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Collineations of finite 2-affine planes

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Abstract We investigate collineations of finite 2-affine planes and show that with the exception of some 2-affine planes of order at most 4 each collineation is induced by a collineation of its projective completion. We further deal in more detail with 2-affine planes of type II and their collineations that fix each long line.

Keywords 2-Affine plane · Affine plane · Projective plane · Collineation · Generalised perspectivity · Elation

Mathematics Subject Classification 51A10 · 51E14 · 51E15 · 51E26

1 Introduction

As a generalisation of affine planes, h-affine planes were introduced in Oehler [11]. These are non-trivial linear spaces such that for each line L and each point $p \notin L$ there are between 1 and h lines through p that do not meet L. Oehler [11] determined all finite 2-affine planes. They fall into four classes: affine planes, affine planes from which a point has been removed, affine planes from which a line and all of its points have been removed, or one of four sporadic planes of order 2, 3 or 4.

In particular, finite 2-affine planes of order $n \geq 5$ can be embedded into affine planes and thus projective planes of the same order n, but uniqueness of the extending plane was not discussed in Oehler [11]. More generally, Beutelspacher and Metsch considered in [5] and [6] non-trivial finite linear spaces of order n and showed that such spaces can be embedded in projective planes of the same order n, provided that the line sizes are big enough, that is,

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n is greater than some quartic polynomial in a, where a is $n + 1$ minus the maximum degree of lines in the linear space. In case of 2-affine planes $a = 2$ and $n \geq 13$ by the formula in Beutelspacher and Metsch [5]. It is also remarked at the end of Beutelspacher and Metsch [6] that the projective plane of order n in which the non-trivial linear space is embeddable is unique up to isomorphism. A similar uniqueness result is obtained in Metsch [10] under different assumptions.

The results in this paper also cover the uniqueness of a projective extension of a 2-affine plane in small orders. The main focus however is to deal with the question of whether and how collineations of a 2-affine plane extend to collineations of the projective extension, which has not been discussed previously.

In Sect. 2 it is shown that the projective completion of a finite 2-affine plane of the first three classes above is unique. In the following section we verify that a collineation of such a finite 2-affine plane uniquely extends to a collineation of its projective completion.

In the remaining two sections we investigate special kinds of collineations, those that fix each 'long line', see the definition of the kernel Δ at the beginning of Sect. 4. This further leads to the introduction, in Definition 1, of the elation group of a 2-affine plane of type II (the third class of 2-affine plane above, see Sect. 2 for a definition). Such 2-affine planes can be regarded as transversal designs $TD_1(2, n-1, n)$. They occur, for example, as internal incidence structures of Laguerre near-planes, see [14, Lemma 2.1]. In extending Laguerre near-planes to Laguerre planes (i.e., extending a transversal design $TD_1(3, n, n)$ to a $TD_1(3, n+1, n)$) it is crucial to understand these 2-affine planes and their collineations. Typically, collineations in the kernel of a 2-affine plane lead to central collineations in its projective completion, except in the case of members of the elation group where one has to make some additional assumptions. A maximal kernel implies that the 2-affine plane extends to a Desarguesian projective plane, see Theorem 5. More specifically, Sect. 4 deals with those finite 2-affine planes of type II and of order at least 5 whereas in the last section we consider 2-affine planes of order 3 and 4. The latter planes have many special features and their collineations do not fit in the pattern of those planes of orders at least 5.

2 Finite 2-affine planes and their projective extensions

A *non-trivial linear space* is a geometry $(P, \mathcal{L})$ consisting of a point set P and a set $\mathcal{L}$ of lines (subsets of P) such that

(J) Two points are on a unique line.
(R) There are three points not on a line.

Two lines are called *parallel* if they are identical or do have no point in common. Given a point p and a line L of a non-trivial linear space we let $\pi(p, L)$ to be the number of lines through p that do not meet L, that is, the number of parallels to L through p. Following [11] a *2-affine* or *bi-affine plane* is a non-trivial linear space $\mathcal{B} = (P, \mathcal{L})$ such that

(P_2) For each line L and each point p not on L one has that $1 \leq \pi(p, L) \leq 2$.

If the point set P of a 2-affine plane $\mathcal{B}$ is finite, the 2-affine plane is called finite. In this case the maximum number of points on a line of $\mathcal{B}$ is called the order of $\mathcal{B}$. Thus, if $\mathcal{B}$ has order $n \geq 2$, then

(L) each line contains at most n points and there is a line with exactly n points on it.

The notion of a proper 2-affine plane of order n captures the restriction of a finite affine plane of order n to the complement of one point or one line. As usual, an *affine plane of* *order* $n \geq 2$ is an incidence structure of points and lines satisfying the axioms (J), (R) and (L) from above and (P_2) being replaced by the usual parallel axiom

(P_1) For each line L and each point p not on L one has that $\pi(p, L) = 1$.

Oehler showed that in a 2-affine plane of order $n \geq 4$ each line has either n or $n - 1$ points. Furthermore, each point is on $n + 1$ lines; see [11, Satz 10]. Both cardinalities of lines occur in a 2-affine plane $\mathcal{B}$ of order n unless the map π from above is constant, that is, $\mathcal{B}$ is *complete*. In the latter case $\mathcal{B}$ is either an affine plane (π has constant value 1) or the 2-affine plane $\mathcal{K}_5$ below (π has constant value 2; this case can only occur in order 2, see [11, Satz 6]). In *proper 2-affine planes* of order n, that is, both cardinalities occur, we call a line of $\mathcal{B}$ with n points a *long line* and one with $n - 1$ points a *short line*. A finite proper 2-affine plane is called of *type I* if through each point there is precisely one short line through it, and it is called of *type II* if each point is on precisely one long line. Note that a type II plane cannot have order 2, because in this case there are only two points which together form a long line and each of the points forms a short line. Thus axiom (R) is not satisfied.

Oehler determined all finite 2-affine planes, see [11, Sätze 6, 14, 19 and 20], see also [15, Theorem 1] for linear spaces with $n^2 - n$ points where each point is on $n + 1$ lines, [3, Theorem 1] for non-trivial finite linear spaces where each line has n or $n - 1$ points and [4] for non-trivial finite linear spaces with two line degrees (not necessarily consecutive numbers).

Theorem 1 *A 2-affine plane of order n is isomorphic to*

- *an affine plane of order n;*
- *an affine plane of order n from which one point has been removed;*
- *an affine plane of order n from which all points on a line have been removed;*
- *one of four sporadic planes of order at most 4.*

Finite 2-affine planes of order n of the first three kinds are embedded into affine planes and thus projective planes of the same order n. We call these kinds of 2-affine planes of *affine type*. We consider their projective completions and show that they are uniquely determined by the 2-affine plane. Points or lines that are adjoined to the 2-affine plane in order to obtain its projective completion are referred to as *ideal points* or *ideal lines*.

Note that 2-affine planes of the second and third kind are of type I and type II, respectively. Conversely, by Oehler [11, Satz 14] a finite 2-affine plane of type I is an affine plane minus one point if the order is at least 4. Likewise, by Oehler [11, Satz 19] a finite 2-affine plane of type II is an affine plane minus one line and all of its points if the order is at least 6. In case of a type II plane we say that a partition of the point set of $\mathcal{B}$ into mutually disjoint long lines is a *v-bundle*, and similarly, a partition into mutually parallel short lines is an *h-bundle*. There are $n - 1$ long lines and they form a v-bundle.

