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An upper bound of the value of t of the support t-designs of extremal binary doubly even self-dual codes

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Abstract Let C be an extremal binary doubly even self-dual code of length n and D_w be the support design of C for a weight w. We introduce the two numbers $\delta(C)$ and $s(C)$: $\delta(C)$ is the largest integer t such that, for all wight, D_w is a t-design; $s(C)$ denotes the largest integer t such that there exists a w such that D_w is a t-design. In this paper, we consider the possible values of $\delta(C)$ and $s(C)$.

1 Introduction

Let C be an extremal binary doubly even self-dual code (Type II code) of length n. Mallows et al. [10] showed that C does not exist for all sufficiently large n. In 1999, it was shown by Zhang [11] that C does not exist if $n = 24m$ ($m \geq 154$), $24m + 8$ ($m \geq 159$), $24m + 16$ ($m \geq 164$). A t-(v, k, λ) design is a pair $\mathcal{D} = (X, \mathcal{B})$, where X is a set of points of cardinality v, and $\mathcal{B}$ a collection of k-element subsets of X called blocks, with the property that any t points are contained in precisely λ blocks. It follows that every i-subset of points ($i \leq t$) is contained in exactly $\lambda_i = \lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}$ blocks. The support $\mathrm{supp}(c)$ of a codeword $c = (c_1, \ldots, c_n) \in C$ is the set of indices of its nonzero coordinates: $\mathrm{supp}(c) = \{i : c_i \neq 0\}$. The support design of

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C for a given nonzero weight w ($w \equiv 0 \pmod 4$ and $4\lfloor n/24 \rfloor + 4 \le w \le n - (4\lfloor n/24 \rfloor + 4)$) is the design for which the points are the n coordinate indices, and the blocks are the supports of all codewords of weight w. Let D_w be the support design of C for a weight w. Then it is known from the Assmus–Mattson theorem [2] that, for all w, D_w is a 5-, 3- and 1-design for $n = 24m, 24m + 8$ and $24m + 16$, respectively.

Let

$$\delta(C) := \max\{t \in \mathbb{N} \mid \forall w; \ D_w \text{ is a } t\text{-design}\},$$
$$s(C) := \max\{t \in \mathbb{N} \mid \exists w; \ \text{s.t. } D_w \text{ is a } t\text{-design}\}.$$

Note that $\delta(C) \le s(C)$.

In this paper, we consider the possible values of $\delta(C)$ and $s(C)$. Our motivations are the following problems.

Problem 1.1 Find an upper bound of $s(C)$.

Problem 1.2 Does the case occur for $\delta(C) < s(C)$?

First, we provide some known results for $\delta(C)$ and $s(C)$. The following theorem gives the lower bound of $\delta(C)$ due to Janusz [9].

Theorem 1.3 *Let C be an extremal binary doubly even self-dual code of length $n = 24m+8r$, $r = 0, 1$ or 2. Then either $\delta(C) \ge 7 - 2r$, or $\delta(C) = 5 - 2r$ and there is no nontrivial weight w such that D_w holds a $(1 + \delta(C))$-design.*

We collect some known results for the support t-design of the minimum weight. Let D_{4m+4}^{24m} be the support t-design of the minimum weight of an extremal binary doubly even self-dual $[24m, 12m, 4m + 4]$ code. By the Assmus–Mattson theorem, D_{4m+4}^{24m} is a $5\text{-}\left(24m, 4m + 4, \binom{5m-2}{m-1}\right)$ design. Suppose that D_{4m+4}^{24m} is a $t\text{-}(24m, 4m + 4, \lambda_t)$ design with $t \ge 6$. Then $\lambda_t = \binom{5m-2}{m-1}\binom{4m-1}{t-5}/\binom{24m-5}{t-5}$ is a nonnegative integer. It is known that if D_{4m+4}^{24m} is a 6-design, then it is a 7-design by a strengthening of the Assmus–Mattson theorem [5]. In 2006, Bannai et al. [4] showed that D_{4m+4}^{24m} is never a 9-design. In [7,8], we showed that D_{4m+4}^{24m} is never an 8-design.

