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Codes with a pomset metric and constructions

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Abstract Brualdi's introduction to the concept of poset metric on codes over $\mathbb{F}_q$ paved a way for studying various metrics on $\mathbb{F}_q^n$. As the support of vector x in $\mathbb{F}_q^n$ is a set and hence induces order ideals and metrics on $\mathbb{F}_q^n$, the poset metric codes could not accommodate Lee metric structure due to the fact that the support of a vector with respect to Lee weight is not a set but rather a multiset. This leads the authors to generalize the poset metric structure on to a pomset (partially ordered multiset) metric structure. This paper introduces pomset metric and initializes the study of codes equipped with pomset metric. The concept of order ideals is enhanced and pomset metric is defined. Construction of pomset codes are obtained and their metric properties like minimum distance and covering radius are determined.

Keywords Multiset · Pomset · Lee weight · Poset codes · Covering radius

Mathematics Subject Classification 06A06 · 94B05 · 94B75

1 Introduction

In 1995, Brualdi [2] introduced the concept of poset metric codes over $\mathbb{F}_q$ by imposing a partial order relation on the set $[n] = \{1, 2, \ldots, n\}$ of coordinates of a vector in $\mathbb{F}_q^n$. Thus, if $P([n], \leq)$ is a poset on $[n]$, a subset $I \subseteq [n]$ is called an order ideal of P if $i \in I$, $j \leq i$ imply that $j \in I$. For a subset A of P, the smallest order ideal containing A is denoted as $\langle A \rangle$. Given a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, the support of x is $supp(x) = \{i : x_i \neq 0\}$.

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The poset weight $w_P(x)$ of x is defined as $w_P(x) = |\langle supp(x)\rangle|$ and $d_P(x, y) = w_P(x - y)$ is a well defined metric on $\mathbb{F}_q^n$. Thus, by varying posets one gets different metrics on $\mathbb{F}_q^n$ like Rosenbloom–Tsfasman (RT)-metric if P is a chain, Hamming metric if P is an antichain and so on. Moreover, many results that hold for the Hamming metric, may fail for a particular poset metric. For instance, the well-known equation $\rho = \lfloor \frac{d-1}{2} \rfloor$, which relates the minimum distance d of a code with its packing radius ρ, is not valid for general posets [4]. Thus, extension of coding theory to poset metric spaces is interesting and has been the subject of study for over two decades [7,8].

Note that the support of x which is defined as $supp(x) = \{i : x_i \neq 0\}$ is a set and hence does not accommodate Lee metric by considering any particular poset. Moreover, the Lee weight of an element $l \in \mathbb{Z}_m$ is defined as $w_L(l) = \min\{l, m - l\}$ whereas the Hamming weight of any $l \neq 0$ is 1. Also the Hamming weight of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_m^n$ is sum of the weights of the non-zero coordinates, so that it counts the number of non-zero positions whereas Lee weight adds the Lee weights of non-zero coordinates in x. Thus, the support of $x \in \mathbb{Z}_m^n$ with respect to Lee weight is to be defined as $supp_L(x) = \{k/i : k = w_L(x_i), k \neq 0\}$ which is a *multiset* (Here, k/i stands for the notation that the position i is counted k times in the multiset). This motivates us in finding a partial order-like relations on multisets.

1.1 On multisets and relations

Unlike in set theory which is well established, the research in multiset theory is in its initial stages. In this subsection, a brief review of basic definitions and notations of multisets and relations on multisets, introduced by Girish and John [5,6], are presented which will be needed for our investigation and findings.

Definition 1 A collection of elements which may contain duplicates is called a *multiset* (in short, *mset*). Formally, if X is a set of elements, a multiset M drawn from the set X is represented by a function count $C_M : X \to \mathbb{W}$ where $\mathbb{W}$ represents the set of non-negative integers. For each $a \in X$, $C_M(a)$ indicates the number of occurrences of the element a in M.

$a \in X$ appearing n times in M is denoted by $a \in^n M$ or $n/a \in M$. The mset M drawn from the set X is represented as $M = \{k_1/a_1, k_2/a_2, \ldots, k_n/a_n\}$. If $C_M(a_i) = k_i$ then we can say $r_i/a_i \in M \ \forall\ 1 \leq r_i \leq k_i$. An mset is called *regular* or *constant* if all its objects occur with the same multiplicity and the common multiplicity is called its *height*.

A *domain* is defined as a set X of elements from which msets are constructed. The *cardinality* of an mset M drawn from its domain X is $|M| = \sum_{a \in X} C_M(a)$. The *support set* of M denoted by M^* is a subset of X and $M^* = \{a \in X : C_M(a) > 0\}$, i.e., M^* is an ordinary set. M^* is also called the *root set* of M.

Definition 2 Let M_1 and M_2 be two msets drawn from a set X. Then M_1 is called a *submultiset* (in short, a *submset*) of M_2 $(M_1 \subseteq M_2)$ if $C_{M_1}(a) \leq C_{M_2}(a)$ for all $a \in X$. M_1 is a *proper submset* of M_2 $(M_1 \subset M_2)$ if $C_{M_1}(a) \leq C_{M_2}(a)$ for all $a \in X$ and there exists at least one $a \in X$ such that $C_{M_1}(a) < C_{M_2}(a)$. Two msets M_1 and M_2 are equal $(M_1 = M_2)$ if $M_1 \subseteq M_2$ and $M_2 \subseteq M_1$.

Let M be an mset and A be a submset of M. An element $a \in A^*$ is said to have full count with respect to M if $C_A(a) = C_M(a)$. If $C_A(a) = C_M(a)\ \forall a \in A^*$, then A is said to have full count with respect to M.

Analogous to the union, intersection and symmetric difference of sets, these operations are also defined in multiset theory [3], in addition to the operations called as sum and subtraction of msets:

Let M_1 and M_2 be two msets with domain X. *Addition* (*sum*) of M_1 and M_2 is a new mset $M = M_1 \oplus M_2$ such that for all $a \in X, C_M(a) = C_{M_1}(a) + C_{M_2}(a)$. *Subtraction* (*difference*) of M_2 from M_1 is an mset $M = M_1 \ominus M_2$ such that for all $a \in X$, $C_M(a) = \max\{C_{M_1}(a) - C_{M_2}(a), 0\}$. The *union* of M_1 and M_2 is an mset M denoted by $M = M_1 \cup M_2$ such that for all $a \in X, C_M(a) = \max\{C_{M_1}(a), C_{M_2}(a)\}$. The *intersection* of M_1 and M_2 is an mset M denoted by $M = M_1 \cap M_2$ such that for all $a \in X$, $C_M(a) = \min\{C_{M_1}(a), C_{M_2}(a)\}$. The *symmetric difference* of M_1 and M_2 is an mset M denoted by $M = M_1 \Delta M_2$ such that for all $a \in X, C_M(a) = |C_{M_1}(a) - C_{M_2}(a)|$.

Definition 3 The *mset space* $[X]^m$ is the set of all msets drawn from X such that no element in an mset occurs more than m times.

Thus, if $M_1, M_2 \in [X]^m$, the *mset sum* can be modified as follows:

$$C_{M_1 \oplus M_2}(a) = \min\{m, C_{M_1}(a) + C_{M_2}(a)\} \text{ for all } a \in X.$$

And for any mset $M \in [X]^m$, the *complement* M^c of M in $[X]^m$ is an element of $[X]^m$ such that $C_{M^c}(a) = m - C_M(a)$ for all $a \in X$.

Notation 1 In [5], while defining the cartesian product of two msets M_1 and M_2, the authors introduced the notation $(m/a, n/b)/k$ which means that a is repeated m times, b is repeated n times and the pair (a, b) is repeated k times. From this, the count of the pair is k. Actually, an element a which is repeated k times is denoted by k/a. To avoid confusion and have coherence, we modified the notation $(m/a, n/b)/k$ to $k/(m/a, n/b)$ which gives the same meaning. In fact $k/(m/a, n/b)$ and $k/(a, b)$ are one and the same except the fact that the former gives an additional information about the counts of a and b. $C_1(a, b)$ denotes the count of the first coordinate in the ordered pair (a, b) and $C_2(a, b)$ denotes the count of the second coordinate in the ordered pair (a, b).

Definition 4 Let M_1 and M_2 be two msets drawn from a set X; then the *cartesian product* of M_1 and M_2 is defined as

$$M_1 \times M_2 = \{mn/(m/a, n/b) : m/a \in M_1, n/b \in M_2\}.$$

Example 1 Consider an mset $M = \{4/a, 2/b\}$. Then $M \times M = \{16/(4/a,\ 4/a), 8/(4/a,\ 2/b), 8/(2/b,\ 4/a), 4/(2/b,\ 2/b)\}$.

Definition 5 A submset R of $M \times M$ is said to be an *mset relation* on M if every member $(m/a, n/b)$ of R has count $C_1(a, b) \cdot C_2(a, b)$.

Thus, if $(m/a, n/b) \in R$ we say "m/a is R-related to n/b" and write "$m/a \ R \ n/b$". Moreover, if $m/a \ R \ n/b$ then we can say $r/a \ R \ s/b \ \forall \ r \leq m, s \leq n$.

Example 2 For the mset M given in Example 1, $S = \{5/(4/a,\ 2/a), 8/(4/a,\ 2/b)\}$ is a submset of $M \times M$ but not an mset relation as $5 \neq 4 \times 2$ whereas $R = \{4/(2/a,\ 2/b), 6/(2/b,\ 3/a)\}$ is an mset relation on M.

Definition 6 An *mset relation* R on an mset M is

(i) *reflexive* iff $m/a \ R \ m/a$ for all m/a in M.
(ii) *antisymmetric* iff $m/a \ R \ n/b$ and $n/b \ R \ m/a$ imply $m = n$ and $a = b$.
(iii) *transitive* iff $m/a \ R \ n/b, n/b \ R \ k/c$ imply $m/a \ R \ k/c$.

Definition 7 Let R be an mset relation on an mset M in $[X]^m$. Then R is called a *partially ordered mset relation* (or *pomset relation*) if it is reflexive, antisymmetric and transitive. The pair (M, R) is known as a *partially ordered multiset* (*pomset*) and it is denoted by $\mathbb{P}$.

Let $\mathbb{P} = (M, R)$ be a pomset and $p = C_M(a)$, $q = C_M(b)$. Suppose that for $a \neq b$, $(p/a,\ q/b) \notin R$ but $(m/a,\ n/b) \in R$ where either $m < p$ or $n < q$ or both. By reflexive property, $p/a\ R\ p/a$, $q/b\ R\ q/b$. As $p/a\ R\ m/a$, $n/b\ R\ q/b$, it follows by transitive property that $(p/a,\ q/b)$ is an element of R, which is not true. So, in $\mathbb{P}$, if $(p/a,\ q/b) \notin R$ then we cannot say $(m/a,\ n/b) \in R$ for any $m < p$ or $n < q$. Thus, every member of a pomset relation R has full count with respect to $M \times M$.