In the third case in Theorem 1 an affine extension is not necessarily unique, even up to isomorphism. For example, if $\mathcal{P}$ is a projective translation plane that is not a Moufang plane, then the translation line A is unique. Removing A and all of its points from $\mathcal{P}$ gives us an affine translation plane $\mathcal{A} = \mathcal{P} \setminus A$. The same removal process applied to a line $L \neq A$ results in an affine plane $\mathcal{A}' = \mathcal{P} \setminus L$, which is not a translation plane. Therefore $\mathcal{A}$ and $\mathcal{A}'$ are not isomorphic. However, both affine planes are affine extensions of the 2-affine plane $\mathcal{B} = \mathcal{P} \setminus (A \cup L) = \mathcal{A} \setminus L = \mathcal{A}' \setminus A$.

Theorem 2 *A 2-affine plane of affine type is embeddable into a unique projective plane of the same order.*

Proof It is well known that an affine plane determines its projective completion up to isomorphism; see, for example [12, 1.7] or [8, Theorem 3.10]. A finite 2-affine plane $\mathcal{B}$ of type I is an affine plane $\mathcal{A}$ from which a point has been removed. Clearly, by adding an ideal point to all short lines in $\mathcal{B}$ the affine plane $\mathcal{A}$ extending $\mathcal{B}$ is uniquely determined, and thus so is the projective completion $\mathcal{P}$.

It remains to verify the Theorem in case of 2-affine planes of type II. Since a projective plane of order $q \leq 5$ is unique (the Desarguesian plane $\mathsf{PG}(2, q)$; see, for example, [13]), there is nothing to prove in these cases. Oehler's proof gives a recipe how to extend a type II 2-affine plane $\mathcal{B}$ of order $n \geq 6$ to an affine plane $\mathcal{A}$, see [11, proof of Satz 18]. (Essentially the same strategy is employed in Totten [15]. The steps involved are also clear from the eventual end result that $\mathcal{B}$ is a projective plane minus two lines.) Start with a short line L. It is shown in Oehler [11, Sect. 5] that L is contained in precisely two h-bundles. Moreover, given a short line L' disjoint to L, there is precisely one h-bundle that contains both L and L'. Choose one of the h-bundles $\mathcal{H}_L$ that contains L. Every line M in $\mathcal{H}_L$ is then contained in exactly one further h-bundle $\mathcal{H}'_M \neq \mathcal{H}_L$. Finally, if $L_1 = L, L_2, \ldots, L_n$ are the lines in $\mathcal{H}_L$, adjoin a new point p_i to each line in $\mathcal{H}'_{L_i}$. Furthermore, $W = \{p_1, \ldots, p_n\}$ also forms a line of the extended incidence structure $\mathcal{A}$. Then $\mathcal{A}$ is an affine plane of order n such that $\mathcal{B}$ is obtained from $\mathcal{A}$ by removing the special line W and all of its points on it. It follows from [11, Lemmata 1 and 2] that the h-bundles $\mathcal{H}'_{L_1}, \ldots, \mathcal{H}'_{L_n}$ are mutually disjoint and form a partition of the collection $\mathcal{L}_s$ of short lines of $\mathcal{B}$.

If we choose the other h-bundle $\mathcal{H}'_L \neq \mathcal{H}_L$ that contains L, we similarly obtain n h-bundles $\mathcal{H}''_{M_i}$ where $M_1 = L, M_2, \ldots, M_n$ are the lines in $\mathcal{H}'_L$ and then an affine plane $\mathcal{A}'$ with new points q_i adjoined to each line in $\mathcal{H}''_{M_i}$ and additional line $W' = \{q_1, \ldots, q_n\}$. Again, the h-bundles $\mathcal{H}''_{M_1}, \ldots, \mathcal{H}''_{M_n}$ form a partition of $\mathcal{L}_s$. Together with the $\mathcal{H}'_{L_i}$ we have accounted for all h-bundles in $\mathcal{B}$. Furthermore, the lines in $\mathcal{H}'_{L_i}$ belong to different h-bundles $\mathcal{H}''_{M_j}$, and vice versa. This means that the lines in an h-bundle $\mathcal{H}''_{M_i}$ are adjoined different points p_j when obtaining $\mathcal{A}$ from $\mathcal{B}$. Hence, each $\mathcal{H}''_{M_i}$ gives rise to a bundle $\tilde{\mathcal{H}}''_{M_i}$ of parallel lines in $\mathcal{A}$.

When we form the projective completion $\mathcal{P}$ of $\mathcal{A}$ we essentially adjoin the points q_i to each line in $\tilde{\mathcal{H}}''_{M_i}$, as we did when extending $\mathcal{B}$ to $\mathcal{A}'$, plus a new point ω, the ideal point of long lines. This shows that the ideal points of $\mathcal{P}$ that are adjoined to $\mathcal{B}$ in order to obtain the projective completion are in one-to-one correspondence with bundles of mutually parallel lines of $\mathcal{B}$. Hence $\mathcal{P}$ is uniquely determined by $\mathcal{B}$. (If we begin with $\mathcal{H}'_L$ we first adjoin the points q_i in order to obtain $\mathcal{A}'$ and then the points p_j and ω to get to $\mathcal{P}$.) □

In general, the 2-affine plane one obtains by removing two lines and all of its points from a projective plane depends on the two lines. However, because the collineation group $\mathsf{PGL}(3, q)$ of $\mathsf{PG}(2, q)$, the Desarguesian projective plane of order q over the Galois field $\mathbb{F}_q$ of order q, is 2-transitive on the line set of $\mathsf{PG}(2, q)$, it does not matter which two lines are removed from $\mathsf{PG}(2, q)$ in order to obtain a 2-affine plane. We denote the resulting 2-affine plane by $\mathcal{B}_{\mathsf{PG}(2,q)}$.

According to [11, Sätze 6 and 20] there are four sporadic 2-affine planes, two of order 2 and one each of order 3 and 4. More specifically, the sporadic planes $(P, \mathcal{L})$ are as follows.

- *The complete finite 2-affine plane* $\mathcal{K}_5$ It has 5 points and the lines are precisely the 2-subsets of P, that is, this geometry is represented by the complete graph on five vertices; see the left diagram in Fig. 1. By Oehler [11, Satz 6] it is the only finite 2-affine plane for which there are always two parallels to a line through a given point.