We investigate the support designs of the non minimum weights and as a corollary, we have upper bound of $s(C)$. This paper is organized as follows. In Sect. 2, we recall the definition and some properties of the harmonic weight enumerators, which were introduced in [3], and which will be our main tool to study the support designs for the non minimum weights. In particular, we will use the fact that the harmonic weight enumerators of Type II codes are related to the invariant rings of some finite subgroups of $GL(2, \mathbb{C})$. By using these facts, we extend the methods of Bachoc [3] and Bannai et al. [4].

In Sect. 3, a proof of our result is given for each cases of lengths $24m, 24m + 8$ and $24m + 16$. By using the methods of the harmonic weight enumerators, we give the results of the support designs for any weights in Propositions 3.1, 3.3 and 3.6. Then, by using these propositions, we apply our previous results [7,8] of the minimum weight to the non minimum weights. Thus our result is the following theorem.

Theorem 1.4 *Let C be an extremal binary doubly even self-dual code of length n.*

(1) *If $n = 24m$, then $\delta(C) = s(C) = 5$ or $\delta(C) = s(C) = 7$.*
(2) *If $n = 24m + 8$, then $\delta(C) = s(C) = 3$ or $5 \le \delta(C) \le s(C) \le 7$.*

(3) If $n = 24m + 16$, then $\delta(C) = s(C) = 1$ or $3 \leq \delta(C) \leq s(C) \leq 5$.

It is still unknown whether $\delta(C) = s(C) = 7$ (in case (1)) or $5 \leq \delta(C) \leq s(C) \leq 7$ (in case (2)) or $3 \leq \delta(C) \leq s(C) \leq 5$ (in case (3)), actually occurs or not. It is an interesting open problem to determine the existence of such examples. We give the details for each m of the above theorem in Sects. 3.1, 3.2 and 3.3.

For Problem 1.1, we conclude that $s(C) \leq 7$ for any extremal Type II code C. For Problem 1.2, if $n = 24m$, we see that $\delta(C) < s(C)$ does not occur by (1) of Theorem 1.4. There is no known example of $\delta(C) < s(C)$. In the process of proving Theorem 1.4, we will see that, if $\delta(C) < s(C)$ occurs, it can only happen in a limited number of cases listed in the following proposition.

Proposition 1.5 *If the case $\delta(C) < s(C)$ occurs, then one of the following holds:*

(1) $n = 24m + 8$, $m = 58$, $\delta(C) = 6$ *and* $s(C) = 7$ *with* $w = n/2$;
(2) $n = 24m + 16$, $m \in \{10, 23, 79, 93, 118, 120, 123, 125, 142\}$, $\delta(C) = 4$ *and* $s(C) = 5$ *with* $w = n/2$.

2 Harmonic weight enumerators

2.1 Harmonic weight enumerators

In this section, we extend a method of the harmonic weight enumerators which were used by Bachoc [3] and Bannai et al. [4]. For the reader's convenience we quote the definitions and properties of discrete harmonic functions from [3,6] (for more information the reader is referred to [3,6]).

Let $\Omega = \{1, 2, \ldots, n\}$ be a finite set (which will be the set of coordinates of the code) and let X be the set of its subsets, while, for all $k = 0, 1, \ldots, n$, X_k is the set of its k-subsets. We denote by $\mathbb{R}X$, $\mathbb{R}X_k$ the free real vector spaces spanned by respectively the elements of X, X_k. An element of $\mathbb{R}X_k$ is denoted by

$$f = \sum_{z \in X_k} f(z)z$$

and is identified with the real-valued function on X_k given by $z \mapsto f(z)$.