Example 3 Consider an mset $M = \{3/1, 3/2, 3/3, 3/4\}$ where $M^* = \{1, 2, 3, 4\}$. Then $R = \{9/(3/1,\ 3/1), 9/(3/2,\ 3/2), 9/(3/3,\ 3/3), 9/(3/4,\ 3/4), 9/(3/2,\ 3/1), 9/(3/3,\ 3/4)\}$ is a pomset relation on M and $\mathbb{P} = (M, R)$ is a pomset whereas $Q = \{9/(3/1,\ 3/1), 9/(3/2,\ 3/2), 9/(3/3,\ 3/3), 9/(3/4,\ 3/4), 4/(2/2,\ 2/1)\}$ is not a pomset relation on M.

Definition 8 Let $\mathbb{P} = (M, R)$ and $m/a \in M$. Then m/a is a maximal element of $\mathbb{P}$ if there exists no $n/b \in M$ $(b \neq a)$ such that $m/a\ R\ n/b$; m/a is a minimal element of $\mathbb{P}$ if there exists no $n/b \in M$ $(b \neq a)$ such that $n/b\ R\ m/a$.

Definition 9 A submset structure $\mathbb{C} = (C \subseteq M, R)$ of $\mathbb{P} = (M, R)$ is a *chain* in $\mathbb{P}$ if every distinct pair of points from C is comparable in $\mathbb{P}$, i.e., $\forall\ m/a, n/b\ (a \neq b)$ in C, either $m/a\ R\ n/b$ or $n/b\ R\ m/a$ in $\mathbb{P}$.

A pomset $\mathbb{P} = (M, R)$ itself is called a chain if every distinct pair of points from M is comparable in $\mathbb{P}$. When $\mathbb{P}$ is a chain, we call $\mathbb{P}$ a *linear mset order* (also a *total mset order*) on M.

Definition 10 A submset structure $\mathbb{A} = (A \subseteq M, R)$ of $\mathbb{P} = (M, R)$ is an *antichain* in $\mathbb{P}$ if every distinct pair of points from A is incomparable in $\mathbb{P}$, i.e., $\forall\ m/a, n/b\ (a \neq b)$ in A, neither $m/a\ R\ n/b$ nor $n/b\ R\ m/a$ in $\mathbb{P}$.

A pomset $\mathbb{P} = (M, R)$ itself is called an antichain if every distinct pair of points from M is incomparable in $\mathbb{P}$.

Now we have enough armour into our fold to define order ideals in pomsets which will pave a way for introducing pomset metric on $\mathbb{Z}_m^n$. We will do this in the next section.

In the remaining part of this paper, we will study various constructions of pomset codes and their metric properties. To study the properties of new codes, especially, their minimum distance and covering radius in terms of the constituent codes, suitable pomset structure is to be imposed on new codes. As far as posets are concerned, there are several ways to create new posets from given posets as in [1,9]. In this paper, we extended these operations on pomsets, that is, we define new pomsets through direct sum of pomsets, ordinal sum of pomsets, puncturing a pomset, extending a pomset and so on (which are dealt with in Sect. 3).

Thus, we could impose a new pomset structure to the codes constructed which enables us to study minimum distance and covering radius. We obtain these parameters for codes constructed through direct sum of codes whereas for codes constructed through $(u, u + v)$-construction, puncturing codes and extension of codes, bounds on these parameters are established. For product codes, we obtain the bounds on the minimum distance and the covering radius by taking constituent pomsets to be combinations of chain and antichain. These form the content of Sects. 4 and 5.

2 Ideals and pomset metric

2.1 Ideals in pomsets

By adopting Definition 37 in [5] and slightly modifying the definition of pre-class of M given in [6], we introduce the definition of order ideal and the order ideal generated by a submset in a pomset. Let $\mathbb{P} = (M, R)$ be a pomset. A submset I of M is called an *order ideal* (or simply an *ideal*) of $\mathbb{P}$ if $m/a \in I$ and $n/b\ R\ k/a$ $(b \neq a)$ for some $k > 0$ imply $n/b \in I$. An *ideal generated by an element* m/a in M is defined by

$$\langle m/a \rangle = \{m/a\} \cup \{n/b \in M : n/b\ R\ k/a \text{ for some } k > 0 \text{ and } b \neq a\}.$$

More precisely, n must be equal to $C_M(b)$ in the light of the discussion followed by Definition 7.

An *ideal generated by a submset* S of M is defined by $\langle S \rangle = \bigcup_{m/a \in S} \langle m/a \rangle$. By $\mathcal{I}(\mathbb{P})$ (resp. $\mathcal{I}^r(\mathbb{P})$) we mean the set of all ideals of $\mathbb{P}$ (resp. of cardinality r).

Remark 1 In the definition of an ideal I of $\mathbb{P}$, n/b is not a maximal element and the count of b in I is same as that in M. Hence, in an ideal of $\mathbb{P} = (M, R)$, non-maximal elements have full count with respect to M.

Example 4 Consider $S = \{1/1, 2/3\} \subseteq M$ from Example 3. Now $\langle S \rangle = \langle 1/1 \rangle \cup \langle 2/3 \rangle = \{1/1, 3/2, 2/3\}$ as $3/2\ R\ 3/1$ implies $3/2\ R\ 1/1$. In this ideal, 1/1 and 2/3 are maximal elements; 3/2 is not a maximal element and so, the count of 2 is same as that in M.

Based on the above definitions, the following results are straightforward consequences:

Proposition 1 *Let $M \in [X]^n$ be an mset defined over X and $\mathbb{P} = (M, R)$ be a pomset. If A and B are any two order ideals in $\mathbb{P}$, then the following holds:*

(a) *$A \cap B$ is an ideal.*
(b) *$A \cup B$ is an ideal.*
(c) *$A \oplus B$ is an ideal if M is a regular mset with height n.*
(d) *$A \oplus A = A$ if A is a submset of a regular mset M with height n such that $C_A(a) = C_M(a)\ \forall a \in A^*$.*

Proposition 2 *Let $M \in [X]^n$ be an mset defined over X and $\mathbb{P} = (M, R)$ be a pomset. If A and B are any two submets of M, then the following holds:*

(a) *$\langle A \cap B \rangle \subseteq \langle A \rangle \cap \langle B \rangle$.*
(b) *$\langle A \cup B \rangle = \langle A \rangle \cup \langle B \rangle$.*
(c) *$\langle A \oplus B \rangle \subseteq \langle A \rangle \oplus \langle B \rangle$ if M is a regular mset with height n.*
(d) *$\langle A \rangle \Delta \langle B \rangle \subseteq \langle A \Delta B \rangle \subseteq \langle A \cup B \rangle$.*

The following propositions are straightforward and show that every pomset has a good stock of order ideals.

Proposition 3 *Let $\mathbb{P} = (M, R)$ be a pomset. Let $0 \leq s \leq r \leq |M|$ and $I \in \mathcal{I}^r(\mathbb{P})$. Then there exists $J \in \mathcal{I}^s(\mathbb{P})$ such that $J \subseteq I$.*

Proposition 4 *Let $\mathbb{P} = (M, R)$ be a pomset. Let $0 \leq r \leq s \leq |M|$ and $I \in \mathcal{I}^r(\mathbb{P})$. Then there exists $J \in \mathcal{I}^s(\mathbb{P})$ such that $I \subseteq J$.*

2.2 Pomset metric on $\mathbb{Z}_m^n$

Consider the space $\mathbb{Z}_m^n$ and the set $X = \{1, 2, \ldots, n\}$. Consider a regular mset M of height $\lfloor \frac{m}{2} \rfloor$ drawn from X, i.e., $M = \{\lfloor \frac{m}{2} \rfloor/1, \lfloor \frac{m}{2} \rfloor/2, \lfloor \frac{m}{2} \rfloor/3, \ldots, \lfloor \frac{m}{2} \rfloor/n\} \in [X]^{\lfloor \frac{m}{2} \rfloor}$. Let $\mathbb{P} = (M, R)$ be a pomset. Let $x = (x_1, x_2, \ldots, x_n)$ be an n tuple in $\mathbb{Z}_m^n$. We define the support of x with respect to Lee weight as

$$supp_L(x) = \{k/i \mid k = w_L(x_i), k \neq 0\}$$

where $w_L(x_i) = \min\{x_i, m - x_i\}$ is the Lee weight of x_i in $\mathbb{Z}_m$.

We define the *pomset weight* of x to be the cardinality of the ideal generated by $supp_L(x)$, that is,

$$w_{Pm}(x) = |\langle supp_L(x)\rangle|.$$

The *pomset distance* between two vectors x, y in $\mathbb{Z}_m^n$ is defined as

$$d_{Pm}(x, y) = w_{Pm}(x - y).$$

The pomset weight of a vector depends on the non-zero coordinate positions, elements in those positions and the pomset structure that is considered. If the pomset is an antichain, then the pomset weight and pomset distance are Lee weight and Lee distance respectively. Here, $|supp_L(x)^*|$ is the Hamming weight of x and $|\langle supp_L(x)\rangle^*|$ is the poset weight of x.

Now we prove that the above pomset distance is indeed a metric on $\mathbb{Z}_m^n$.

Theorem 1 *If $\mathbb{P}$ is a pomset on a regular mset $M = \{\lfloor \frac{m}{2} \rfloor/1, \lfloor \frac{m}{2} \rfloor/2, \ldots, \lfloor \frac{m}{2} \rfloor/n\}$, then the pomset distance $d_{Pm}(.,.)$ is a metric on $\mathbb{Z}_m^n$.*

Proof Clearly $d_{Pm}(u, v) \geq 0$, and $d_{Pm}(u, v) = 0$ iff $u = v$. Let $u, v \in \mathbb{Z}_m^n$. As $w_L(a) = w_L(-a)$ for any $a \in \mathbb{Z}_m$, $supp_L(u-v) = supp_L(v-u)$. Hence $w_{Pm}(u-v) = w_{Pm}(v-u)$. Thus $d_{Pm}(u, v) = d_{Pm}(v, u)$. As $d_{Pm}(u, v) = w_{Pm}(u - v) = w_{Pm}(u - w + w - v)$, to prove the triangle inequality, it suffices to show that the pomset weight satisfies the inequality $w_{Pm}(x + y) \leq w_{Pm}(x) + w_{Pm}(y)$ for all $x, y \in \mathbb{Z}_m^n$. Clearly $supp_L(x + y) \subseteq supp_L(x) \oplus supp_L(y)$. Since $\langle supp_L(x+y)\rangle \subseteq \langle supp_L(x) \oplus supp_L(y)\rangle$, from Proposition 2 (c), we have $w_{Pm}(x + y) \leq |\langle supp_L(x) \oplus supp_L(y)\rangle| \leq |\langle supp_L(x)\rangle \oplus \langle supp_L(y)\rangle| \leq |\langle supp_L(x)\rangle| + |\langle supp_L(y)\rangle|$. □

We call the metric $d_{Pm}(.,.)$ on $\mathbb{Z}_m^n$ as *pomset metric*. If $\mathbb{Z}_m^n$ is endowed with a pomset metric, then we call a subset $\mathcal{C}$ of $\mathbb{Z}_m^n$ a *pomset code* of length n. A *linear pomset code* $\mathcal{C}$ of length n is a submodule of $\mathbb{Z}_m^n$. If the pomset metric corresponds to a pomset $\mathbb{P}$, then $\mathcal{C}$ is called a $\mathbb{P}$-code. *Minimum pomset distance* $d_{Pm}(\mathcal{C})$ of a $\mathbb{P}$-code $\mathcal{C}$ is the smallest pomset distance between distinct codewords of $\mathcal{C}$. We denote pomset code $\mathcal{C}$ of length n, cardinality K and minimum distance $d_{Pm}(\mathcal{C})$ by (n, K, d_{Pm}).