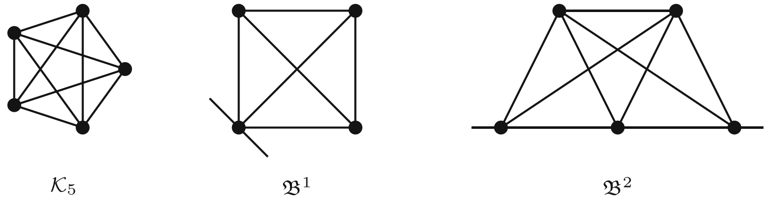

Fig. 1 The sporadic 2-affine planes of orders ≤ 3

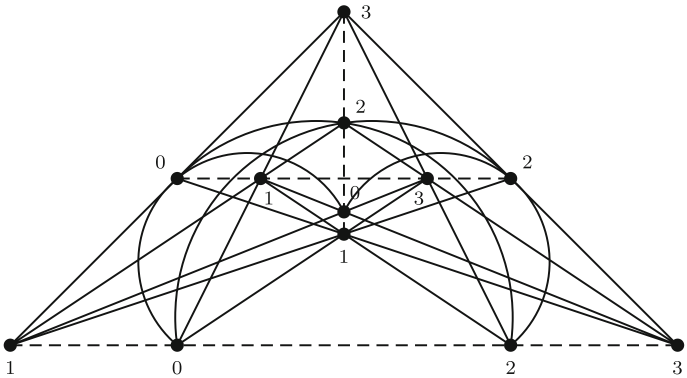

Fig. 2 The Shrikhande plane $\mathcal{S}$. The three lines of length 4 are *dashed*. Four lines of length 3 are shown as *circular arcs*

This 2-affine plane can be embedded in $\mathsf{PG}(2, 4)$ by taking P to be the points of a conic $\mathcal{C}$ in $\mathsf{PG}(2, 4)$. Lines of $\mathcal{K}_5$ are the traces on P of secant lines to $\mathcal{C}$ of $\mathsf{PG}(2, 4)$.

- *The 2-affine plane* $\mathfrak{B}^1$ It has 4 points, each 2-subset of P is a line and there is precisely one line with a single point; see the middle diagram in Fig. 1. (So it is the affine plane of order 2 with an extra line of length 1.)
- *The 2-affine plane* $\mathfrak{B}^2$ It has 5 points and one line L_3 of size 3. The other lines are precisely the 2-subsets of P not entirely contained in L_3; see the right diagram in Fig. 1. It is readily seen that $\mathfrak{B}^2$ can be embedded in $\mathsf{PG}(2, 3)$ as follows. Choose two points p_1, p_2 and a line L through one but not the other. Then P consists of all points on L and the line through p_1 and p_2 except for the two points p_1, p_2. Lines then are the traces on P of lines of $\mathsf{PG}(2, 3)$ provided they contain at least two points.
- *The Shrikhande plane* $\mathcal{S}$ It has twelve points, three lines of length 4 and 16 lines of length 3. This 2-affine plane can be embedded in $\mathsf{PG}(2, 5)$ as follows. Choose three non-concurrent lines L_1, L_2, L_3. Then P consists of all the points of $\mathsf{PG}(2, 5)$ that are on precisely one of these three lines (so points of intersection $L_i \cap L_j$, $i \neq j$, do not belong to P). Lines of $\mathcal{S}$ are the traces of lines of $\mathsf{PG}(2, 5)$ on P provided these contain at least two points.

  An algebraic description of this plane is obtained as follows; compare Sect. 5. The three lines of length 4 are $\{a_0, a_1, a_2, a_3\}$, $\{b_0, b_1, b_2, b_3\}$ and $\{c_0, c_1, c_2, c_3\}$. The points a_i, b_j, c_k form a short line if and only if $i + j + k \equiv 0 \pmod 4$. For a pictorial representation see Fig. 2 where only the indices of points are shown.

  This 2-affine plane is called the Shrikhande plane in Totten [15] and labelled $\mathfrak{B}^3$ in Oehler [11]. For incidence matrices of $\mathcal{S}$ see [11, p. 435] and [15, Fig. 1].

3 Collineations of finite 2-affine planes

As usual in incidence geometry, a collineation of a 2-affine plane $\mathcal{B}$ is a permutation of the point set that maps lines onto lines. All collineations of $\mathcal{B}$ form a group with respect to composition, the collineation group $\Gamma = \mathrm{Aut}(\mathcal{B})$ of $\mathcal{B}$. More generally, an isomorphism between two 2-affine planes is a bijection between the point sets that takes lines in one to lines in the other.

If $\mathcal{P}$ is a projective plane and L_1 and L_2 are two lines of $\mathcal{P}$, then any collineation γ of $\mathcal{P}$ that fixes each of L_1 and L_2 or interchanges L_1 and L_2 induces a collineation γ' of the 2-affine plane $\mathcal{B} = \mathcal{P} \setminus (L_1 \cup L_2)$ obtained from $\mathcal{P}$ by removing L_1 and L_2 and all of their points. If $\mathcal{A}_1 = \mathcal{P} \setminus L_1$ is the affine plane obtained from $\mathcal{P}$ by removing the line L_1 and all of its points, then $\mathcal{A}_1$ is an affine extension of $\mathcal{B}$, but γ' does not extend to a collineation of $\mathcal{A}_1$ in case γ interchanges L_1 and L_2. However, for the projective completions we have.

Theorem 3 *Each isomorphism between finite 2-affine planes of affine type extends to a unique isomorphism of their projective completions.*

Proof It is well known that an isomorphism between affine planes uniquely extends to an isomorphism of their projective completions; see, for example [12, 1.8]. In case of 2-affine planes of type I an isomorphism φ uniquely extends to an isomorphism of their affine completions by defining that the ideal point in one plane that has been added to all short lines is taken to the corresponding ideal point in the other 2-affine plane. In turn, one has a unique extension of φ to an isomorphism of the projective completions of the 2-affine planes.

It remains to verify that an isomorphism φ from a 2-affine plane $\mathcal{B}_1 = (P_1, \mathcal{L}_1)$ to a 2-affine plane $\mathcal{B}_2 = (P_2, \mathcal{L}_2)$, both of affine type II, extends to an isomorphism of their projective completions $\mathcal{P}_1 = (\tilde{P}_1, \tilde{\mathcal{L}}_1)$ and $\mathcal{P}_2 = (\tilde{P}_2, \tilde{\mathcal{L}}_2)$. Obviously, the ideal point ω_1 of long lines in $\mathcal{B}_1$ must be taken to the ideal point ω_2 of long lines in $\mathcal{B}_2$. As outlined in the proof of Theorem 2 the ideal points $\neq \omega_i$ that are adjoined to $\mathcal{B}_i$ in order to obtain the projective completion $\mathcal{P}_i$ are in one-to-one correspondence with h-bundles (partitions of P_i into mutually parallel lines) of $\mathcal{B}_i$. Furthermore, every short line of $\mathcal{B}_i$ belongs to precisely two h-bundles.

Since an h-bundle of $\mathcal{B}_1$ is obviously taken by φ to an h-bundle in $\mathcal{B}_2$ there is a unique extension $\tilde{\varphi}$ of φ to a bijection from $\tilde{P}_1$ to $\tilde{P}_2$. We still have to verify that $\varphi(x_1), \varphi(x_2), \varphi(x_3)$ are collinear in $\mathcal{P}_2$ whenever x_1, x_2, x_3 are collinear points in $\mathcal{P}_1$. This is clearly the case when at least two of the three points belong to $\mathcal{B}_1$.