Such an element $f \in \mathbb{R}X_k$ can be extended to an element $\tilde{f} \in \mathbb{R}X$ by setting, for all $u \in X$,

$$\tilde{f}(u) = \sum_{z \in X_k, z \subset u} f(z).$$

If an element $g \in \mathbb{R}X$ is equal to some $\tilde{f}$, for $f \in \mathbb{R}X_k$, we say that g has degree k. The differentiation γ is the operator defined by linearity from

$$\gamma(z) = \sum_{y \in X_{k-1}, y \subset z} y$$

for all $z \in X_k$ and for all $k = 0, 1, \ldots n$, and Harm_k is the kernel of γ:

$$\text{Harm}_k = \ker(\gamma|_{\mathbb{R}X_k}).$$

Theorem 2.1 ([6]). *A set $\mathcal{B} \subset X_k$ of blocks is a t-design if and only if $\sum_{b \in \mathcal{B}} \tilde{f}(b) = 0$ for all $f \in \text{Harm}_k$, $1 \leq k \leq t$.*

In [3], the harmonic weight enumerator associated to a binary linear code C was defined as follows:

Definition 2.2 Let C be a binary code of length n and let $f \in \text{Harm}_k$. The harmonic weight enumerator associated to C and f is

$$W_{C,f}(x, y) = \sum_{c \in C} \tilde{f}(c) x^{n-\text{wt}(c)} y^{\text{wt}(c)}.$$

Let G be the subgroup of $\text{GL}(2, \mathbb{C})$ generated by elements

$$T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

We consider the group $G = \langle T_1, T_2 \rangle$ together with the characters χ_k defined by

$$\chi_k(T_1) = (-1)^k, \quad \chi_k(T_2) = i^{-k}.$$

We denote by $I_G = \mathbb{C}[x, y]^G$ the ring of polynomial invariants of G and by I_{G,χ_k} the ring of relative invariants of G with respect to the character χ_k. Let $P_8 = x^8 + 14x^4 y^4 + y^8$, $P_{12} = x^2 y^2 (x^4 - y^4)^2$, $P_{18} = xy(x^8 - y^8)(x^8 - 34x^4 y^4 + y^8)$, $P_{24} = x^4 y^4 (x^4 - y^4)^4$, $P_{30} = P_{12} P_{18}$ and

$$I_{G,\chi_k} = \begin{cases} \langle P_8, P_{24} \rangle & \text{if } k \equiv 0 \pmod 4 \\ P_{12} \langle P_8, P_{24} \rangle & \text{if } k \equiv 2 \pmod 4 \\ P_{18} \langle P_8, P_{24} \rangle & \text{if } k \equiv 3 \pmod 4 \\ P_{30} \langle P_8, P_{24} \rangle & \text{if } k \equiv 1 \pmod 4 \end{cases}.$$

Then the structure of these invariant rings is described as follows:

Theorem 2.3 ([3]). *Let C be an extremal binary doubly even self-dual code of length n, and let $f \in \text{Harm}_k$. Then we have $W_{C,f}(x, y) = (xy)^k Z_{C,f}(x, y)$. Moreover, the polynomial $Z_{C,f}(x, y)$ is of degree $n - 2k$ and is in I_{G,χ_k}, the space of the relative invariants of G with respect to the character χ_k.*

We recall the slightly more general definition of the notion of a T-design, for a subset T of $\{1, 2, \ldots, n\}$: a set $\mathcal{B}$ of blocks is called a T-design if and only if $\sum_{b \in \mathcal{B}} \tilde{f}(b) = 0$ for all $f \in \text{Harm}_k$ and for all $k \in T$. By Theorem 2.1, a t-design is a $T = \{1, \ldots, t\}$-design. Let $W_{C,f} = \sum_{i=0}^{n} c_f(i) x^{n-i} y^i$. Then we note that D_w is a T-design if and only if $c_f(w) = 0$ for all $f \in \text{Harm}_j$ with $j \in T$. The following theorem is called a strengthening of the Assmus–Mattson theorem.

Theorem 2.4 ([5]). *Let D_w be the support design of an extremal binary doubly even self-dual code of length n.*

- *If $n \equiv 0 \pmod{24}$, D_w is a $\{1, 2, 3, 4, 5, 7\}$-design.*
- *If $n \equiv 8 \pmod{24}$, D_w is a $\{1, 2, 3, 5\}$-design.*
- *If $n \equiv 16 \pmod{24}$, D_w is a $\{1, 3\}$-design.*

We remark that Bachoc gave an alternative proof of a strengthening of the Assmus–Mattson theorem in [3, Theorem 4.2].