Example 5 Let $\mathcal{C} = \{(0, 0, 0, 0), (1, 3, 0, 2), (2, 0, 0, 4), (3, 3, 0, 0), (4, 0, 0, 2), (5, 3, 0, 4)\} \subset \mathbb{Z}_6^4$ be a $\mathbb{P}$-code for the pomset $\mathbb{P}$ given in Example 3. The support of $u = (5, 3, 0, 4)$ with respect to Lee weight is $supp_L(u) = \{1/1, 3/2, 2/4\}$ and $\langle supp_L(u)\rangle = \{1/1, 3/2, 3/3, 2/4\}$. Thus, $w_{Pm}(u) = 9$, $d_{Pm} = 6$.

Let u be a vector in $\mathbb{Z}_m^n$ and r be a non-negative integer. The *pomset ball* with center u and radius r is the set

$$B_r(u) = \{v \in \mathbb{Z}_m^n : d_{Pm}(u, v) \leq r\}$$

of all vectors in $\mathbb{Z}_m^n$ whose pomset distance to u is less than or equal to r.

In Sects. 4 and 5, it is necessary to specify the parameters of code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ with respect to Hamming and RT metrics as well. Note that the Hamming weight of x, $w_H(x)$, is the number of non-zero coordinate positions in x and the RT weight of x, $w_\rho(x)$, is the maximum coordinate position that is non-zero in x. $d_H(x, y) = w_H(x - y)$ and $d_\rho(x, y) = w_\rho(x - y)$ are the Hamming and RT distances between two vectors x and y. We use the notations $d_H(\mathcal{C})$ and $d_\rho(\mathcal{C})$ to denote the minimum Hamming distance and the minimum RT distance of $\mathcal{C}$ respectively and $D_H(\mathcal{C})$ and $D_\rho(\mathcal{C})$ to denote the maximum Hamming distance and the maximum RT distance of $\mathcal{C}$ respectively. The maximum weight of a code $\mathcal{C}$, denoted by $W(\mathcal{C})$, is the maximum of weights of all codewords of $\mathcal{C}$. The maximum distance of a code $\mathcal{C}$, denoted by $D(\mathcal{C})$, is the greatest distance between codewords of $\mathcal{C}$. For linear codes, $D(\mathcal{C})$ is the same as $W(\mathcal{C})$.

3 Construction of pomsets

For a given pomset $\mathbb{P} = (M, R)$, we define a pomset $\widetilde{\mathbb{P}} = (M, \widetilde{R})$ as follows:

$$\mathbb{P} \text{ and } \widetilde{\mathbb{P}} \text{ have the same underlying mset}$$

and

$$m/a \ \widetilde{R}\ n/b \text{ in } \widetilde{\mathbb{P}} \text{ if and only if } n/b \ R \ m/a \text{ in } \mathbb{P}.$$

The pomset $\widetilde{\mathbb{P}}$ is called the *dual pomset* of $\mathbb{P}$. If $\mathbb{P}$ is a chain or an antichain then $\widetilde{\mathbb{P}}$ is also a chain or an antichain respectively. Moreover, it is obvious to see that the order ideals of $\widetilde{\mathbb{P}}$ are precisely the complements of the order ideals of $\mathbb{P}$, i.e., $\mathcal{I}(\widetilde{\mathbb{P}}) = \{I^c | I \in \mathcal{I}(\mathbb{P})\}$.

Remark 2 For all practical purposes and foregoing discussions, whenever we consider a chain pomset $\mathbb{P} = (M, R)$, we regard the elements of M in such a manner that $p/i \ R \ q/j$ for $i < j$.

Pomsets beget pomsets Given any two pomsets, one can construct a new pomset by what we call as direct sum, ordinal sum, direct product and ordinal product of pomsets. In what follows, we describe how we achieve them.

(a) *Direct sum of pomsets* Let $\mathbb{P}_1 = (M_1, R_1)$ and $\mathbb{P}_2 = (M_2, R_2)$ be two pomsets with $M_1^* = [n_1]$ and $M_2^* = [n_2]$ respectively. Now consider an mset M with $M^* = [n_1 + n_2]$ and

$$C_M(i) = \begin{cases} C_{M_1}(i) & \text{if } i \leq n_1, \\ C_{M_2}(i - n_1) & \text{if } i > n_1. \end{cases}$$

Define an mset relation R on M in the following way. Given $p/i, q/j \in M$, we say

$$p/i \ R \ q/j \iff \begin{cases} i, j \leq n_1 & \text{and } \ p/i \ R_1 \ q/j \quad \text{or} \\ i, j > n_1 & \text{and } \ p/(i - n_1) \ R_2 \ q/(j - n_1). \end{cases}$$

We can easily see that $\mathbb{P} = (M, R)$ is a pomset and term it as the *direct sum* of $\mathbb{P}_1$ and $\mathbb{P}_2$ denoted by $\mathbb{P}_1 \oplus \mathbb{P}_2$.

If the constituent pomsets $\mathbb{P}_1$ and $\mathbb{P}_2$ are chains then $\mathbb{P}$ is not a chain but it is a disjoint union of two chains of sizes $|M_1|$ and $|M_2|$ respectively. In fact, $\mathbb{P}$ can never be a chain by its construction. But if $\mathbb{P}_1$ and $\mathbb{P}_2$ are antichains then $\mathbb{P}$ is also an antichain.

(b) *Ordinal sum of pomsets* Let $\mathbb{P}_1 = (M_1, R_1)$ and $\mathbb{P}_2 = (M_2, R_2)$ be two pomsets with $M_1^* = [n_1]$ and $M_2^* = [n_2]$ respectively. Now consider an mset M with $M^* = [n_1 + n_2]$ and

$$C_M(i) = \begin{cases} C_{M_1}(i) & \text{if } i \leq n_1, \\ C_{M_2}(i - n_1) & \text{if } i > n_1. \end{cases}$$

Define an mset relation R on M in the following way. Given $p/i, q/j \in M$, we say

$$p/i \ R \ q/j \iff \begin{cases} i, j \le n_1 & \text{and } \ p/i \ R_1 \ q/j \ \text{ or} \\ i, j > n_1 & \text{and } \ p/(i - n_1) \ R_2 \ q/(j - n_1) \ \text{ or} \\ i \le n_1 < j. \end{cases}$$

Clearly, $\mathbb{P} = (M, R)$ is a pomset and we term it as the *ordinal sum* of $\mathbb{P}_1$ and $\mathbb{P}_2$ denoted by $\mathbb{P}_1 + \mathbb{P}_2$.

From this construction, we can observe that $\mathbb{P}$ can never be an antichain. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are chains then $\mathbb{P}$ must be a chain.

(c) *Direct product of pomsets* Let $\mathbb{P}_1 = (M_1, R_1)$ and $\mathbb{P}_2 = (M_2, R_2)$ be two pomsets with $M_1^* = [n_1]$ and $M_2^* = [n_2]$ respectively. Now consider an mset M as $M_1 \times M_2$. Given $k_1/(p/i_1, \ q/j_1), k_2/(r/i_2, \ s/j_2) \in M_1 \times M_2$, define an mset relation R on M as:

$$k_1/(p/i_1, \ q/j_1) \ R \ k_2/(r/i_2, \ s/j_2) \iff p/i_1 \ R_1 \ r/i_2 \text{ and } q/j_1 \ R_2 \ s/j_2.$$

One can easily show that $\mathbb{P} = (M, R)$ is a pomset and is called as the *direct product* of $\mathbb{P}_1$ and $\mathbb{P}_2$ denoted by $\mathbb{P}_1 \otimes \mathbb{P}_2$.

(d) *Ordinal product of pomsets* Let $\mathbb{P}_1 = (M_1, R_1)$ and $\mathbb{P}_2 = (M_2, R_2)$ be two pomsets with $M_1^* = [n_1]$ and $M_2^* = [n_2]$ respectively. Now consider an mset M as $M_1 \times M_2$. Given $k_1/(p/i_1, \ q/j_1), k_2/(r/i_2, \ s/j_2) \in M_1 \times M_2$, define an mset relation R on M as:

$$k_1/(p/i_1, \ q/j_1) \ R \ k_2/(r/i_2, \ s/j_2) \iff \begin{cases} i_1 = i_2 \ \text{ and } \ q/j_1 \ R_2 \ s/j_2 \ \text{ or} \\ p/i_1 \ R_1 \ r/i_2 \ \text{ where } \ i_1 \ne i_2. \end{cases}$$

Similarly, it is easy to show that $\mathbb{P} = (M, R)$ is a pomset and is called as the *ordinal product* of $\mathbb{P}_1$ and $\mathbb{P}_2$ denoted by $\mathbb{P}_1 \times \mathbb{P}_2$.

Observe that, if S is a submset of M, then R is a pomset relation on S.

If $\mathbb{P}$ is either the direct or the ordinal product of $\mathbb{P}_1$ and $\mathbb{P}_2$, its structure depends up on the constituent pomsets. By considering $\mathbb{P}_1$ and $\mathbb{P}_2$ as combinations of chain and antichain, for example, and by representing M by $n_1 \times n_2$ matrix, we shall analyse the structure of $\mathbb{P}$, as follows:

Let $k_1/a, k_2/b \in M$ where $a = (p/i_1, \ q/j_1)$ and $b = (r/i_2, \ s/j_2)$. Consider $\mathbb{P}_1$ to be an antichain. If $\mathbb{P}_2$ is also an antichain then, for both $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$ and $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$, the elements $k_1/a, k_2/b$ are not comparable in $\mathbb{P}$ unless $i_1 = i_2$ and $j_1 = j_2$ as p/i_1 (q/j_1) and r/i_2 (s/j_2) are not comparable in $\mathbb{P}_1$ $(\mathbb{P}_2)$ for $i_1 \ne i_2$ $(j_1 \ne j_2)$. If $\mathbb{P}_2$ is a chain then q/j_1 and s/j_2 are comparable for any j_1, j_2. Moreover, p/i_1 and r/i_2 are comparable for $i_1 = i_2$ but not for $i_1 \ne i_2$. Hence, any two elements in M are comparable only if they are from the same row. This is true for both direct and ordinal product of pomsets.