So now assume that x_1 and x_2 are ideal points of $\mathcal{P}_1$, both $\neq \omega_1$. These points belong to different ideal lines of $\mathcal{P}_1$ if and only if they are on the extension $\tilde{L}$ of a short line L in $\mathcal{B}_1$, that is, if and only if there are corresponding h-bundles in $\mathcal{B}_1$ that have a short line in common (the line L). The latter property is preserved under φ. Hence, $\tilde{\varphi}(\tilde{L})$ where L is a short line of $\mathcal{B}_1$ with extension $\tilde{L}$ is a line of $\mathcal{P}_2$. On the other hand, if x_1 and x_2 are ideal points $\neq \omega_1$ on the same ideal line of $\mathcal{P}_1$, then so are their images in $\mathcal{P}_2$ under $\tilde{\varphi}$. This shows that $\tilde{\varphi}$ preserves collinearity. Hence $\tilde{\varphi}$ is an isomorphism of projective planes. □

Corollary 1 *Each collineation of a finite 2-affine plane $\mathcal{B}$ of affine type extends to a unique collineation of the projective completion of $\mathcal{B}$.*

From the description of the three sporadic 2-affine planes of order at most 3 it is evident that

- The collineation group of $\mathcal{K}_5$ is the symmetric group S_5 on five letters; since in the collineation group $\mathsf{PGL}(3, 4)$ of $\mathsf{PG}(2, 4)$ the stabilizer of a conic of $\mathsf{PG}(2, 4)$ is isomorphic to S_5, each collineation of $\mathcal{K}_5$ extends to a collineation of $\mathsf{PG}(2, 4)$.

- The collineation group of $\mathfrak{B}^1$ is the symmetric group S_3 on three letters, because the unique point that forms the line of length 1 must be fixed by any collineation.
- The collineation group of $\mathfrak{B}^2$ is isomorphic to $S_2 \times S_3$; since in the collineation group $\mathsf{PGL}(3, 3)$ of $\mathsf{PG}(2, 3)$ the stabilizer of two points and a line through one of them is isomorphic to $S_2 \times S_3$, each collineation of $\mathfrak{B}^2$ extends to a collineation of $\mathsf{PG}(2, 3)$.

As for the Shrikhande plane $\mathcal{S}$ the stabilizer of a triangle in the collineation group $\mathsf{PGL}(3, 5)$ of $\mathsf{PG}(2, 5)$ induces a group of collineations of $\mathcal{S}$. As an abstract group this stabilizer is a semi-direct product $(C_4 \times C_4) \rtimes S_3$ of $C_4 \times C_4$ by S_3 where C_4 denotes the cyclic group of order 4, and thus has order 96. However, in Corollary 3 we shall see that the collineation group of $\mathcal{S}$ has order 192. Thus not every collineation of $\mathcal{S}$ extends to a collineation of $\mathsf{PG}(2, 5)$.

For example, in Fig. 2, one obtains a collineation γ of $\mathcal{S}$ if all points with labels 0 or 2 are fixed and points labelled 1 or 3 are interchanged on each long line. The line all of whose points have label 0 and the line all of whose points have label 2 are parallel in $\mathcal{S}$. The extensions in $\mathsf{PG}(2, 5)$ of the latter two lines intersect in a point p not belonging to $\mathcal{S}$. This point p and the three points $p_{ij} = L_i \cap L_j, i \neq j$ (in the notation of the geometric construction of $\mathcal{S}$ at the end of Sect. 2 from three non-concurrent lines L_1, L_2, L_3 in $\mathsf{PG}(2, 5)$) form a quadrangle Q in $\mathsf{PG}(2, 5)$. Furthermore, each point in Q must be fixed by any extension $\tilde{\gamma}$ of γ to a collineation of $\mathsf{PG}(2, 5)$. Therefore, the fixed configuration of $\tilde{\gamma}$ is a subplane of $\mathsf{PG}(2, 5)$, which of course must be the entire plane. Thus $\tilde{\gamma} = id$—a contradiction. Therefore, γ cannot be extended to a collineation of $\mathsf{PG}(2, 5)$.

4 Collineations of finite 2-affine planes of type II and order at least 5

Throughout this section $\mathcal{B} = (P, \mathcal{L})$ denotes a finite 2-affine plane of type II and order $n \geq 5$. Hence there is a unique long line (of length n) through each point $p \in P$, denoted by $[p]$, so that the long lines form a partition of P (a v-bundle). Let $\mathcal{L}_v$ be the collection of long lines and $\mathcal{L}_s$ be the collection of short lines (of lengths $n-1$) of $\mathcal{B}$. Furthermore, the 2-affine plane $\mathcal{B}$ is of affine type, that is, $\mathcal{B}$ has a projective completion $\mathcal{P}$ of order n. We denote by ω the ideal point of lines in $\mathcal{P}$ coming from long lines of $\mathcal{B}$, and by $\tilde{L}$ the line in $\mathcal{P}$ whose trace on $\mathcal{B}$ is L. Finally, we let L_1 and L_2 be the lines adjoined to obtain $\mathcal{P}$ from $\mathcal{B}$.

The cases of order 3 and 4 are special and will be dealt with in Sect. 5.

The collineation group Γ of $\mathcal{B}$ acts on the set $\mathcal{L}_v$ of long lines; the kernel of this action is denoted by Δ, that is,

$$\Delta = \{\gamma \in \Gamma \mid \gamma(x) \in [x] \text{ for all } x \in P\}.$$

If γ is a collineation of $\mathcal{B}$, then its extension to $\mathcal{P}$ is denoted by $\tilde{\gamma}$. Note that $\tilde{\gamma}$ fixes ω and permutes the lines L_1 and L_2, that is, $\tilde{\gamma}$ either fixes both L_1 and L_2 or interchanges the two lines.

Lemma 1 *Given a short line $L \in \mathcal{L}_s$ and a point $x \notin L$ in $\mathcal{B}$, the stabilizer $\Delta_{L,x}$ of L and x in Δ is trivial: $\Delta_{L,x} = \{id\}$.*

Proof $\delta \in \Delta_{L,x}$ extends to a collineation $\tilde{\delta}$ of the projective plane $\mathcal{P}$ by Corollary 1. Note that $\tilde{\delta}$ fixes all points on L and also ω.

Assume first that $\tilde{\delta}$ fixes both L_1 and L_2. Then $\tilde{\delta}$ is a central collineation with centre ω and axis the line induced by L. But $\tilde{\delta}$ also fixes the point x, which is neither the centre nor on the axis. Hence $\tilde{\delta} = id$ by Hughes and Piper [8, Theorem 4.9]. Thus δ must be the identity.

We now assume that $\tilde{\delta}$ interchanges L_1 and L_2. Then $\tilde{\delta}^2$ fixes both L_1 and L_2. As before we see that $\tilde{\delta}^2 = id$. Hence $\tilde{\delta}$ is an involution. By Baer [1], see also [8, Theorems 4.3 and 4.4], $\tilde{\delta}$ is either a central collineation or a Baer involution. In the former case, the centre is ω and the axis $\tilde{L}$, the line that induces L, and we have a contradiction to the assumption that $\tilde{\delta}$ interchanges L_1 and L_2. The latter case cannot occur because $\tilde{\delta}$ has $n-1$ fixed points on $\tilde{L}$ but $n-1 \neq \sqrt{n}+1$ since $n \geq 5$. □

Corollary 2 *The order of* Δ_L *where* $L \in \mathcal{L}_s$ *divides* $n-1$ *and* $|\Delta| \leq (n-1)n^2$.