2.2 Harmonic weight enumerators of extremal Type II codes

In this section, we give the explicit description of the harmonic weight enumerators of extremal Type II codes for the particular cases, which will be needed in the proof of our theorems in Sect. 3. We set $n = 24m + 8r$ the length of a code C.

Case $t = 4$ and $r = 2$ Let us assume that $t = 4$, and C is an extremal binary doubly even self-dual code of length $n = 24m + 16$. Then by the Theorem 2.3 we have $W_{C,f}(x, y) = c(f)(xy)^4 Z_{C,f}(x, y)$, where $c(f)$ is a linear function from Harm$_t$ to $\mathbb{R}$ and $Z_{C,f}(x, y) \in I_{G,\chi_0}$. By Theorem 2.3, $Z_{C,f}(x, y)$ can be written in the following form:

$$Z_{C,f}(x, y) = \sum_{i=0}^{m} a_i P_8^{3(m-i)+1} P_{24}^i.$$

Since the minimum weight of C is $4m + 4$, we have $a_i = 0$ for $i \neq m$. Therefore, $W_{C,f}(x, y)$ can be written in the following form:

$$\begin{aligned} W_{C,f}(x, y) &= c(f)(xy)^4 P_8 P_{24}^m \\ &= c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m}(x^8 + 14x^4y^4 + y^8). \end{aligned} \tag{2.1}$$

The other cases are as follows.

Case $t = 5$ and $r = 2$ $W_{C,f}(x, y)$ can be written in the following form:

$$\begin{aligned} W_{C,f}(x, y) &= c(f)(xy)^5 Z_{C,f}(x, y) \\ &= c(f)(xy)^5 P_{30} \sum_{i=0}^{m-1} a_i P_8^{3(m-i)-3} P_{24}^i. \end{aligned}$$

If C is extremal, then

$$\begin{aligned} W_{C,f}(x, y) &= c(f)(xy)^5 P_{30} P_{24}^{m-1} \\ &= c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-1}(x^4 + y^4) \ (x^8 - 34x^4y^4 + y^8). \end{aligned} \tag{2.2}$$

Case $t = 6$ and $r = 1, 2$ $W_{C,f}(x, y)$ can be written in the following form:

$$\begin{aligned} W_{C,f}(x, y) &= c(f)(xy)^6 Z_{C,f}(x, y) \\ &= c(f)(xy)^6 P_{12} \sum_{i=0}^{m-1} a_i P_8^{3(m-i)-3+r} P_{24}^i. \end{aligned}$$

If C is extremal, then

$$\begin{aligned} W_{C,f}(x, y) &= c(f)(xy)^6 P_{12} P_8^r P_{24}^{m-1} \\ &= c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-2}(x^8 + 14x^4y^4 + y^8)^r. \end{aligned} \tag{2.3}$$

Case $t = 7$ and $r = 1$ $W_{C,f}(x, y)$ can be written in the following form:

$$\begin{aligned} W_{C,f}(x, y) &= c(f)(xy)^7 Z_{C,f}(x, y) \\ &= c(f)(xy)^7 P_{18} \sum_{i=0}^{m-1} a_i P_8^{3(m-i)-3} P_{24}^i. \end{aligned}$$

If C is extremal, then

$$W_{C,f}(x,y) = c(f)(xy)^7 P_{18} P_{24}^{m-1}$$
$$= c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-3}(x^4 + y^4) \; (x^8 - 34x^4y^4 + y^8). \quad (2.4)$$

Case $t = 8$ and $r = 0, 1$ $W_{C,f}(x,y)$ can be written in the following form:

$$W_{C,f}(x,y) = c(f)(xy)^8 Z_{C,f}(x,y)$$
$$= c(f)(xy)^8 \sum_{i=0}^{m-1} a_i P_8^{3(m-i)-2+r} P_{24}^i.$$

If C is extremal, then

$$W_{C,f}(x,y) = c(f)(xy)^8 P_8^{r+1} P_{24}^{m-1}$$
$$= c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-4}(x^8 + 14x^4y^4 + y^8)^{r+1}. \quad (2.5)$$

2.3 Coefficients of the harmonic weight enumerators of extremal Type II codes

As we mentioned in Sect. 2.1, it is important for the support designs of a code C whether the coefficients of $W_{C,f}(x,y)$ are zero or not. Therefore, we investigate it and show the following lemmas.