Now, consider $\mathbb{P}_1$ to be a chain. Let $\mathbb{P}_2$ be taken as an antichain. Since the roles of $\mathbb{P}_1$ and $\mathbb{P}_2$ are interchanged when compared to the previous case, $\mathbb{P}_1 \otimes \mathbb{P}_2$ must be a disjoint union of n_2 chains. When $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$, it is obvious to see that each row in M is an antichain and each column is a chain. Moreover, if $k_1'/c \in M$ where $c = (p'/i, \ q'/j)$, then for any $k_2'/d \in M$ with $d = (r'/k, \ s'/l)$ such that $k \ne i$, p'/i and r'/k are comparable with respect to $\mathbb{P}_1$ and thus, k_1'/c and k_2'/d are comparable.

Now, letting $\mathbb{P}_2$ also to be a chain, each row and each column in M will be a chain when $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$. Moreover, let $k_1'/c \in M$ where $c = (p'/i, \ q'/j)$. Then, for any element $k_2'/d \in M$ where $d = (r'/k, \ s'/l)$ in the (k, l)-cell, k_1'/c and k_2'/d are not comparable when $k > i$ but $l < j$; they are not comparable when $k < i$ but $l > j$ too. This ensures that each element in the cells (k, l) is related to the element in the cell (i, j) if $k \le i$ and $l \le j$; and the element in the (i, j) cell is related to each element in the cells (k, l) if $k \ge i$ and $l \ge j$. Now, in the case of the ordinal product, any two elements in M are comparable by definition.

Table 1 Product of pomsets

$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_1 \otimes \mathbb{P}_2$	$\mathbb{P}_1 \times \mathbb{P}_2$
Antichain	Antichain	Antichain	Antichain
	Chain	Disjoint union of n_1 chains	Disjoint union of n_1 chains
Chain	Antichain	Disjoint union of n_2 chains	$\star_1$
	Chain	$\star_2$	Chain

$\star_1$ Each row is an antichain and each column is a chain such that the element in the (i, j) cell is comparable with all the elements in the cells (k, l) such that $k \neq i$

$\star_2$ Each row and each column is a chain such that the element in the (i, j) cell is not comparable with any of the elements in the cells (k, l) for which $k > i$ but $l < j$ and $k < i$ but $l > j$

We summarize the above results as Table 1.

Example 6 Let M_1 be a regular mset with $M_1^* = [2] = \{1, 2\}$ and height 2 and M_2 be a regular mset with $M_2^* = [3] = \{1', 2', 3'\}$ and height 2. Define $R_1 = \{2/(2/i,\ 2/i)\}_{i \in M_1^*} \cup \{2/(2/1,\ 2/2)\}$ and $R_2 = \{2/(2/i,\ 2/i)\}_{i \in M_2^*}$ as pomset relations on M_1 and M_2 respectively. Here, $\mathbb{P}_1 = (M_1, R_1)$ is a chain and $\mathbb{P}_2 = (M_2, R_2)$ is an antichain. Consider a submset $M \subseteq M_1 \times M_2$ given in matrix representation as follows:

$$M = \begin{pmatrix} 2/(2/1,\ 2/1') & 2/(2/1,\ 2/2') & 2/(2/1,\ 2/3') \\ 2/(2/2,\ 2/1') & 2/(2/2,\ 2/2') & 2/(2/2,\ 2/3') \end{pmatrix} = \begin{pmatrix} 2/1 & 2/2 & 2/3 \\ 2/4 & 2/5 & 2/6 \end{pmatrix}, \text{ say.}$$

If $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$ on M, then one can easily see that it is a disjoint union of 3 chains (which are columns of M).

Given any pomset $\mathbb{P}$, one can puncture it or extend it to obtain another pomset as described below.

(e) *Puncturing pomsets* Let $\mathbb{P} = (M_1, R_1)$ be a pomset with $M_1^* = [n]$. Now construct a new mset M by deleting the i^{th} element from M_1. Consider M such that $M^* = [n-1]$ and

$$C_M(j) = \begin{cases} C_{M_1}(j) & \text{for } j < i, \\ C_{M_1}(j+1) & \text{for } j \geq i. \end{cases}$$

Given $p/j, q/l \in M$, define an mset relation R on M as follows:

$$p/j\ R\ q/l \iff \begin{cases} j, l < i & \text{and } p/j\ R_1\ q/l \quad \text{or} \\ j, l \geq i & \text{and } p/(j+1)\ R_1\ q/(l+1) \quad \text{or} \\ j < i, l \geq i & \text{and } p/j\ R_1\ q/(l+1) \quad \text{or} \\ j \geq i, l < i & \text{and } p/(j+1)\ R_1\ q/l. \end{cases}$$

$\mathbb{P}^{\circ} = (M, R)$ is a pomset. If $\mathbb{P}$ is a chain (antichain) then $\mathbb{P}^{\circ}$ is also a chain (antichain).

(f) *Extending pomsets* Let $\mathbb{P} = (M_1, R_1)$ be a pomset with $M_1^* = [n]$. Now construct a new mset M with $M^* = [n+1]$ from M_1 by inserting an element with count $k > 0$ either in the beginning or at the end or in the interior of M_1. If the new element of M is k/i, then the count function on M is defined as

$$C_M(j) = \begin{cases} C_{M_1}(j) & \text{for } j < i, \\ k > 0 & \text{for } j = i, \\ C_{M_1}(j-1) & \text{for } j > i. \end{cases}$$

Define an mset relation R on M in the following way. Given $p/j, q/l \in M$, we say

$$p/j\ R\ q/l \Longleftrightarrow \begin{cases} j, l < i & \text{and } p/j\ R_1\ q/l \ \text{ or} \\ j, l > i & \text{and } p/(j-1)\ R_1\ q/(l-1) \ \text{ or} \\ j < i, l > i & \text{and } p/j\ R_1\ q/(l-1) \ \text{ or} \\ j > i, l < i & \text{and } p/(j-1)\ R_1\ q/l \ \text{ or} \\ j = l = i. & \end{cases}$$

It is easy to see that $\widehat{\mathbb{P}} = (M, R)$ is a pomset.

4 Construction of codes in pomset metric

In this section, we combine any two $\mathbb{P}$-codes by direct sum construction, $(u|u+v)$-construction and as product codes. Later, discussion on codes arrived through puncturing and extension is given. The pomset structure that could be imposed on the resultant codes will have its effect on the minimum distance and covering radius.

(A) *Direct sum of codes* For $i \in \{1, 2\}$, let $\mathcal{C}_i$ be an $(n_i, K_i, d_{P_i m})$ $\mathbb{P}_i$-code over the ring $\mathbb{Z}_m$. Then the *direct sum* of $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined as

$$\mathcal{C}_1 \oplus \mathcal{C}_2 = \{(u, v) | u \in \mathcal{C}_1, v \in \mathcal{C}_2\}$$

and is a pomset code of length $n_1 + n_2$ and cardinality $K_1 K_2$.

Proposition 5 *Let $\mathcal{C}_1$ be an $(n_1, K_1, d_{P_1 m})$ $\mathbb{P}_1$-code and $\mathcal{C}_2$ be an $(n_2, K_2, d_{P_2 m})$ $\mathbb{P}_2$-code both over the ring $\mathbb{Z}_m$. Then their direct sum $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$ is an $(n_1 + n_2, K_1 K_2, d_{Pm})$ $\mathbb{P}$-code for some pomset $\mathbb{P}$. If $\mathbb{P}$ is a direct sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then $d_{Pm} = \min\{d_{P_1 m}, d_{P_2 m}\}$. If $\mathbb{P}$ is an ordinal sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then $d_{Pm} = d_{P_1 m}$.*

Proof Let $x, y \in \mathcal{C}$. Then $x = (u, v)$ and $y = (u', v')$ for some $u, u' \in \mathcal{C}_1$ and $v, v' \in \mathcal{C}_2$. If $\mathbb{P}$ is a direct sum of $\mathbb{P}_1$ and $\mathbb{P}_2$, then $d_{Pm}(x, y) = w_{Pm}(x - y) = w_{P_1 m}(u - u') + w_{P_2 m}(v - v')$ and it is $w_{P_1 m}(u - u')$ when $v = v'$ or $w_{P_2 m}(v - v')$ when $u = u'$. Thus, $d_{Pm} = \min\{d_{P_1 m}, d_{P_2 m}\}$. If $\mathbb{P}$ is an ordinal sum, then $w_{Pm}(x - y) = n_1 \lfloor \frac{m}{2} \rfloor + d_{P_2 m}$ when $v \neq v'$ and hence $d_{Pm} = d_{P_1 m}$. □

(B) *$(u|u+v)$-construction of codes* For $i \in \{1, 2\}$, let $\mathcal{C}_i$ be an $(n, K_i, d_{P_i m})$ $\mathbb{P}_i$-code over the ring $\mathbb{Z}_m$. The $(u|u+v)$-construction produces the code

$$\mathcal{C} = \{(u, u + v) | u \in \mathcal{C}_1, v \in \mathcal{C}_2\}$$

of length $2n$ and cardinality $K_1 K_2$.

Proposition 6 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two $\mathbb{P}_1$- and $\mathbb{P}_2$-codes of same length with parameters $(n, K_1, d_{P_1 m})$ and $(n, K_2, d_{P_2 m})$ respectively over the ring $\mathbb{Z}_m$. Then $(u|u+v)$-construction produces a $(2n, K_1 K_2, d_{Pm})$ $\mathbb{P}$-code for some pomset $\mathbb{P}$. If $\mathbb{P}$ is a direct sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then either $d_{Pm} \geq \min\{d_{P_1 m}, d_{P_2 m}\}$ or $d_{Pm} \geq \min\{d_{P_2 m}, d_{P_1 m} + d_3, d_{P_1 m} + d_4\}$, whereas, if $\mathbb{P}$ is an ordinal sum then either $d_{Pm} \geq d_{P_1 m}$ or $d_{Pm} = n\lfloor \frac{m}{2} \rfloor + \min\{d_{P_2 m}, d_3, d_4\}$. Here, d_3 and d_4 are minimum $\mathbb{P}_2$-distances respectively of the codes $\mathcal{C}_1$ and $\{u + v | u \in \mathcal{C}_1, v \in \mathcal{C}_2\}$.*

Proof Let $x = (u, u + v)$, $y = (u', u' + v') \in \mathcal{C}$ be such that $x \neq y$ for some $u, u' \in \mathcal{C}_1$ and $v, v' \in \mathcal{C}_2$. Then $x - y = (u - u', u - u' + v - v')$. If $\mathbb{P}$ is a a direct sum of $\mathbb{P}_1$ and $\mathbb{P}_2$, then $d_{Pm}(x, y) = w_{P_1 m}(u - u') + w_{P_2 m}(u - u' + v - v')$. Thus, $d_{Pm}(x, y) = d_{P_2 m}$ when $u = u'$,