Proof Let $x \in L$. According to Lemma 1 each point $y \in [x] \setminus \{x\}$ has an orbit of size $t = |\Delta_L|$ under Δ_L; thus t divides $|[x] \setminus \{x\}| = n-1$. Since the orbit $\Delta(L)$ of L has size at most $|\mathcal{L}| = n^2$, we obtain $|\Delta| = t|\Delta_L| \leq (n-1)n^2$. □

Remark 1 In fact, the order of Δ is a divisor of $(n-1)n^2$; compare Lemma 4.

Remark 2 Lemma 1 and Corollary 2 are no longer valid in case of the three type II 2-affine planes of order 3 and 4; see Sect. 5. In each of these planes there are points x not on $L \in \mathcal{L}_s$ such that $\Delta_{L,x}$ has order 2. Moreover, the kernel Δ of the 2-affine planes $\mathcal{B}_{\mathsf{PG}(2,3)}$ and $\mathcal{B}_{\mathsf{PG}(2,4)}$ has order 36 and 96, respectively, which is twice the bound given in Corollary 2.

An element of Δ induces a collineation of the projective completion that fixes ω and almost looks like a central collineation with centre ω. Collineations of this kind are known as generalised elations or generalised homologies. More precisely, a *generalised perspectivity* of a projective plane $\mathcal{P}$ is a collineation γ such that all fixed points of γ are on a line A and all fixed lines of γ pass through a point c; see [2] and [9]. A and c are called the *axis* and *centre*, respectively, of γ. Clearly, every perspectivity (or central collineation) is a generalised perspectivity. As for perspectivities, a generalised perspectivity γ is called a *generalised elation* or a *generalised homology* if $c \in A$ or $c \notin A$, respectively.

Lemma 2 *Let* $\delta \in \Delta_L \setminus \{id\}$ *where* L *is a short line of* $\mathcal{B}$. *Then* δ *induces a homology* $\tilde{\delta}$ *in the projective completion* $\mathcal{P}$ *of* $\mathcal{B}$ *with centre* ω *and axis* $\tilde{L}$.

Proof $\tilde{\delta}$ fixes all lines that come from long lines of $\mathcal{B}$ and the line $\tilde{L}$. If $\tilde{\delta}$ also fixes both remaining lines L_1 and L_2 through ω, then $\tilde{\delta}$ is a homology of $\mathcal{P}$.

Suppose that L_1 and L_2 are interchanged by $\tilde{\delta}$. Then $\tilde{\delta}$ is a generalised homology that has precisely one orbit of length 2 on its axis. Consider a line $K \neq L_1, \tilde{L}$ that passes through $p_1 = L_1 \cap \tilde{L}$. This line K comes from a short line $K' \neq L$ of $\mathcal{B}$. If $x \in K' \cap \delta(K')$, then, because $\delta([x]) = [x]$, one finds that $\{\delta(x)\} = \delta(K' \cap [x]) = \delta(K') \cap \delta([x]) = \delta(K') \cap [x] = \{x\}$ so that $\delta(x) = x$—a contradiction to Lemma 1. In $\mathcal{P}$ this means that K and $\tilde{\delta}(K)$ intersect in a point of $L_1 \cup L_2$. However, $\tilde{\delta}(a_1) \in L_2 \cap \tilde{L} = \{a_2\}$ so that $\tilde{\delta}(K)$ is a line through a_2 different from $\tilde{L}$ and L_2. But any two distinct lines in a projective plane intersect in a point. The point of intersection of K and $\tilde{\delta}(K)$ is neither a_1 nor a_2, and it can neither be on L_1 (since $K \neq L_1$) nor on L_2 (otherwise $\tilde{\delta}(K) = L_2$). Hence we have a contradiction, and the case that L_1 and L_2 are interchanged is not possible. See also [9, Corollary 14]. □

Remark 3 Lemma 2 is neither valid in the 2-affine plane $\mathcal{B}_{\mathsf{PG}(2,3)}$ nor in $\mathcal{B}_{\mathsf{PG}(2,4)}$. For example, in Fig. 4, which gives a pictorial representation of $\mathcal{B}_{\mathsf{PG}(2,4)}$, one obtains a collineation γ of $\mathcal{B}_{\mathsf{PG}(2,4)}$ if all points with labels 00 or 11 are fixed and points labelled 01 or 10 are interchanged on each long line. Since the three points with label 00 form a short line L, γ belongs to Δ_L. However, the six fixed points of γ plus the point ω form a Baer subplane of $\mathsf{PG}(2,4)$ and $\tilde{\gamma}$ is a Baer involution of $\mathsf{PG}(2,4)$. For $\mathcal{B}_{\mathsf{PG}(2,3)}$ see Sect. 5.

In order for $\delta \in \Delta$ to induce an elation in the projective completion, we need that δ fixes no short line. Indeed, in this case, we have the following.

Lemma 3 *Let $\delta \in \Delta$ be such that δ fixes no short line of $\mathcal{B}$. Then δ induces a generalised elation $\tilde{\delta}$ with centre ω in the projective completion $\mathcal{P}$ of $\mathcal{B}$. Furthermore, if the order n of $\mathcal{B}$ is odd, then $\tilde{\delta}$ is an elation of $\mathcal{P}$.*

Proof $\tilde{\delta}$ fixes each line of $\mathcal{P}$ that comes from a long line of $\mathcal{B}$. Since by assumption δ fixes no short line, $\tilde{\delta}$ cannot fix a line not passing through ω. Hence, the fixed lines of $\tilde{\delta}$ all pass through ω, and there are at least $n-1$ of them. In particular, ω must be the centre of $\tilde{\delta}$. If $\tilde{\delta}$ also fixes both remaining lines L_1 and L_2 through ω, then $\tilde{\delta}$ is an elation of $\mathcal{P}$ and thus a generalised elation.

If L_1 and L_2 are interchanged by $\tilde{\delta}$, then $\tilde{\delta}$ fixes exactly $n-1$ lines and so by Baer [1, Theorem 1.1] or [8, Theorem 13.3] there must also be precisely $n-1$ points fixed by $\tilde{\delta}$. One of them is ω and the others are neither on L_1 nor on L_2. Furthermore, the remaining $n-2$ fixed points must all be on a line through ω otherwise δ fixes a short line. Hence $\tilde{\delta}$ moves exactly two points on its axis. It follows that $\tilde{\delta}^2$ is an elation of $\mathcal{P}$ with centre ω and axis coming from a long line of $\mathcal{B}$. In case the order n of $\mathcal{P}$ is odd, the order m of $\tilde{\delta}^2$ is also odd. But then $\gamma = \tilde{\delta}^m$ is an involution and a generalised elation that is not an elation (because L_1 and L_2 are still interchanged by γ). Thus γ is either a central collineation or a Baer involution, neither of which is possible (the latter because because $n \geq 5$); compare also [16, Corollary 7 or Theorem 11]. □

Remark 4 In the proof of Lemma 3 the case that $\tilde{\delta}$ interchanges L_1 and L_2 could not be eliminated if the order n is even. In this situation $\tilde{\delta}$ has exactly one orbit of size 2 on its axis, and all other points are fixed. By Vedder [16, Theorem 10], $\tilde{\delta}$ has order 4.