Lemma 2.5 *Let $Q = (x^4 - y^4)^\alpha (x^8 + 14x^4y^4 + y^8)^\beta$ with $0 \le \alpha \le 652$ and $\beta = 1, 2$.*

(1) *In the case $\beta = 1$, if the coefficients of $(x^4)^{\alpha+2-i}(-y^4)^i$ in Q are equal to 0 for $0 \le i \le \frac{\alpha+2}{2}$, then $(\alpha, i) = (14, 1), (223, 15)$.*

(2) *In the case $\beta = 2$, the coefficients of $(x^4)^{\alpha+4-i}(-y^4)^i$ in Q are equal to 0 for $0 \le i \le \frac{\alpha+4}{2}$, then $(\alpha, i) = (28, 1)$.*

Proof We have checked numerically using computer.

We note that C does not exist if $n = 24m$ ($m \ge 154$), $24m + 8$ ($m \ge 159$), $24m + 16$ ($m \ge 164$). Then $0 \le \alpha \le 652$ satisfy the condition for $m \le 163$ in Eqs. 2.1–2.5. $\quad \square$

Lemma 2.6 *Let $R = (x^4 - y^4)^\alpha (x^4 + y^4)(x^8 - 34x^4y^4 + y^8)$ with $1 \le \alpha \le 652$. If the coefficients of $(x^4)^{\alpha+3-i}(-y^4)^i$ in R are equal to 0 for $0 \le i \le \frac{\alpha+3}{2}$, then $\alpha = 2i - 3$.*

Proof We have checked numerically using computer. $\quad \square$

3 Proof of Theorems

3.1 Case for $n = 24m$

In this section, we consider the case of length $n = 24m$. Let D_w^{24m} be the support t-design of weight w of an extremal binary doubly even self-dual $[24m, 12m, 4m + 4]$ code ($m \le 153$). By Theorem 1.3 and [7,8, Theorem 1.1], we remark that if there exists w' such that $D_{w'}^{24m}$ becomes a 6-design, then D_w^{24m} is a 7-design for any w, and m must be in the set $\{15, 52, 55, 57, 59, 60, 63, 90, 93, 104, 105, 107, 118, 125, 127, 135, 143, 151\}$.

For $t \ge 8$, we give the following proposition.

Proposition 3.1 *Let D_w^{24m} be the support t-design of weight w of an extremal binary doubly even self-dual code of length $n = 24m$. Then all D_w^{24m} are 8-designs simultaneously, or none of D_w^{24m} is an 8-design.*

Proof If $r = 0$ in the Eq. 2.5, we have

$$W_{C,f}(x, y) = c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-4}(x^8 + 14x^4y^4 + y^8).$$

We recall that C does not exist if $n = 24m$ $(m \geq 154)$ [10,11]. By Lemma 2.5 (1), the coefficients of x^i with $i \equiv 0$ (mod 4) and $4m + 4 \leq i \leq n - (4m + 4)$ are all nonzero if $c(f) \neq 0$ or zero if $c(f) = 0$ for $m \leq 153$. Therefore, all D_w^{24m} are 8-designs simultaneously, or none of D_w^{24m} is an 8-design.   □

We apply [7,8, Theorem 1.1] to Proposition 3.1. In [7,8, Theorem 1.1], we showed that D_{4m+4}^{24m} cannot be an 8-design, so we obtain the following theorem.

Theorem 3.2 *D_w^{24m} is never an 8-design for any w.*

Thus the proof of Theorem 1.4 (1) is completed.

3.2 Case for $24m + 8$

In this section, we state the cases of length $n = 24m + 8$. Let D_w^{24m+8} be the support t-design of weight w of an extremal binary doubly even self-dual $[24m + 8, 12m + 4, 4m + 4]$ code $(m \leq 158)$. By Theorem 1.3 and [7,8, Theorem 4.3(1)], we remark that if there exists w' such that $D_{w'}^{24m+8}$ becomes a 4-design, then D_w^{24m+8} is a 5-design for any w, and m must be in the set $\{15, 35, 45, 58, 75, 85, 90, 95, 113, 115, 120, 125\}$.