$d_{Pm}(x, y) \geq d_{P_1m} + d_3$ when $v = v'$ and $d_{Pm}(x, y) \geq d_{P_1m} + d_4$ when $u - u' + v - v' \neq 0$. But, only if $u - u' + v - v' = 0$ $d_{Pm}(x, y) \geq d_{P_1m}$. As the last case is not guaranteed, either $d_{Pm} \geq \min\{d_{P_1m}, d_{P_2m}\}$ or $d_{Pm} \geq \min\{d_{P_2m}, d_{P_1m} + d_3, d_{P_1m} + d_4\}$. If $\mathbb{P}$ is an ordinal sum of $\mathbb{P}_1$ and $\mathbb{P}_2$, then $d_{Pm}(x, y) = n\lfloor \frac{m}{2} \rfloor + d_{P_2m}$ when $u = u'$, $d_{Pm}(x, y) = n\lfloor \frac{m}{2} \rfloor + d_3$ when $v = v'$ and $d_{Pm}(x, y) = n\lfloor \frac{m}{2} \rfloor + d_4$ when $u - u' + v - v' \neq 0$. But $d_{Pm}(x, y) \geq d_{P_1m}$ only if $u - u' + v - v' = 0$. Thus, either $d_{Pm} \geq d_{P_1m}$ or $d_{Pm} = n\lfloor \frac{m}{2} \rfloor + \min\{d_{P_2m}, d_3, d_4\}$. □

(C) *Product codes* For $i \in \{1, 2\}$, let $\mathcal{C}_i$ be an (n_i, K_i, d_{P_im}) $\mathbb{P}_i$-code over the ring $\mathbb{Z}_m$. The *product* of $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined as

$$\mathcal{C}_1 \bigotimes \mathcal{C}_2 = \{\underline{\mathbf{c}} = u \otimes v | u \in \mathcal{C}_1, v \in \mathcal{C}_2\}$$

where $u \otimes v = (u_i v_j | 1 \leq i \leq n_1, 1 \leq j \leq n_2)$ and is a code of length $n_1 n_2$. The codewords of *product code* can be represented by $n_1 \times n_2$ matrices: if $u = (u_1, u_2, \ldots, u_{n_1})$ and $v = (v_1, v_2, \ldots, v_{n_2})$ then

$$u \otimes v = \begin{pmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_{n_2} \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_{n_2} \\ \vdots & \vdots & \cdots & \vdots \\ u_{n_1} v_1 & u_{n_1} v_2 & \cdots & u_{n_1} v_{n_2} \end{pmatrix}.$$

Note that, if one of the constituent code is $\{0\}$ then $\mathcal{C}$ is also $\{0\}$. In this paper, we consider only the codes $\mathcal{C}_i \neq \{0\}$ for $i = 1, 2$. Observe also that $\mathcal{C}_2 \bigotimes \mathcal{C}_1 = \{\underline{\mathbf{c}}^T : \underline{\mathbf{c}} \in \mathcal{C}_1 \bigotimes \mathcal{C}_2\}$. Moreover, $\mathcal{C} = \mathcal{C}_1 \bigotimes \mathcal{C}_2$ is a $\mathbb{P}$-code for some pomset $\mathbb{P}$.

Example 7 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two codes over the field $\mathbb{Z}_5$ generated by matrices $G_1 = [2\ 3]$ and $G_2 = [1\ 0\ 2]$ respectively. Consider $\mathcal{C}_1$ and $\mathcal{C}_2$ with respect to the pomsets $\mathbb{P}_1$ and $\mathbb{P}_2$ given in Example 4 respectively. Then

$$\mathcal{C} = \mathcal{C}_1 \bigotimes \mathcal{C}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 4 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 4 \\ 3 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 3 \\ 1 & 0 & 2 \end{pmatrix} \right\}$$

is a $\mathbb{P}$-code. Let $c = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & 4 \end{pmatrix}$. Then $supp_L(c) = \{2/1, 1/3, 2/4, 1/6\}$. If $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$ then $\langle supp_L(c) \rangle = \{2/1, 2/3, 2/4, 1/6\}$ and $w_{Pm}(c) = 7$. If $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$ then $\langle supp_L(c) \rangle = \{2/1, 2/2, 2/3, 2/4, 1/6\}$ and $w_{Pm}(c) = 9$.

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be pomsets on $M_1 = \{\lfloor \frac{m}{2} \rfloor/1, \lfloor \frac{m}{2} \rfloor/2, \ldots, \lfloor \frac{m}{2} \rfloor/n_1\}$ and $M_2 = \{\lfloor \frac{m}{2} \rfloor/1, \lfloor \frac{m}{2} \rfloor/2, \ldots, \lfloor \frac{m}{2} \rfloor/n_2\}$ respectively. Let M be a regular submset of $M_1 \times M_2$ of height $\lfloor \frac{m}{2} \rfloor$ such that $M^* = (M_1 \times M_2)^*$. Then the mset relation R defined as in direct product and ordinal product in Sect. 3 is a pomset relation on M.

Let's first consider the case where $\mathbb{P}$ is a direct product of $\mathbb{P}_1$ and $\mathbb{P}_2$ on M. To illustrate, if we take $\mathbb{P}_1$ to be a chain and $\mathbb{P}_2$ as an antichain then by Table 1, $\mathbb{P}$ is a disjoint union of n_2 chains. Thus the ideal generated by an element in the (i, j) cell is the union of that element and the mset of the elements in the cells (k, l) where $k < i$ and $l = j$. Similarly, if $\mathbb{P}_1$ and $\mathbb{P}_2$ both are chains, the ideal generated by an element in the (i, j) cell is the union of that element and the mset of the elements in the cells (k, l) in M where $k < i$ and $l < j$; $k = i$ and $l < j$; $k < i$ and $l = j$.

With similar arguments, we established bounds for minimum distance of product codes in the following proposition:

Proposition 7 *Let $\mathcal{C}_1 \subseteq \mathbb{Z}_m^{n_1}$ and $\mathcal{C}_2 \subseteq \mathbb{Z}_m^{n_2}$ be two linear $\mathbb{P}_1$- and $\mathbb{P}_2$-codes of minimum distance $d_{P_1 m}(\mathcal{C}_1)$ and $d_{P_2 m}(\mathcal{C}_2)$ respectively (where m is prime). Then the product code $\mathcal{C} = \mathcal{C}_1 \bigotimes \mathcal{C}_2$ is a $\mathbb{P}$-code for some pomset $\mathbb{P}$, with minimum distance $d_{Pm}(\mathcal{C})$. If $\mathbb{P}$ is a direct product of $\mathbb{P}_1$ and $\mathbb{P}_2$ then the following hold:*

(a) *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are antichains then $d_H(\mathcal{C}_1)d_H(\mathcal{C}_2) \leq d_{Pm}(\mathcal{C}) \leq d_H(\mathcal{C}_1)d_H(\mathcal{C}_2)\lfloor \frac{m}{2} \rfloor$.*
(b) *If $\mathbb{P}_1$ is an antichain and $\mathbb{P}_2$ is a chain then $d_H(\mathcal{C}_1)(d_\rho(\mathcal{C}_2) - 1)\lfloor \frac{m}{2} \rfloor + d_{P_1 m}(\mathcal{C}_1) \leq d_{Pm}(\mathcal{C}) \leq d_H(\mathcal{C}_1)d_\rho(\mathcal{C}_2)\lfloor \frac{m}{2} \rfloor$.*
(c) *If $\mathbb{P}_1$ is a chain and $\mathbb{P}_2$ is an antichain then $d_H(\mathcal{C}_2)(d_\rho(\mathcal{C}_1) - 1)\lfloor \frac{m}{2} \rfloor + d_{P_2 m}(\mathcal{C}_2) \leq d_{Pm}(\mathcal{C}) \leq d_\rho(\mathcal{C}_1)d_H(\mathcal{C}_2)\lfloor \frac{m}{2} \rfloor$.*
(d) *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are chains then $d_{Pm}(\mathcal{C}) = (d_{P_1 m}(\mathcal{C}_1) - 1)d_\rho(\mathcal{C}_2) + d_{P_2 m}(\mathcal{C}_2)$.*

Proof Let $\underline{\mathbf{c}} \in \mathcal{C}$. Then $\underline{\mathbf{c}} = u \otimes v$, $u \in \mathcal{C}_1$ and $v \in \mathcal{C}_2$.

(a) Since $\mathbb{P}$ is an antichain, the proof is obvious.
(b) Since $\mathbb{P}$ is a disjoint union of n_1 chains, to find the pomset weight of $\underline{\mathbf{c}}$, we need the number of non-zero rows in $\underline{\mathbf{c}}$ and RT weight of each such row. From this, the proof follows.
(c) The Table 1 and the discussion preceding this proposition complete the proof.
(d) Now $\mathbb{P}$ is such that each row and each column is a chain. Thus, to find the pomset weight of $\underline{\mathbf{c}}$ in $\mathcal{C}$, we need a cell (i, j) that has non-zero entry such that entries in the cells (k, l) are all zero for $k = i, l > j$ and $k \geq i + 1$ (when $i \neq n_1$), $l \geq 1$. We obtain it by RT weight of constituent codewords of $\underline{\mathbf{c}}$. Hence proved. □

Now, let's consider the case where $\mathbb{P}$ is an ordinal product of $\mathbb{P}_1$ and $\mathbb{P}_2$ on M. If $\mathbb{P}_1$ is a chain and $\mathbb{P}_2$ is an antichain then from the Table 1, the ideal generated by an element in the (i, j) cell is the union of that element and the mset of all the elements in the cells (k, l) in M for which $k < i$ but for every l.

From the Table 1, for some cases, the ordinal product coincides with the direct product. From this and the above discussion, we proved the following proposition:

Proposition 8 *Let $\mathcal{C}_1 \subseteq \mathbb{Z}_m^{n_1}$ and $\mathcal{C}_2 \subseteq \mathbb{Z}_m^{n_2}$ be two linear $\mathbb{P}_1$- and $\mathbb{P}_2$-codes of minimum distance $d_{P_1 m}(\mathcal{C}_1)$ and $d_{P_2 m}(\mathcal{C}_2)$ respectively (where m is prime). Then the product code $\mathcal{C} = \mathcal{C}_1 \bigotimes \mathcal{C}_2$ is a $\mathbb{P}$-code for some pomset $\mathbb{P}$, with minimum distance $d_{Pm}(\mathcal{C})$. If $\mathbb{P}$ is an ordinal product of $\mathbb{P}_1$ and $\mathbb{P}_2$ then the following hold:*

(a) *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are antichains then $d_H(\mathcal{C}_1)d_H(\mathcal{C}_2) \leq d_{Pm}(\mathcal{C}) \leq d_H(\mathcal{C}_1)d_H(\mathcal{C}_2)\lfloor \frac{m}{2} \rfloor$.*
(b) *If $\mathbb{P}_1$ is an antichain and $\mathbb{P}_2$ is a chain then $d_H(\mathcal{C}_1)(d_\rho(\mathcal{C}_2) - 1)\lfloor \frac{m}{2} \rfloor + d_{P_1 m}(\mathcal{C}_1) \leq d_{Pm}(\mathcal{C}) \leq d_H(\mathcal{C}_1)d_\rho(\mathcal{C}_2)\lfloor \frac{m}{2} \rfloor$.*
(c) *If $\mathbb{P}_1$ is a chain and $\mathbb{P}_2$ is an antichain then $d_{Pm}(\mathcal{C}) = (d_{P_1 m}(\mathcal{C}_1) - 1)n_2 + d_{P_2 m}(\mathcal{C}_2)$.*
(d) *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are chains then $d_{Pm}(\mathcal{C}) = (d_{P_1 m}(\mathcal{C}_1) - 1)n_2 + d_{P_2 m}(\mathcal{C}_2)$.*

(D) *Puncturing codes* Let $\mathcal{C}$ be an (n, K, d_{Pm}) $\mathbb{P}$-code over the ring $\mathbb{Z}_m$. We can puncture $\mathcal{C}$ by deleting the coordinate i from each codeword. Puncturing of $\mathcal{C}$ is defined as

$$\mathcal{C}^\circ = \{(u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n) \in \mathbb{Z}_m^{n-1} | (u_1, u_2, \ldots, u_n) \in \mathcal{C}\}$$

and is a code of length $n - 1$ and cardinality K'. If there exists pair of codewords of $\mathcal{C}$ that coincide in all positions except at i^{th} position then $K' < K$; otherwise $K' = K$.

