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New optimal [52, 26, 10] self-dual codes

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Abstract We classify up to equivalence all optimal binary self-dual [52, 26, 10] codes having an automorphism of order 3 with 10 fixed points. We achieve this using a method for constructing self-dual codes via an automorphism of odd prime order. We study also codes with an automorphism of order 3 with 4 fixed points. Some of the constructed codes have new values $\beta = 8, 9$, and 12 for the parameter in their weight enumerator.

Keywords Self-dual codes · Automorphism · Classification

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1 Introduction

A linear $[n, k]$ *code* C is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the finite field of q elements and q is a prime power. *The weight* of a codeword $v \in C$ (denoted by $\mathrm{wt}(v)$) is the number of the non-zero coordinates of v. The *minimum weight* d of C is the minimum nonzero weight of any codeword in C and the code is called an $[n, k, d]_q$ code. A matrix whose rows form a basis of C is called a *generator matrix* of this code. We denote a generator matrix of the code C by $\mathrm{gen}(C)$. For every $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in \mathbb{F}_2^n$, $u \cdot v = \sum_{i=1}^n u_i v_i$ defines the *inner product* in $\mathbb{F}_2^n$. The *dual code* of C is $C^\perp = \{v \in \mathbb{F}_2^n \mid u \cdot v = 0, \forall\, u \in C\}$. If $C \subseteq C^\perp$ then C is termed *self-orthogonal*, and if $C = C^\perp$, C is *self-dual*. A binary code is *even* if all its codewords have even weight. Self-dual binary codes are even. A self-dual binary code with all codewords of weight divisible by 4 is called *doubly-even*; a self-dual code with some codeword of weight not divisible by 4 is called *singly-even*.

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Let $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$ be the finite field with four elements, where $\overline{\omega} = \omega^2 = \omega + 1$. The *Hermitian inner product* in $\mathbb{F}_4^n$ is given by $u \cdot v = \sum_{i=1}^{n} u_i v_i^2$ and we denote by $C^{\perp H}$ the dual of C under Hermitian inner product. C is *Hermitian self-dual* if $C = C^{\perp H}$.

The weight enumerator $W(y)$ of a code C is defined as $W(y) = \sum_{i=0}^{n} A_i y^i$, where A_i is the number of codewords of weight i in C.

Let S_n be the symmetric group of degree n. For a permutation $\sigma \in S_n$ and $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ define $x\sigma \in \mathbb{F}_2^n$ by $(x\sigma)_i = x_{i\sigma^{-1}}$. If C is a binary code and $c\sigma \in C$ for all $c \in C$, σ is called *an automorphism* of C. The set of all automorphisms of C forms a group called the *automorphism group* of C (denoted by $\mathrm{Aut}(C)$). Two binary codes are called *equivalent* if one can be obtained from the other by a permutation of the coordinates.

The largest possible minimum weights of singly-even self-dual codes of lengths up to 72 are determined in [2]. It was also shown in [11] that the minimum weight d of a binary self-dual code of length n is bounded by

$$d \leq \begin{cases} 4\lfloor \frac{n}{24} \rfloor + 4, & \text{if } n \not\equiv 22 \pmod{24}; \\ 4\lfloor \frac{n}{24} \rfloor + 6, & \text{if } n \equiv 22 \pmod{24}. \end{cases} \tag{1}$$

We call a self-dual code meeting this upper bound *extremal*. A self-dual code which has the largest minimum weight among all self-dual codes of a given length is named *optimal*.

For example, applying (1) to a putative binary self-dual code of length 52, we have for its minimum distance the inequality $d \leq 12$. Since an extremal binary self-dual [52, 26, 12] code doesn't exist any binary self-dual [52, 26, 10] code is optimal.

The weight enumerators of codes (or putative codes) of lengths up to 72 with the highest possible minimal distance are presented in [2]. For [52, 26, 10] self-dual codes there are two possible weight enumerators:

$$W_{52,1}(y) = 1 + 250y^{10} + 7980y^{12} + 42800y^{14} + \ldots,$$
$$W_{52,2}(y) = 1 + (442 - 16\beta)y^{10} + (6188 + 64\beta)y^{12} + 53040y^{14} + \ldots,$$

where $0 \leq \beta \leq 12$, $\beta \neq 11$ [1]. Codes exist for $W_{52,1}$ and for $W_{52,2}$ when $\beta = 1, \ldots 7$, 12 [6].

Remark The value $\beta = 12$ is from [15] where unfortunately is a typo in the generation parameters. Using the same starting code indeed we obtain codes with $A_{10} = 250$ but of type $W_{52,1}$ only. Thus we have the following.

Open problem *Construct a code with weight enumerator $W_{52,2}$ for every value of the parameter that have not previously arisen, i.e. find a code with $W_{52,2}$ for $\beta = 8, 9, 10$ and 12.*

All binary self-dual [52, 26, 10] codes with an automorphism of order 7 are constructed in [7]. Recently in [12] all self-dual [52, 26, 10] codes with an automorphism of order 13 are classified. In this paper, we are interested in binary self-dual [52, 26, 10] codes with an automorphism of order 3. The case of automorphism of order 5 is open.

2 Construction method

We apply a method for constructing binary self-dual codes possessing an automorphism of odd prime order from [5,13].