For $t \geq 6$, we give the following proposition. We call w the middle weight if $w = n/2$.

Proposition 3.3 *Let D_w^{24m+8} be the support t-design of weight w of an extremal binary doubly even self-dual code of length $n = 24m + 8$.*

(1) (i) *Assume that $m \neq 4$. Then all D_w^{24m+8} are 6-designs simultaneously, or none of D_w^{24m+8} is a 6-design.*
   (ii) *Assume that $m = 4$.*
      Then D_w^{104} is a $\begin{cases} \{1, 2, 3, 5\}\text{-design} & \text{if } w \neq 24 \\ \{1, 2, 3, 5, 6\}\text{-design} & \text{if } w = 24. \end{cases}$
(2) (i) *Assume that $w \neq 12m + 4$. Then all D_w^{24m+8} are 7-designs simultaneously, or none of D_w^{24m+8} is a 7-design.*
   (ii) *D_{12m+4}^{24m+8} is a $\{1, 2, 3, 5, 7\}$-design.*
(3) (i) *Assume that $m \neq 8$. Then all D_w^{24m+8} are 8-designs simultaneously, or none of D_w^{24m+8} is an 8-design.*
   (ii) *Assume that $m = 8$.*
      Then D_w^{200} is a $\begin{cases} \{1, 2, 3, 5\}\text{-design} & \text{if } w \neq 40 \\ \{1, 2, 3, 5, 8\}\text{-design} & \text{if } w = 40. \end{cases}$

Proof (1) If $r = 1$ in the Eq. 2.3, we have

$$W_{C,f}(x, y) = c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-2}(x^8 + 14x^4y^4 + y^8).$$

By Lemma 2.5 (1), if $m \neq 4$, the coefficients of x^i with $i \equiv 0$ (mod 4) and $4m + 4 \leq i \leq n - (4m + 4)$ are all nonzero or zero at the same time. Therefore, if $m \neq 4$, all D_w^{24m+8} are 6-designs simultaneously, or none of D_w^{24m+8} is a 6-design.

Let $m = 4$. By Lemma 2.5 (1), if $i \neq 24$, the coefficients of x^i with $i \equiv 0$ (mod 4) and $20 \leq i \leq 84$ are all nonzero or zero at the same time. Also, the coefficient of x^{24} is equals to 0. Therefore, if $w \neq 24$, D_w^{104} is a $\{1, 2, 3, 5\}$-design. Also, D_{24}^{104} is a $\{1, 2, 3, 5, 6\}$-design.

(2) By the Eq. 2.4, we have

$$W_{C,f}(x, y) = c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-3}(x^4 + y^4)(x^8 - 34x^4y^4 + y^8).$$

By Lemma 2.6, if $i \neq 12m + 4$, the coefficients of x^i with $i \equiv 0 \pmod 4$ and $4m + 4 \leq i \leq n - (4m + 4)$ are all nonzero or zero at the same time. Therefore, if $w \neq 12m + 4$, then all D_w^{24m+8} are 7-designs simultaneously, or none of D_w^{24m+8} is a 7-design.

We consider the case that w is the middle weight. By Lemma 2.6, the coefficient of x^{12m+4} is equals to 0. Hence D_{12m+4}^{24m+8} is a $\{1, 2, 3, 5, 7\}$-design.

(3) If $r = 1$ in the Eq. 2.5, we have

$$W_{C,f}(x, y) = c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-4}(x^8 + 14x^4y^4 + y^8)^2.$$

By Lemma 2.5 (2), if $m \neq 8$, the coefficients of x^i with $i \equiv 0 \pmod 4$ and $4m + 4 \leq i \leq n - (4m + 4)$ are all nonzero or zero at the same time. Therefore, if $m \neq 8$, then all D_w^{24m+8} are 8-designs simultaneously, or none of D_w^{24m+8} is an 8-design.