Proposition 9 *Let $\mathcal{C}$ be an (n, K, d_{Pm}) $\mathbb{P}$-code over the ring $\mathbb{Z}_m$. Then the punctured code $\mathcal{C}^\circ$ is an $(n - 1, K', d_{P^\circ m})$ $\mathbb{P}^\circ$-code for some pomset $\mathbb{P}^\circ$. If $\mathbb{P}^\circ$ is a punctured pomset of $\mathbb{P}$ then $d_{P^\circ m} \leq d_{Pm}$.*

Proof Let $\mathcal{C}^\circ$ be obtained by deleting the coordinate i from each codeword of $\mathcal{C}$. Let c_1°, c_2° be any two distinct codewords of $\mathcal{C}^\circ$ obtained by puncturing $c_1, c_2 \in \mathcal{C}$ respectively. Let $u^\circ = c_1^\circ - c_2^\circ$ where $u = c_1 - c_2 = (u_1, u_2, \ldots, u_n)$, say. Suppose that $u_i = 0$. Then $k/i \notin supp_L(u)$ for all $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$. If $\lfloor \frac{m}{2} \rfloor / i$ is related to some element of $supp_L(u)$ then $\lfloor \frac{m}{2} \rfloor / i \in \langle supp_L(u) \rangle$; otherwise $\lfloor \frac{m}{2} \rfloor / i \notin \langle supp_L(u) \rangle$. Thus, $w_{P^\circ m}(u^\circ) = w_{Pm}(u) - \lfloor \frac{m}{2} \rfloor$ or $w_{P^\circ m}(u^\circ) = w_{Pm}(u)$. On the other hand, if $u_i \neq 0$ then $k/i \in supp_L(u)$ where $1 \leq w_L(u_i) = k \leq \lfloor \frac{m}{2} \rfloor$. If k/i is not a maximal element in $\langle supp_L(u) \rangle$, then $\lfloor \frac{m}{2} \rfloor / i \in \langle supp_L(u) \rangle$ and hence, $w_{P^\circ m}(u^\circ) = w_{Pm}(u) - \lfloor \frac{m}{2} \rfloor$. If k/i is a maximal element in $\langle supp_L(u) \rangle$ then $w_{P^\circ m}(u^\circ) \leq w_{Pm}(u) - k < w_{Pm}(u)$. □

(E) *Extending codes* Let $\mathcal{C}$ be an (n, K, d_{Pm}) $\mathbb{P}$-code over the ring $\mathbb{Z}_m$. We can create longer codes by adding a coordinate. Extension of $\mathcal{C}$ is defined as

$$\widehat{\mathcal{C}} = \{(u_1, u_2, \ldots, u_n, u_{n+1}) \in \mathbb{Z}_m^{n+1} | (u_1, u_2, \ldots, u_n) \in \mathcal{C}\}$$

and is a code of length $n+1$. Extension of $\mathcal{C}$ can also be obtained by adding an overall parity check and is given by

$$\widetilde{\mathcal{C}} = \{(u_1, u_2, \ldots, u_n, u_{n+1}) \in \mathbb{Z}_m^{n+1} | (u_1, u_2, \ldots, u_n) \in \mathcal{C} \text{ with } u_1 + u_2 + \cdots + u_{n+1} = 0\}.$$

Proposition 10 *Let $\mathcal{C}$ be an (n, K, d_{Pm}) $\mathbb{P}$-code over the ring $\mathbb{Z}_m$. Then the extended code $\widehat{\mathcal{C}}$ ($\widetilde{\mathcal{C}}$) is an $(n+1, mK, d_{\widehat{P}m})$ $((n+1, K, d_{\widehat{P}m}))$ $\widehat{\mathbb{P}}$-code for some pomset $\widehat{\mathbb{P}}$. If $\widehat{\mathbb{P}}$ is an extended pomset of $\mathbb{P}$ then*

$$d_{\widehat{P}m} = \begin{cases} d_{Pm} & \text{if } \bar{0} \notin \mathcal{C}; \\ 1 & \text{if } \bar{0} \in \mathcal{C}. \end{cases}$$

(In case of $\widetilde{\mathcal{C}}$, $d_{Pm} \leq d_{\widehat{P}m} \leq d_{Pm} + \lfloor \frac{m}{2} \rfloor$).

Proof Let $\widehat{u} \in \widehat{\mathcal{C}}$ be obtained by adding a coordinate $u_{n+1} \in \mathbb{Z}_m$ to $u \in \mathcal{C}$. Then $w_{\widehat{P}m}(\widehat{u}) = w_{Pm}(u) + w_L(u_{n+1})$, $w_{\widehat{P}m}(\widehat{u}) = 1$. If $u = \bar{0}$, choose $u_{n+1} = 1$; otherwise, choose $u_{n+1} = 0$. The result follows. If $\widetilde{u} \in \widetilde{\mathcal{C}}$ is obtained by adding an overall parity u_{n+1} to $u \in \mathcal{C}$, and since $w_{\widehat{P}m}(\widetilde{u}) = w_{Pm}(u) + w_L(u_{n+1})$, it follows that $d_{Pm} \leq d_{\widehat{P}m} \leq d_{Pm} + \lfloor \frac{m}{2} \rfloor$. □

5 Covering radius

This section deals with the study of covering radius of pomset codes constructed in the last section.

Definition 11 Let $\mathcal{C}$ be a pomset code of length n over the ring $\mathbb{Z}_m$. Then the covering radius of $\mathcal{C}$ is the maximum pomset distance of any word in $\mathbb{Z}_m^n$ from the code $\mathcal{C}$. Mathematically it can be expressed as

$$\rho(\mathcal{C}) = \max_{x \in \mathbb{Z}_m^n} \{d_{Pm}(x, \mathcal{C})\} = \max_{x \in \mathbb{Z}_m^n} \{\min\{d_{Pm}(x, c) | c \in \mathcal{C}\}\}$$

so that for each $x \in \mathbb{Z}_m^n$, there exists a $c \in \mathcal{C}$ such that $x \in B_{\rho(\mathcal{C})}(c)$.

If $\mathcal{C}$ is any linear code of length n over $\mathbb{Z}_m$ and u is in $\mathbb{Z}_m^n$, the coset of $\mathcal{C}$ determined by u and denoted by $u + \mathcal{C}$, is $\{u + c : c \in \mathcal{C}\}$. The weight of the coset $u + \mathcal{C}$ is the minimum of weights of all elements in it. We know that in coding theory, a coset leader is a word of minimum weight in any particular coset. Thus, the weight of a coset is the weight of the coset leader in that coset. Moreover, it is well-known that $\rho(\mathcal{C})$ is the largest value of the weights of all cosets of $\mathcal{C}$.

The following result is concerned with the direct sum of linear codes and its coset leaders.

Proposition 11 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two linear pomset codes of length n_1 and n_2 respectively over the ring $\mathbb{Z}_m$. If u and v are coset leaders of some cosets of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, then (u, v) is a coset leader of some coset of the direct sum code $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$ in $\mathbb{Z}_m^{n_1+n_2}$.*

Proof Let $w = (u, v)$. Then $w \in x + \mathcal{C}$ for some $x \in \mathbb{Z}_m^{n_1+n_2}$. Let $y \in x + \mathcal{C}$. Since $\mathcal{C}$ is linear, $y = w + c$ for some $c = (c_1, c_2) \in \mathcal{C}$. Now, $y = (u + c_1, v + c_2)$. If we consider $\mathcal{C}$ with respect to direct sum of pomsets then $w_{Pm}(y) = w_{P_1 m}(u + c_1) + w_{P_2 m}(v + c_2) \geq w_{P_1 m}(u) + w_{P_2 m}(v) = w_{Pm}(w)$. On the other hand, if $\mathcal{C}$ is considered with respect to ordinal sum of pomsets, then the following two cases arises:

Case 1 Suppose $v \neq \bar{0}$. Here $w_{Pm}(w) = n_1 \lfloor \frac{m}{2} \rfloor + w_{P_2 m}(v)$. Since $w_{P_2 m}(v+c) \geq w_{P_2 m}(v)$ for all $c \in \mathcal{C}_2$, $v + c \neq \bar{0}$. Hence, $w_{Pm}(y) = n_1 \lfloor \frac{m}{2} \rfloor + w_{P_2 m}(v + c_2) \geq n_1 \lfloor \frac{m}{2} \rfloor + w_{P_2 m}(v) = w_{Pm}(w)$.

Case 2 Suppose $v = \bar{0}$. Then $w_{Pm}(w) = w_{P_1 m}(u)$. If $v + c_2 \neq \bar{0}$ then $w_{Pm}(y) = n_1 \lfloor \frac{m}{2} \rfloor + w_{P_2 m}(v + c_2) > w_{P_1 m}(u) = w_{Pm}(w)$; otherwise $w_{Pm}(y) = w_{P_1 m}(u + c_1) \geq w_{P_1 m}(u) = w_{Pm}(w)$.