The product of a planar collineation σ (that is, the geometry of fixed points and fixed lines of σ is a subplane) and an elation η which has both its centre and axis in the subplane fixed by σ but does not stabilise this subplane is a generalised elation that is not an elation. Conversely, it is shown in Vedder [16, Proposition 1] that every generalised elation with axis A and centre c is such a composition if the projective plane is (c, A)-transitive.

For example, in $\mathsf{PG}(2,4)$ with the usual homogeneous coordinates over the Galois field $\mathbb{F}_4 = \{0, 1, a, a+1\} = \mathbb{F}_2(a)$ of order 4 where $a^2+a+1=0$ the following collineation γ given by $\gamma(x:y:z) = (x^2 : y^2+az^2 : z^2)$ is a generalised elation with centre $(0:1:0)$ and axis $[0:0:1]$. (A point $(x:y:z)$ is on the line $[u:v:w]$ if and only if $ux+vy+wz=0$. Thus $\gamma([u:v:w]) = [u^2 : v^2 : av^2+w^2]$.) The fixed points of γ are $(0:1:0)$, $(1:0:0)$ and $(1:1:0)$, and the fixed lines of γ are $[0:0:1)]$, $[1:0:0]$ and $[1:0:1]$. The points $(1:a:0)$ and $(1:a+1:0)$ on the axis are interchanged by γ and so are the lines $[1:0:a]$ and $[1:0:a+1]$ through the centre. Furthermore, γ has order 4. Considering the 2-affine plane obtained from $\mathsf{PG}(2,4)$ by removing the lines $[1:0:a]$ and $[1:0:a+1]$ and all of its points shows that the statement in Lemma 3 about γ extending to an elation of $\mathcal{P}$ does not carry over to the case $n=4$.

The assumption made in Lemma 3 that the collineation fixes no short line leads, more formally, to the following definition of an elation group of a 2-affine plane of type II.

Definition 1 In a finite 2-affine plane of type II we call

$$E := \{\delta \in \Delta \mid \delta \text{ fixes no short line}\} \cup \{id\}$$

the *elation group* of the 2-affine plane.

Lemma 4 *The elation group E of $\mathcal{B}$ is a normal subgroup of the collineation group Γ and acts semi-regularly on $\mathcal{L}_s$. In particular, the order of E divides n^2, and E acts regularly on $\mathcal{L}_s$ if and only if E has order n^2.*

Proof E contains the identity and with δ its inverse δ^{-1} also belongs to E. Let $\gamma, \delta \in E$ where $\gamma \neq \delta^{-1}$. Suppose that $\gamma\delta \notin E$. Then there is a short line L fixed by $\gamma\delta \neq id$. Thus the stabilizer Δ_L is non-trivial. Let $B = \Delta(L) \subseteq \mathcal{L}_s$ be the orbit of L under Δ. If $B = \{L\}$, then every $\eta \in \Delta$ fixes L so that $E = \{id\}$—a contradiction to $\gamma\delta \neq id$. Hence B must be non-trivial.

Since a stabilizer Δ_K is conjugate to Δ_L for each $K \in B$ and because $\Delta_{L,K} \leq \Delta_{L,x} = \{id\}$ for each point $x \in K \setminus L$ by Lemma 1, we find that Δ is a Frobenius group operating on B (see [7, p. 37]). From [7, Theorems 2.7.5 and 2.7.6], see also [7, Theorem 4.5.1], it now follows that

$$E^{(B)} = \Delta \setminus \left(\bigcup_{K \in B} (\Delta_K \setminus \{id\}) \right),$$

is a (normal) subgroup of Δ, that $E^{(B)}$ acts regularly on B and that $\Delta = E^{(B)} \cdot \Delta_L$. But $E \subseteq E^{(B)}$ so that $\gamma, \delta \in E^{(B)}$ and thus $\gamma\delta \in E^{(B)}$. However, $id \neq \gamma\delta \in \Delta_L$ in contradiction to the regular action of $E^{(B)}$ on B. This shows that $\gamma\delta \in E$. Hence E is a subgroup of Δ.

Since Δ is normal in Γ, the conjugate $\gamma E \gamma^{-1}$ of E, where $\gamma \in \Gamma$, is again a subgroup of Δ. But a collineation δ fixes a line $L \in \mathcal{L}_s$ if and only if $\gamma\delta\gamma^{-1}$ fixes the short line $\gamma(L)$. Thus $\gamma E \gamma^{-1} = E$.

Since no $\eta \in E \setminus \{id\}$ fixes a short line, all orbits of E on $\mathcal{L}_s$ have the same size $|E|$, i.e. E acts semi-regularly on $\mathcal{L}_s$. Thus $|E|$ divides $n^2 = |\mathcal{L}_s|$. The statement about the regularity of E now directly follows. □

We call a 2-affine plane $\mathcal{B}$ of type II an *elation 2-affine plane* if the elation group E of $\mathcal{B}$ acts regularly on $\mathcal{L}_s$ or equivalently, by Lemma 4, has order n^2 where n is the order of $\mathcal{B}$. So $\mathcal{B}$ is an elation 2-affine plane if and only if E has maximal order. With this notion we can deal with the unresolved case of even order in Lemma 3.

Theorem 4 *If $\mathcal{B}$ is an elation 2-affine plane, then the projective completion $\mathcal{P}$ of $\mathcal{B}$ is a dual translation plane. Thus the order of $\mathcal{B}$ is a prime power and the elation group E of $\mathcal{B}$ is elementary abelian. Every collineation in E extends to an elation of $\mathcal{P}$.*

Proof We consider the map $\varphi : E \to \mathrm{Aut}(\mathcal{P})$ given by $\delta \mapsto \tilde{\delta}$, that is, φ takes $\delta \in E$ to its extension onto $\mathcal{P}$. This map is a homomorphism, which is injective by Corollary 1. As seen in the proof of Lemma 3 each $\tilde{\delta}^2 = \varphi(\delta^2)$ is a central collineation with centre ω. Moreover, E has a normal subgroup N of index at most 2 (the kernel of the action on the lines though ω).

Suppose that $N = E$, that is, each $\tilde{\delta}$ where $\delta \in E$ is a central collineation with centre ω. Then the elation group of $\mathcal{P}$ with common centre ω has order n^2 and thus is transitive on the lines not passing through ω. Hence, $\mathcal{P}$ is a dual translation plane, n is a prime power (see [12, 12.7]) and E is elementary abelian (cp. [12, 8.1]).

If n is odd, then $N = E$ by Lemma 3, and we are done as seen above.

Finally assume that $n > 4$ is even and that $N \neq E$. Then N is a normal subgroup of E of index 2. Every collineation $\delta \in E \setminus N$ induces a proper generalised elation $\tilde{\delta}$ in $\mathcal{P}$. Furthermore, $\tilde{\delta}$ interchanges the two lines L_1 and L_2. It therefore has an axis A through ω and $n - 2$ fixed points on A.

