Both of these appear rather rare, although examples have been found in [15] and [16]. The results from [17] also could possibly be used to find new examples.
5. The constructions in this paper, [14,15,17], and [16] have all been made through partial difference sets. The constructions and proofs could be put into the context of strongly regular graphs. It is possible that there could be new resulting graphs to be constructed in this manner since the existence of a strongly regular graph does not automatically imply the existence of a corresponding partial difference set.

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