Thus, $w_{Pm}(y) \geq w_{Pm}(w)$ for all $y \in x + \mathcal{C}$. Hence proved. □

Theorem 2 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two linear $\mathbb{P}_1$- and $\mathbb{P}_2$-codes of length n_1 and n_2 respectively over the ring $\mathbb{Z}_m$. Let $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$ be a $\mathbb{P}$-code.*

(a) *If $\mathbb{P}$ is a direct sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then $\rho(\mathcal{C}) = \rho(\mathcal{C}_1) + \rho(\mathcal{C}_2)$.*
(b) *If $\mathbb{P}$ is an ordinal sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then $\rho(\mathcal{C}) = n_1 \lfloor \frac{m}{2} \rfloor + \rho(\mathcal{C}_2)$.*

Proof (a) Let $y = (y_1, y_2) \in \mathbb{Z}_m^{n_1+n_2}$ where $y_1 \in \mathbb{Z}_m^{n_1}$ and $y_2 \in \mathbb{Z}_m^{n_2}$. Then $y_1 = x_1 + c_1$ for some $c_1 \in \mathcal{C}_1$ and $y_2 = x_2 + c_2$ for some $c_2 \in \mathcal{C}_2$ such that x_1 and x_2 are respective coset leaders. Now $y = x + c$ where $x = (x_1, x_2) \in \mathbb{Z}_m^{n_1+n_2}$, $c = (c_1, c_2) \in \mathcal{C}$ so that

$$d_{Pm}(y, c) = w_{Pm}(y - c) = w_{Pm}(x) \leq \rho(\mathcal{C}_1) + \rho(\mathcal{C}_2). \tag{1}$$

Thus, for each vector $y \in \mathbb{Z}_m^{n_1+n_2}$, one can find at least one codeword c in $\mathcal{C}$ such that $d_{Pm}(y, c) \leq \rho(\mathcal{C}_1) + \rho(\mathcal{C}_2)$. If u and v are the coset leaders of largest weight in $\mathbb{Z}_m^{n_1}$ and $\mathbb{Z}_m^{n_2}$ respectively, then the vector $w = (w_1, w_2)$ where $w_1 = u + c_1$, $w_2 = v + c_2$ for some $c_1 \in \mathcal{C}_1$ and $c_2 \in \mathcal{C}_2$ will be such that $d_{Pm}(w, c') = w_{Pm}(w - c') = w_{Pm}((u, v) + (c_1, c_2) - c') \geq w_{Pm}(u, v) = \rho(\mathcal{C}_1) + \rho(\mathcal{C}_2)$ for any $c' \in \mathcal{C}$ (due to Proposition 11). In fact, for this w, the codeword $c = (c_1, c_2)$ is closest at pomset distance $\rho(\mathcal{C}_1) + \rho(\mathcal{C}_2)$. Hence, $\rho(\mathcal{C}) = \rho(\mathcal{C}_1) + \rho(\mathcal{C}_2)$.

(b) If $\mathbb{P}$ is an ordinal sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then inequality (1) becomes $d_{Pm}(y, c) \leq n_1 \lfloor \frac{m}{2} \rfloor + \rho(\mathcal{C}_2)$. Similar arguments like those in (a), will yield the desired result. □

Corollary 1 *If $\mathcal{C}_2 = \mathbb{Z}_m^{n_2}$ then $\rho(\mathcal{C}) = \rho(\mathcal{C}_1)$ irrespective of whether $\mathbb{P}$ being direct sum or ordinal sum of $\mathbb{P}_1$ and $\mathbb{P}_2$.*

Thus, in case of direct sum of codes, we could determine its covering radius in terms of that of the constituent codes. In what follows, for the pomset codes obtained through $(u, u + v)$-construction, puncturing and extension, we will establish upper bounds on their covering radius.

Theorem 3 *For $i \in \{1, 2\}$, let $\mathcal{C}_i$ be a linear $\mathbb{P}_i$-code of length n over the ring $\mathbb{Z}_m$. Let $\mathcal{C} = \{(u, u + v)|u \in \mathcal{C}_1, v \in \mathcal{C}_2\}$ be a $\mathbb{P}$-code.*

(a) If $\mathbb{P}$ is a direct sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then $\rho(\mathcal{C}) \leq \rho(\mathcal{C}_1) + \rho(\mathcal{C}_2)$.
(b) If $\mathbb{P}$ is an ordinal sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then $\rho(\mathcal{C}) \leq n_1\lfloor \frac{m}{2} \rfloor + \rho(\mathcal{C}_2)$.

Proof (a) Let $y = (y_1, y_2) \in \mathbb{Z}_m^{2n}$ where $y_1, y_2 \in \mathbb{Z}_m^n$. Then $y_1 = x_1+c_1$ and $y_2 = x_2+c_2$ for some $c_1, c_2 \in C_1$ such that x_1 and x_2 are coset leaders. Since $\mathcal{C}_1$ is linear, $c_2 - c_1 = c_3 \in \mathcal{C}_1$. Now $y = (y_1, y_2) = (x_1+c_1, x_2+c_2) = (x_1+c_1, x_2+c_1+c_3)$. Let $x_3 = x_2+c_3$. Now x_3 must be in some coset of $\mathcal{C}_2$ with coset leader x_4. So, $x_3 = x_4 + c_4$ for some $c_4 \in \mathcal{C}_2$ and $y = (x_1 + c_1, x_4 + c_4 + c_1) = (x_1, x_4) + (c_1, c_1 + c_4) = x + c, x \in \mathbb{Z}_m^{2n}$ and $c \in \mathcal{C}$. Now $d_{Pm}(y, c) = w_{Pm}(x) = w_{P_1m}(x_1) + w_{P_2m}(x_4) \leq \rho(\mathcal{C}_1) + \rho(\mathcal{C}_2)$. Thus, for each $y \in \mathbb{Z}_m^{2n}$, one can find at least one codeword c in $\mathcal{C}$ such that $d_{Pm}(y, c) \leq \rho(\mathcal{C}_1)+\rho(\mathcal{C}_2)$.
(b) The proof follows easily. □

Corollary 2 *If $\mathcal{C}_1 \subseteq \mathcal{C}_2$ then the following hold:*

(a) *If $\mathbb{P}$ is a direct sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ then $\rho(\mathcal{C}) = \rho(\mathcal{C}_1) + \rho(\mathcal{C}_2)$.*
(b) *Let $\mathbb{P}$ be an ordinal sum of $\mathbb{P}_1$ and $\mathbb{P}_2$. If $\mathcal{C}_2 \neq \mathbb{Z}_m^n$ then $\rho(\mathcal{C}) = n\lfloor \frac{m}{2} \rfloor+\rho(\mathcal{C}_2)$; otherwise, $\rho(\mathcal{C}) = \rho(\mathcal{C}_1)$.*

Theorem 4 *Let $\mathcal{C}^\circ$ be a punctured $\mathbb{P}^\circ$-code of a linear $\mathbb{P}$-code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ where $\mathbb{P}^\circ$ is a punctured pomset of $\mathbb{P}$. Then $\rho(\mathcal{C}^\circ) \leq \rho(\mathcal{C})$.*

Proof Let $y \in \mathbb{Z}_m^n$. y must be in some coset of $\mathcal{C}$ and hence $y = x + c$ for some $c \in \mathcal{C}$ such that x is a coset leader. Now, puncture the vectors x and c on i^{th} coordinate and denote the resultant vectors by x° and c° respectively. Clearly, $x^\circ \in \mathbb{Z}_m^{n-1}$ and $c^\circ \in \mathcal{C}^\circ$. Then $y^\circ = x^\circ + c^\circ \in \mathbb{Z}_m^{n-1}$ and $d_{P^\circ m}(y^\circ, c^\circ) = w_{P^\circ m}(x^\circ) \leq w_{Pm}(x) \leq \rho(\mathcal{C})$. As y and hence y° are arbitrary, $\rho(\mathcal{C}^\circ) \leq \rho(\mathcal{C})$. □

Theorem 5 *Let $\widehat{\mathcal{C}}$ $(\widetilde{\mathcal{C}})$ be an extended $\widehat{\mathbb{P}}$-code of a linear $\mathbb{P}$-code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ where $\widehat{\mathbb{P}}$ is an extended pomset of $\mathbb{P}$. Then $\rho(\widehat{\mathcal{C}}) \leq \rho(\mathcal{C})$. (In case of $\widetilde{\mathcal{C}}$, $\rho(\mathcal{C}) \leq \rho(\widetilde{\mathcal{C}}) \leq \rho(\mathcal{C}) + \lfloor \frac{m}{2} \rfloor$).*

Proof Let $y' = (y, y_{n+1}) \in \mathbb{Z}_m^{n+1}$ where $y \in \mathbb{Z}_m^n$. y must be in some coset of $\mathcal{C}$ with coset leader x. So, $y = x + c$ for some $c \in \mathcal{C}$. $y' = (x + c, y_{n+1}) = (x, 0) + (c, y_{n+1}) = x' + \widehat{c}$, where $x' = (x, 0) \in \mathbb{Z}_m^{n+1}$ and $\widehat{c} = (c, y_{n+1}) \in \widehat{\mathcal{C}}$. $d_{\widehat{P}m}(y', \widehat{c}) = w_{\widehat{P}m}(x') = w_{\widehat{P}m}(x, 0) = w_{Pm}(x) \leq \rho(\mathcal{C})$. Thus, for each $y' \in \mathbb{Z}_m^{n+1}$, one can find at least one codeword $\widehat{c}$ in $\widehat{\mathcal{C}}$ such that $d_{\widehat{P}m}(y', \widehat{c}) \leq \rho(\mathcal{C})$ and hence $\rho(\widehat{\mathcal{C}}) \leq \rho(\mathcal{C})$. For the case of $\widetilde{\mathcal{C}}$, let $\widetilde{c} = (c, c_{n+1})$ be the extended codeword of c in $\widetilde{\mathcal{C}}$ where $d_{Pm}(y, c) \leq \rho(\mathcal{C})$. As, $d_{\widehat{P}m}(y', \widetilde{c}) = w_{\widehat{P}m}(y' - \widetilde{c}) = w_{\widehat{P}m}(y - c, y_{n+1} - c_{n+1}) = w_{Pm}(y - c) + w_L(y_{n+1} - c_{n+1})$, $\rho(\mathcal{C}) \leq \rho(\widetilde{\mathcal{C}}) \leq \rho(\mathcal{C}) + \lfloor \frac{m}{2} \rfloor$. □

In the following, lower and upper bounds for covering radius of product code are established with respect to the pomset $\mathbb{P}$ where $\mathbb{P}$ is either direct or ordinal product of constituent pomsets. The bounds are arrived at in terms of some of the fundamental parameters of the constituent codes and by considering the codes with respect to combinations of chain and antichain pomsets.

Theorem 6 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two linear $\mathbb{P}_1$- and $\mathbb{P}_2$-codes of length n_1 and n_2 respectively over the field $\mathbb{Z}_m$ (where m is prime). Let $\mathcal{C}$ be a product code of $\mathcal{C}_1$ and $\mathcal{C}_2$ with respect to a pomset $\mathbb{P}$. If $\mathbb{P}$ is a direct product of $\mathbb{P}_1$ and $\mathbb{P}_2$ then the following hold:*

(a) *If $\mathbb{P}_1$ is a chain and $\mathbb{P}_2$ is an antichain then*

(i) $\rho(\mathcal{C}) \geq \begin{cases} (d_\rho(\mathcal{C}_1)-1)d_H(\mathcal{C}_2)\lfloor \frac{m}{2}\rfloor + d_{P_2 m}(\mathcal{C}_2) & \text{if } d_\rho(\mathcal{C}_1)\neq 1,\\ \rho(\mathcal{C}_2) & \text{if } d_\rho(\mathcal{C}_1)=1.\end{cases}$

(ii) $\rho(\mathcal{C}) \leq \begin{cases} (n_1-1)n_2\lfloor \frac{m}{2}\rfloor + \rho(\mathcal{C}_2) & \text{if } D_\rho(\mathcal{C}_1)=n_1,\\ n_1 n_2\lfloor \frac{m}{2}\rfloor & \text{if } D_\rho(\mathcal{C}_1)\neq n_1.\end{cases}$

(b) *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are chains then*

(i) $\rho(\mathcal{C}) \geq \begin{cases} (d_\rho(\mathcal{C}_1)-1)d_\rho(\mathcal{C}_2)\lfloor \frac{m}{2}\rfloor + d_{P_2 m}(\mathcal{C}_2) & \text{if } d_\rho(\mathcal{C}_1)\neq 1,\\ \rho(\mathcal{C}_2) & \text{if } d_\rho(\mathcal{C}_1)=1.\end{cases}$

(ii) $\rho(\mathcal{C}) \leq \begin{cases} (n_1-1)\lfloor \frac{m}{2}\rfloor R + \rho(\mathcal{C}_2) & \text{if } D_\rho(\mathcal{C}_1)=n_1,\\ n_1 n_2\lfloor \frac{m}{2}\rfloor & \text{if } D_\rho(\mathcal{C}_1)\neq n_1.\end{cases}$

Here, R is the covering radius of $\mathcal{C}_2$ with respect to RT-metric.