Since N has order $\frac{1}{2}n^2 > n$ and consists of elations of $\mathcal{P}$, this subgroup contains elations with different axes. Hence N is abelian; compare [8, Theorem 4.14]. Let $\beta \in N$ with axis $B \neq A$ so that $B \cap A = \{\omega\}$. The commutator $[\tilde{\delta}, \tilde{\beta}] = \tilde{\delta}\tilde{\beta}\tilde{\delta}^{-1}\tilde{\beta}^{-1}$ also is an elation with axis B. Moreover, because $n > 4$, there is a point x on A such that both x and $\tilde{\beta}(x)$ are fixed by $\tilde{\delta}$. But then $[\tilde{\delta}, \tilde{\beta}]$ fixes $\tilde{\beta}(x) \notin B$ as well. Hence, $[\tilde{\delta}, \tilde{\beta}] = id$, that is, $\tilde{\delta}$ and $\tilde{\beta}$ commute and thus δ and β commute on $\mathcal{B}$.

There are at least $\frac{1}{2}n^2 - n = \frac{1}{4}n^2 + \frac{1}{4}n(n-4) > \frac{1}{4}n^2$ elations in N whose axis is different from A. Therefore the subgroup generated by all these elations equals N. This means that δ commutes with each element of N, and because N is abelian, E must also be abelian.

Let a_1 and a_2 be the two points on A that get moved by δ. Then $\{a_1, a_2\}$ is invariant under E. Otherwise, some $\delta' \in E$ takes (at least) one of a_1 or a_2 to a fixed point of δ and this a_i is fixed itself by δ—a contradiction. Finally, choose a point $b \notin A$. The stabiliser $E_{a_1,b}$ has order at least $\frac{n^2}{2n} = \frac{1}{2}n > 2$, and thus is non-trivial. But then the stabilizer in E of the line through a_1 and b is also non-trivial in contradiction to the regularity of E on $\mathcal{L}_s$ by Lemma 4. This finally shows that the case $n > 4$ even and $N \neq E$ cannot occur. We therefore always have $N = E$, and the statement follows. □

Theorem 5 *If the kernel of $\mathcal{B}$ has maximal order $(n-1)n^2$, then $\mathcal{B}$ is an elation 2-affine plane, the projective completion $\mathcal{P}$ of $\mathcal{B}$ is Desarguesian, and $\mathcal{B}$ is isomorphic to $\mathcal{B}_{\mathsf{PG}(2,n)}$.*

Proof Let $\mathcal{B}$ be a finite type II 2-affine plane of order $n \geq 5$ such that $|\Delta| = (n-1)n^2$. Then $\mathcal{B}$ is of affine type. Since there are n^2 short lines, Corollary 2 shows that Δ is transitive on $\mathcal{L}_s$ and that $|\Delta_L| = n-1$ for each short line L. As in the proof of Lemma 4 for the group $E^{(B)}$, and because now $B = \mathcal{L}_s$ so that $E = E^{(\mathcal{L}_s)}$, one finds that Δ is a Frobenius group operating on $\mathcal{L}_s$, that the elation group E is the Frobenius kernel acting regularly on $\mathcal{L}_s$ and that $\Delta = E \cdot \Delta_L$. In particular, $\mathcal{B}$ is an elation 2-affine plane. Hence, by Theorem 4, n is a prime power and $\mathcal{P}$ is a dual translation plane with translation centre ω, that is, $\mathcal{P}$ is (ω, ω)-transitive.

By Lemma 2 each collineation in Δ_L extends to a homology of $\mathcal{P}$ with centre ω and axis $\tilde{L}$. Since Δ_L has order $n-1$, we see that Δ_L acts transitively on $[x] \setminus \{x\}$ where $x \in L$. Thus $\mathcal{P}$ is $(\omega, \tilde{L})$-transitive. Hence, $\mathcal{P}$ is $(\omega, \tilde{K})$-transitive for any line $\tilde{K}$ of $\mathcal{P}$. The dualisation of [12, 3.5.49] shows that $\mathcal{P}$ is Desarguesian. □

5 The 2-affine planes of type II of order at most 4

As mentioned in Sect. 2, a 2-affine plane of order 2 cannot be of type II.

From the pictorial representation of the type II 2-affine plane $\mathcal{B}_{\mathsf{PG}(2,3)}$ of order 3 in Fig. 3 one sees that the kernel and collineation group of $\mathcal{B}_{\mathsf{PG}(2,3)}$ are isomorphic to $S_3 \times S_3$ and $(S_3 \times S_3) \rtimes C_2$, respectively, and thus have order 36 and 72. A collineation δ in the kernel that fixes each point of one long line A and exactly one point c on the other long line, fixes a short line L through c and a point $x \notin L$. Hence, $\Delta_{L,x}$ is non-trivial. In fact, this stabilizer is generated by δ and has order 2.

Note that δ extends to a homology of $\mathsf{PG}(2,3)$ with centre c and axis $\tilde{A}$—a different behaviour than the one postulated in Lemma 2. On the other hand, Lemma 2 remains valid if the collineation in Δ_L fixes exactly the two points on L.

There are two type II 2-affine planes of order 4. One is the 2-affine plane $\mathcal{B}_{\mathsf{PG}(2,4)}$ and the other is the Shrikhande plane $\mathcal{S}$. Both planes have geometric representations in terms of embeddings in $\mathsf{PG}(2,4)$ and $\mathsf{PG}(2,5)$, respectively. We give an algebraic description of the two planes, which then will explain the labels attached to points in Figs. 2 and 4.

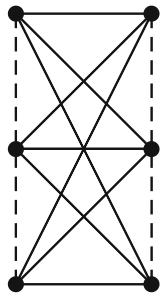

Fig. 3 The 2-affine plane of type II and order 3. The two long lines are *dashed*

Let $(G, +)$ be a group of order 4 with identity element 0 and let $P = G \times \{1, 2, 3\}$. Then P has 12 points and we describe a 2-affine plane $\mathcal{B}_G$ of type II on P. The long lines of $\mathcal{B}_G$ are $G \times \{i\}$ where $i \in \{1, 2, 3\}$. Moreover, $\{(g_1, 1), (g_2, 2), (g_3, 3)\}$ is a short line of $\mathcal{B}_G$ where $g_i \in G$, $i = 1, 2, 3$, if and only if $g_1 + g_2 + g_3 = 0$. For brevity we write this line as (g_1, g_2, g_3).

From this definition it is clear that $\mathcal{B}_G$ is a non-trivial linear space, each long line meets each short line in exactly one point, and the long lines partition P. Given a short line $L = (g_1, g_2, g_3)$ and a point $p = (h_i, i)$ (where $i \in \{1, 2, 3\}$ and $h_i \in G$) not on L (so that $h_i \neq g_i$), a line (h_1, h_2, h_3) is a parallel to L through p if and only if $h_j \neq g_j, g_j + g_i - h_i$ where $i \neq j \in \{1, 2, 3\}$. Since $h_1 + h_2 + h_3 = 0$, having chosen h_j uniquely determines h_k where $\{i, j, k\} = \{1, 2, 3\}$. Note that this condition is independent of the choice of $j \neq i$ because then also $h_k \neq g_k, g_k + g_i - h_i$. Since we have two choices for h_j in G (as $g_j \neq g_j + g_i - h_i$), we see that there are precisely two short lines through p that are parallel to L. Thus we have shown.