Let $m = 8$. By Lemma 2.5 (2), if $i \neq 40$, the coefficients of x^i with $i \equiv 0 \pmod 4$ and $36 \leq i \leq 164$ are all nonzero or zero at the same time. Also, the coefficient of x^{40} is equals to 0. Therefore, if $w \neq 40$, D_w^{200} is a $\{1, 2, 3, 5\}$-design. Also, D_{40}^{200} is a $\{1, 2, 3, 5, 8\}$-design.   □

Remark 3.4 In Lemma 2.5 (1), the solution $(\alpha, i) = (223, 15)$ corresponds to the polynomial $Q = (x^4 - y^4)^{223}(x^8 + 14x^4y^4 + y^8)$. In the case $t = 9$ and $r = 1$, if C is extremal, then the harmonic weight enumerator is

$$\begin{aligned} W_{C,f}(x, y) &= c(f)(xy)^9 P_{30} P_8 P_{24}^{m-2} \\ &= c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-5}(x^4 + y^4) \\ &\quad \times (x^8 + 14x^4y^4 + y^8)(x^8 - 34x^4y^4 + y^8). \end{aligned} \tag{3.1}$$

The polynomial Q is contained in the case of $m = 57$ in the Eq. 3.1. By a computation, the coefficients of x^i in the Eq. 3.1 with $i \equiv 0 \pmod 4$ and $4m + 4 \leq i \leq n - (4m + 4)$ are not equal to 0. Thus the solution $(\alpha, i) = (223, 15)$ does not give a design.

We apply [7,8, Theorem 4.3(1)] to Proposition 3.3. In [7,8, Theorem 4.3(1)], we showed the following: if D_{4m+4}^{24m+8} becomes a 6-design, then m must be 58; if D_{4m+4}^{24m+8} becomes a 7-design, then m must be 58; D_{4m+4}^{24m+8} cannot be an 8-design, so we obtain the following theorem.

Theorem 3.5 *Let D_w^{24m+8} be the support t-design of weight w of an extremal binary doubly even self-dual $[24m + 8, 12m + 4, 4m + 4]$ code ($m \leq 158$).*

(1) (i) *In the case $w \neq 12m + 4$. If D_w^{24m+8} becomes a 6-design, then m must be 58. If D_w^{24m+8} becomes a 7-design, then m must be 58.*

(ii) *In the case $w = 12m + 4$. If D_{12m+4}^{24m+8} becomes a 6-design, then D_{12m+4}^{24m+8} becomes a 7-design and m must be 58.*

(2) *D_w^{24m+8} is never an 8-design for any w.*

Thus the proof of Theorem 1.4 (2) is completed.

3.3 Case for $24m + 16$

In this section, we state the cases of length $n = 24m + 16$. Let D_w^{24m+16} be the support t-design of weight w of an extremal binary doubly even $[24m + 16, 12m + 8, 4m + 4]$ code ($m \leq 163$). By Theorem 1.3 and [7,8, Theorem 4.3 (2)], we remark that if there exists w' such that $D_{w'}^{24m+16}$ becomes a 2-design, then D_w^{24m+16} is a 3-design for any w, and m must be in the set $\{5, 10, 20, 23, 25, 35, 44, 45, 50, 55, 60, 70, 72, 75, 79, 80, 85, 93, 95, 110, 118, 120, 121, 123, 125, 130, 142, 144, 145, 149, 150, 155, 156, 157, 160, 163\}$.

For $t \geq 4$, we give the following proposition.

Proposition 3.6 *Let D_w^{24m+16} be the support t-design of weight w of an extremal binary doubly even self-dual code of length $n = 24m + 16$.*

(1) *All D_w^{24m+16} are 4-designs simultaneously, or none of D_w^{24m+16} is a 4-design.*
(2) (i) *Assume that $w \neq 12m + 8$. Then all D_w^{24m+16} are 5-designs simultaneously, or none of D_w^{24m+16} is a 5-design.*
     (ii) *D_{12m+8}^{24m+16} is a $\{1, 2, 3, 5\}$-design.*
(3) *All D_w^{24m+16} are 6-designs simultaneously, or none of D_w^{24m+16} is a 6-design.*

Proof (1) If $r = 2$ in the Eq. 2.1, we have

$$W_{C,f}(x, y) = c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m}(x^8 + 14x^4y^4 + y^8).$$

By Lemma 2.5 (1), the coefficients of x^i with $i \equiv 0 \pmod 4$ and $4m + 4 \leq i \leq n - (4m + 4)$ are all nonzero or zero at the same time. Therefore, all D_w^{24m+16} are 4-designs simultaneously, or none of D_w^{24m+16} is a 4-design.