(c) *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are antichains then*

$$\rho(\mathcal{C}) \geq \max\{(d_H(\mathcal{C}_1)-1)d_{P_2 m}(\mathcal{C}_2), (d_H(\mathcal{C}_2)-1)d_{P_1 m}(\mathcal{C}_1)\}.$$

(d) *If $\mathbb{P}_1$ is an antichain and $\mathbb{P}_2$ is a chain then*

(i) $\rho(\mathcal{C}) \geq \begin{cases} (d_\rho(\mathcal{C}_2)-1)d_H(\mathcal{C}_1)\lfloor \frac{m}{2}\rfloor + d_{P_1 m}(\mathcal{C}_1) & \text{if } d_\rho(\mathcal{C}_2)\neq 1,\\ \rho(\mathcal{C}_1) & \text{if } d_\rho(\mathcal{C}_2)=1.\end{cases}$

(ii) $\rho(\mathcal{C}) \leq \begin{cases} (n_2-1)n_1\lfloor \frac{m}{2}\rfloor + \rho(\mathcal{C}_1) & \text{if } D_\rho(\mathcal{C}_2)=n_2,\\ n_1 n_2\lfloor \frac{m}{2}\rfloor & \text{if } D_\rho(\mathcal{C}_2)\neq n_2.\end{cases}$

Proof (a) Now the pomset $\mathbb{P}$ is a disjoint union of n_2 chains from Table 1. Let $Z = (Z_1, Z_2, \ldots, Z_{n_1})^T \in \mathbb{Z}_m^{n_1\times n_2}$ where $Z_i \in \mathbb{Z}_m^{n_2}$. For each $Z_i \in \mathbb{Z}_m^{n_2}$, there exists a codeword V_i in $\mathcal{C}_2$ such that $d_{P_2 m}(Z_i, V_i) \leq \rho(\mathcal{C}_2)$. Let $u = (u_1, u_2, \ldots, u_{n_1})$ be a codeword in $\mathcal{C}_1$. Then, we have codewords $\underline{\mathbf{c}}_i = (u_1 V_i, u_2 V_i, \ldots, u_{n_1} V_i)^T$ in $\mathcal{C}$ for each i. (i) Suppose that $Z_1 \neq 0$, $Z_i = 0$ for $2 \leq i \leq n_1$. Let u be a minimum RT weight codeword with RT-weight $w_\rho(u) = s$. If $s \neq 1$ then $d_{Pm}(Z, \underline{\mathbf{c}}_1) = (s-1)w_H(u_s V_1)\lfloor \frac{m}{2}\rfloor + w_{P_2 m}(u_s V_1) \geq (d_\rho(\mathcal{C}_1)-1)d_H(\mathcal{C}_2)\lfloor \frac{m}{2}\rfloor + d_{P_2 m}(\mathcal{C}_2)$. Otherwise, $d_{Pm}(Z, \underline{\mathbf{c}}_1) = w_{P_2 m}(Z_1 - u_1 V_1) \geq \rho(\mathcal{C}_2)$. (ii) Now suppose that $Z_{n_1} \neq 0$ and u be a codeword with maximum RT weight. If $w_\rho(u) = n_1$ then we have a codeword $\underline{\mathbf{c}}'$ in $\mathcal{C}$ as $(u_1 V_{n_1}, u_2 V_{n_1}, \ldots, V_{n_1})^T$ and $d_{Pm}(Z, \underline{\mathbf{c}}') = (n_1-1)w_H(Z_{n_1} - V_{n_1})\lfloor \frac{m}{2}\rfloor + d_{P_2 m}(Z_{n_1}, V_{n_1}) \leq (n_1-1)n_2\lfloor \frac{m}{2}\rfloor + \rho(\mathcal{C}_2)$. If $w_\rho(u) \neq n_1$ then $d_{Pm}(Z, \underline{\mathbf{c}}_{n_1}) \leq n_1 n_2 \lfloor \frac{m}{2}\rfloor$.

(b) Now $\mathbb{P}$ is such that each row and each column is a chain. But the process of the proof is same as that in (a) and hence the desired bounds are achieved.

(c) Let $u = (u_1, u_2, \ldots, u_{n_1})$ be a codeword in $\mathcal{C}_1$ of minimum Hamming weight such that $u_i \neq$ for an i. Then, we have a codeword $\underline{\mathbf{c}}$ in $\mathcal{C}$ as $(u_1 v, u_2 v, \ldots, u_{n_1} v)^T$ where $0 \neq v \in \mathcal{C}_2$. There exists a $Z = (Z_1, Z_2, \ldots, Z_{n_1})^T$ in $\mathbb{Z}_m^{n_1\times n_2}$ such that $Z_i = u_i v \neq 0$ and $Z_j = 0$ for all $j \neq i$. Now $d_{Pm}(Z, \underline{\mathbf{c}}) \geq (d_H(\mathcal{C}_1)-1)d_{P_2 m}(\mathcal{C}_2)$. Now choose a codeword y from $\mathcal{C}_2$ of minimum Hamming weight such that $y_j \neq 0$ for a j. Now we have a codeword $\underline{\mathbf{c}}'$ in $\mathcal{C}$ as $x \otimes y$ where $0 \neq x \in \mathcal{C}_1$. There exists a Z' in $\mathbb{Z}_m^{n_1\times n_2}$ such that its j^{th} column is same as that in $\underline{\mathbf{c}}'$ and the remaining columns are zero. Then $d_{Pm}(Z', \underline{\mathbf{c}}') \geq (d_H(\mathcal{C}_2)-1)d_{P_1 m}(\mathcal{C}_1)$.

(d) Since $\mathbb{P}$ is a disjoint union of n_1 chains row-wise, the proof is similar to the case (a). □

Now the corresponding bounds for the case where $\mathbb{P}$ is an ordinal product of $\mathbb{P}_1$ and $\mathbb{P}_2$ are given as the following theorem whose proof we omit due to similarity with Theorem 6.

Theorem 7 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two linear $\mathbb{P}_1$- and $\mathbb{P}_2$-codes of length n_1 and n_2 respectively over the field $\mathbb{Z}_m$ (where m is prime). Let $\mathcal{C}$ be a product code of $\mathcal{C}_1$ and $\mathcal{C}_2$ with respect to a pomset $\mathbb{P}$. If $\mathbb{P}$ is an ordinal product of $\mathbb{P}_1$ and $\mathbb{P}_2$ then the following hold:*

(a) *If $\mathbb{P}_1$ is a chain and $\mathbb{P}_2$ is an antichain then*

(i) $\rho(\mathcal{C}) \geq \begin{cases} (d_\rho(\mathcal{C}_1) - 1)n_2\lfloor \frac{m}{2} \rfloor + d_{P_2m}(\mathcal{C}_2) & \text{if } d_\rho(\mathcal{C}_2) \neq 1, \\ \rho(\mathcal{C}_2) & \text{if } d_\rho(\mathcal{C}_2) = 1. \end{cases}$

(ii) $\rho(\mathcal{C}) \leq \begin{cases} (n_1 - 1)n_2\lfloor \frac{m}{2} \rfloor + \rho(\mathcal{C}_2) & \text{if } D_\rho(\mathcal{C}_1) = n_1, \\ n_1 n_2 \lfloor \frac{m}{2} \rfloor & \text{if } D_\rho(\mathcal{C}_1) \neq n_1. \end{cases}$

(b) *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are chains then*

(i) $\rho(\mathcal{C}) \geq \begin{cases} (d_\rho(\mathcal{C}_1) - 1)n_2\lfloor \frac{m}{2} \rfloor + d_{P_2m}(\mathcal{C}_2) & \text{if } d_\rho(\mathcal{C}_2) \neq 1, \\ \rho(\mathcal{C}_2) & \text{if } d_\rho(\mathcal{C}_2) = 1. \end{cases}$

(ii) $\rho(\mathcal{C}) \leq \begin{cases} (n_1 - 1)n_2\lfloor \frac{m}{2} \rfloor + \rho(\mathcal{C}_2) & \text{if } D_\rho(\mathcal{C}_1) = n_1, \\ n_1 n_2 \lfloor \frac{m}{2} \rfloor & \text{if } D_\rho(\mathcal{C}_1) \neq n_1. \end{cases}$

(c) *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are antichains then*

$$\rho(\mathcal{C}) \geq \max\{(d_H(\mathcal{C}_1) - 1)d_{P_2m}(\mathcal{C}_2), (d_H(\mathcal{C}_2) - 1)d_{P_1m}(\mathcal{C}_1)\}.$$

(d) *If $\mathbb{P}_1$ is an antichain and $\mathbb{P}_2$ is a chain then*

(i) $\rho(\mathcal{C}) \geq \begin{cases} (d_\rho(\mathcal{C}_2) - 1)d_H(\mathcal{C}_1)\lfloor \frac{m}{2} \rfloor + d_{P_1m}(\mathcal{C}_1) & \text{if } d_\rho(\mathcal{C}_2) \neq 1, \\ \rho(\mathcal{C}_1) & \text{if } d_\rho(\mathcal{C}_2) = 1. \end{cases}$

(ii) $\rho(\mathcal{C}) \leq \begin{cases} (n_2 - 1)n_1\lfloor \frac{m}{2} \rfloor + \rho(\mathcal{C}_1) & \text{if } D_\rho(\mathcal{C}_2) = n_2, \\ n_1 n_2 \lfloor \frac{m}{2} \rfloor & \text{if } D_\rho(\mathcal{C}_2) \neq n_2. \end{cases}$

6 Conclusion

The poset weight of a vector x defined as the cardinality of the ideal generated by the support of x is a generalization to weights such as Hamming weight and RT weight. As the poset weight of x does not accommodate Lee weight, the support of the vector with respect to Lee weight (which is a multiset) is defined in this work. By defining order ideal in pomsets, a new metric called pomset metric is introduced which generalizes posets in general and gives rise to Lee metric if the underlying pomset is an antichain. The construction methods in posets are extended to arrive at new pomsets which are subsequently imposed upon the constructed pomset codes obtained through direct sum, $(u, u + v)$-construction, puncturing, extension and direct product of codes. Basic parameters such as minimum distance and covering radius are studied, and bounds are established upon the new pomset codes. Moreover, bounds for minimum distance and covering radius of product codes are established by considering the constituent codes with respect to various combinations of chain and antichain pomsets.

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