Proposition 1 *$\mathcal{B}_G$ is a 2-affine plane of type II and order 4.*

Up to isomorphism there are two groups of order 4, the cyclic group C_4 and the Klein 4-group $C_2 \times C_2$. As indicated in Sect. 2, the Shrikhande plane $\mathcal{S}$ is isomorphic to $\mathcal{B}_{C_4}$. Furthermore, $\mathcal{B}_{\mathsf{PG}(2,4)}$ is isomorphic to $\mathcal{B}_{C_2 \times C_2}$. Pictorial representations of these planes are given in Figs. 2 and 4 where points are labelled by group elements.

Proposition 2 *$\mathcal{B}_G$ is an elation 2-affine plane. The kernel $\Delta_{\mathcal{B}_G}$ of $\mathcal{B}_G$ is the semi-direct product of $G \times G$ by the automorphism group* $\mathrm{Aut}(G)$ *of G and the collineation group $\Gamma_{\mathcal{B}_G}$ of $\mathcal{B}_G$ is the semi-direct product of $\Delta_{\mathcal{B}_G}$ by S_3.*

Proof Let $\pi \in S_3$ be a permutation of the index set $\{1, 2, 3\}$. Then the map γ_π defined by $\gamma_\pi((g, i)) = (g, \pi(i))$ is a collineation of $\mathcal{B}_G$; a line (g_1, g_2, g_3) is taken to the line $(g_{\pi^{-1}(1)}, g_{\pi^{-1}(2)}, g_{\pi^{-1}(3)})$ by γ_π. Clearly, each γ_π fixes the line $(0, 0, 0)$ and the collection $\{\gamma_\pi \mid \pi \in S_3\}$ is a subgroup of the stabilizer of $(0, 0, 0)$ in $\Gamma_{\mathcal{B}_G}$ and acts 3-transitively on the points of $(0, 0, 0)$.

Let $h_1, h_2 \in G$ and define $h_3 = -h_1 - h_2$. Then the map γ_{h_1,h_2} defined by $\gamma_{h_1,h_2}((g, i)) = (g + h_i, i)$ is a collineation of $\mathcal{B}_G$; a line (g_1, g_2, g_3) is taken to the line $(g_1 + h_1, g_2 + h_2, g_3 + h_3)$ by γ_{h_1,h_2}. Clearly, γ_{h_1,h_2} fixes no line unless $h_1 = h_2 = 0$, that is, $\gamma_{h_1,h_2} = id$. It readily follows that the collection $E_G = \{\gamma_{h_1,h_2} \mid h_1, h_2 \in G\}$ is a subgroup of $\Delta_{\mathcal{B}_G}$ of order 16 that acts regularly on the set of short lines of $\mathcal{B}_G$. Thus E_G is the elation group of $\mathcal{B}_G$. In particular, $\mathcal{B}_G$ is an elation 2-affine plane.

Let $\alpha \in \mathrm{Aut}(G)$ be an automorphism of G. Then the map γ_α defined by $\gamma_\alpha((g, i)) = (\alpha(g), i)$ is a collineation of $\mathcal{B}_G$; a line (g_1, g_2, g_3) is taken to the line $(\alpha(g_1), \alpha(g_2), \alpha(g_3))$

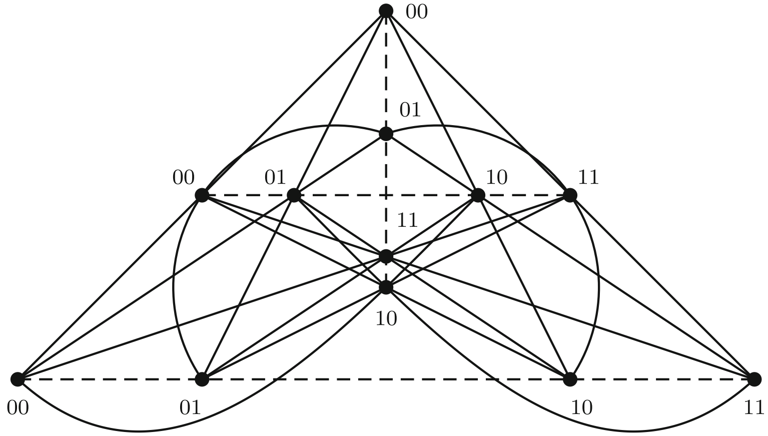

Fig. 4 The 2-affine plane $\mathcal{B}_{\mathsf{PG}(2,4)} \cong \mathcal{B}_{C_2 \times C_2}$. The three lines of length 4 are *dashed*. Four lines of length 3 are shown as *circular/cubic arcs*

by γ_α. Clearly, γ_α belongs to the kernel of $\mathcal{B}_G$ and fixes the line $(0, 0, 0)$. Moreover, $\{\gamma_\alpha \mid \alpha \in \mathrm{Aut}(G)\}$ is a subgroup of the stabilizer of $(0, 0, 0)$ in $\Delta_{\mathcal{B}_G}$.

Now, let γ be a collineation of $\mathcal{B}_G$. By using a collineation γ_π where $\pi \in S_3$ we may assume that γ fixes each long line, that is, γ belongs to $\Delta_{\mathcal{B}_G}$. We can now apply a collineation γ_{h_1,h_2} in the elation group in order to ensure that γ fixes the line $(0, 0, 0)$. Under these assumptions each σ_i, $i = 1, 2, 3$, defined by $\sigma_i(g) = \gamma((g, i))$ is a permutation of G that fixes 0. Since γ is a collineation one obtains that $\sigma_1(g_1) + \sigma_2(g_2) + \sigma_3(g_3) = 0$ for all $g_1, g_2, g_3 \in G$ such that $g_1 + g_2 + g_3 = 0$. Hence

$$-\sigma_3(-g_1 - g_2) = \sigma_1(g_1) + \sigma_2(g_2),$$

for all $g_1, g_2 \in G$. When $g_1 = g$ and $g_2 = 0$ one finds that $-\sigma_3(-g) = \sigma_1(g)$ for all $g \in G$. One similarly obtains $-\sigma_3(-g) = \sigma_2(g)$ for all $g \in G$. Hence $\sigma_1 = \sigma_2$. Then

$$\sigma_1(g_1 + g_2) = -\sigma_3(-g_1 - g_2) = \sigma_1(g_1)\sigma_1(g_2)$$

for all $g_1, g_2 \in G$. Thus σ_1 is an automorphism of G, and so $\sigma_3(g) = -\sigma_1(-g) = \sigma_1(g)$. This shows that $\gamma = \gamma_{\sigma_1}$.

In summary, we have found that $\Gamma_{\mathcal{B}_G}$ is generated by all the γ_π, γ_{h_1,h_2} and γ_α. Obviously, $\Gamma_{\mathcal{B}_G}/\Delta_{\mathcal{B}_G} \cong S_3$ and $\Delta_{\mathcal{B}_G}/E_{\mathcal{B}_G} \cong \mathrm{Aut}(G)$. □

Corollary 3 *The collineation groups of $\mathcal{S}$ and $\mathcal{B}_{\mathsf{PG}(2,4)}$ are of order 192 and 576, respectively. The respective orders of the kernels of these planes are 32 and 96.*

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