(2) By the Eq. 2.2, we have

$$W_{C,f}(x, y) = c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-1}(x^4 + y^4)(x^8 - 34x^4y^4 + y^8).$$

By Lemma 2.6, if $i \neq 12m + 8$, the coefficients of x^i with $i \equiv 0 \pmod 4$ and $4m + 4 \leq i \leq n - (4m + 4)$ are all nonzero or zero at the same time. Therefore, if $w \neq 12m + 8$, then all D_w^{24m+16} are 5-designs simultaneously, or none of D_w^{24m+16} is a 5-design.

We consider the case that w is the middle weight. By Lemma 2.6, the coefficient of x^{12m+8} is equals to 0. Hence D_{12m+8}^{24m+16} is a $\{1, 2, 3, 5\}$-design.

(3) If $r = 2$ in the Eq. 2.3, we have

$$W_{C,f}(x, y) = c(f)x^{4m+4}y^{4m+4}(x^4 - y^4)^{4m-2}(x^8 + 14x^4y^4 + y^8)^2.$$

By Lemma 2.5 (2), the coefficients of x^i with $i \equiv 0 \pmod 4$ and $4m + 4 \leq i \leq n - (4m + 4)$ are all nonzero or zero at the same time. Therefore, all D_w^{24m+16} are 6-designs simultaneously, or none of D_w^{24m+16} is a 6-design.   □

We apply [7,8, Theorem 4.3 (2)] to Proposition 3.6. In [7,8, Theorem 4.3 (2)], we showed the following: if D_{4m+4}^{24m+16} becomes a 4-design, then m must be in the set $\{10, 23, 79, 93, 118, 120, 123, 125, 142\}$; if D_{4m+4}^{24m+16} becomes a 5-design, then m must be in the set $\{23, 79, 93, 118, 120, 123, 125, 142\}$; D_{4m+4}^{24m+16} cannot be a 6-design, so we obtain the following theorem.

Theorem 3.7 *Let D_w^{24m+16} be the support t-design of weight w of an extremal binary doubly even self-dual $[24m + 16, 12m + 8, 4m + 4]$ code ($m \leq 163$).*

(1) (i) *In the case $w \neq 12m + 8$. If D_w^{24m+16} becomes a 4-design, then m must be in the set* {10, 23, 79, 93, 118, 120, 123, 125, 142}. *If D_w^{24m+16} becomes a 5-design, then m must be in the set* {23, 79, 93, 118, 120, 123, 125, 142}.

  (ii) *In the case $w = 12m + 8$. If D_{12m+8}^{24m+16} becomes a 4-design, then D_{12m+8}^{24m+16} becomes a 5-design and m must be in the set* {10, 23, 79, 93, 118, 120, 123, 125, 142}.

(2) D_w^{24m+16} *is never a 6-design for any w.*

Thus the proof of Theorem 1.4 (3) is completed.

Remark 3.8 Let $\mathcal{D} = (X, \mathcal{B})$ be a t-design. The complementary design of $\mathcal{D}$ is $\bar{\mathcal{D}} = (X, \bar{\mathcal{B}})$, where $\bar{\mathcal{B}} = \{X \setminus B : B \in \mathcal{B}\}$. If $\mathcal{D} = \bar{\mathcal{D}}$, $\mathcal{D}$ is called a self-complementary design. Let $D_{n/2}$ be the support t-design of the middle weight of an extremal binary doubly even self-dual code of length n. It is easily seen that $D_{n/2}$ is self-complement.

Alltop [1] proved that if $\mathcal{D}$ is a t-design with an even integer t and self-complementary, then $\mathcal{D}$ is also a $(t + 1)$-design. Hence $D_{n/2}$ is a $\{1, 3, 5, \ldots, 2s + 1\}$-design. Thus Alltop's theorem gives an alternative proof of Propositions 3.3 (2) (ii) and 3.6 (2) (ii).

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