Let C be a binary self-dual code of length n with an automorphism σ of prime order $p \geq 3$ with exactly c independent p-cycles and $f = n - cp$ fixed points in its decomposition. We may assume that

$$\sigma = (1, 2, \ldots, p)(p+1, p+2, \ldots, 2p)\cdots(p(c-1)+1, p(c-1)+2, \ldots, pc), \quad (2)$$

and shortly say that σ is of *type* $p - (c, f)$. Let $\Omega_1, \ldots, \Omega_c$ are the cycles of σ and $\Omega_{c+1}, \ldots, \Omega_{c+f}$—the fixed points. Define

$$F_\sigma(C) = \{v \in C \mid v\sigma = v\},$$
$$E_\sigma(C) = \{v \in C \mid \mathrm{wt}(v|\Omega_i) \equiv 0 \pmod 2, i = 1, \ldots, c+f\},$$

where $v|\Omega_i$ is the restriction of v on Ω_i.

Theorem 1 [5] $C = F_\sigma(C) \oplus E_\sigma(C)$, $\dim(F_\sigma) = \frac{c+f}{2}$, $\dim(E_\sigma) = \frac{c(p-1)}{2}$.

According to the above theorem the code C has a generator matrix

$$\mathrm{gen}\ C = \begin{pmatrix} \mathrm{gen}\ F_\sigma \\ \mathrm{gen}\ E_\sigma \quad O \end{pmatrix}, \quad (3)$$

where O is the $\frac{c(p-1)}{2} \times f$ zero matrix.

We have that $v \in F_\sigma(C)$ iff $v \in C$ and v is constant on each cycle. Let $\pi : F_\sigma(C) \to \mathbb{F}_2^{c+f}$ be the projection map where if $v \in F_\sigma(C)$, $(v\pi)_i = v_j$ for some $j \in \Omega_i$, $i = 1, 2, \ldots, c+f$.

Denote by $E_\sigma(C)^*$ the code $E_\sigma(C)$ with the last f coordinates deleted. So $E_\sigma(C)^*$ is a self-orthogonal binary code of length pc. For v in $E_\sigma(C)^*$ we let $v|\Omega_i = (v_0, v_1, \ldots, v_{p-1})$ correspond to the polynomial $v_0 + v_1x + \cdots + v_{p-1}x^{p-1}$ from $\mathcal{P}$, where $\mathcal{P}$ is the set of even-weight polynomials in $\mathbb{F}_2[x]/\langle x^p - 1\rangle$. Thus we obtain the map $\varphi : E_\sigma(C)^* \to \mathcal{P}^c$. $\mathcal{P}$ is a cyclic code of length p with generator polynomial $x - 1$.

Theorem 2 [14] *A binary* $[n, n/2]$ *code* C *with an automorphism* σ *is self-dual if and only if the following two conditions hold:*

(i) $C_\pi = \pi(F_\sigma(C))$ *is a binary self-dual code of length* $c + f$,
(ii) *for every two vectors* $u, v \in C_\varphi = \varphi(E_\sigma(C)^*)$ *we have* $\sum_{i=1}^{c} u_i(x)v_i(x^{-1}) = 0$.

Theorem 3 [4] *Let* C *be a binary self-dual code having an automorphism* σ *from (2). Let* A_i, B_i, *and* D_i *are the coefficients in the weight enumerators of* C, F_σ, *and* E_σ, *respectively. Then*

$$D_i \equiv 0 \pmod p, \quad A_i \equiv B_i \pmod p. \quad (4)$$

3 Optimal [52, 26, 10] self-dual codes

According to [6, Table 2] for a [52, 26, 10] binary self-dual code there are two possible types for an automorphism of order 3: $3 - (14, 10)$ and $3 - (16, 4)$.

3.1 Codes with an automorphism of type $3 - (14, 10)$

Let C be a binary self-dual code of length 52 with an automorphism σ of type $3 - (14, 10)$. Using Theorem 2 and the fact that the minimum weight of C is 10 we can conclude that C_π is a $[24, 12, \geq 4]$ binary self-dual code. There are exactly 30 inequivalent such codes: 4 decomposable e_8^3, $e_{16} \oplus e_8$, $f_{16} \oplus e_8$, e_{12}^2 and 26 indecomposable codes, labeled A_{24} to Z_{24} [10].

Coordinate positions from 43 to 52 correspond to the fixed points of C, so each choice for these fixed points can lead to a different subcode F_σ. For any 4-weight vector in C_π at most 2 nonzero coordinates may be fixed points. An examination of the vectors of weight 4 in all 30 codes eliminates 26 of them. The four remaining codes are G_{24}, X_{24}, Y_{24} and Z_{24} with generator matrices

$$G_1 = \begin{pmatrix} 100000000000101011100011\\ 010000000000111110010010\\ 001000000000110100101011\\ 000100000000110001110110\\ 000010000000110011011001\\ 000001000000011001101101\\ 000000100000001100110111\\ 000000010000101101111000\\ 000000001000010110111100\\ 000000000100001011011110\\ 000000000010101110001101\\ 000000000001010111000111 \end{pmatrix}, G_2 = \begin{pmatrix} 111000000000000100000000\\ 000111000000000010000000\\ 000000111000000001000000\\ 000000000111000000100000\\ 000000000000110000111111\\ 110000000010000001100010\\ 101000000000010000110100\\ 000110000010000001010100\\ 000101000000100000100110\\ 000000100010000011001001\\ 000000010000100010010101\\ 000000000100100001111000 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 111100000000000000000000\\ 000011110000000000000000\\ 101000000000110000001001\\ 110000000001010010000010\\ 000010100000110000010010\\ 000011000001010001000001\\ 000000001111111111111111\\ 000000000000110000111111\\ 000000000001010111000111\\ 000000000011111001001001\\ 000000000101111010010010\\ 000000000001101000000111 \end{pmatrix}, G_4 = \begin{pmatrix} 111111110000000000000000\\ 111100001111000000000000\\ 111100000000101110000000\\ 111100000000010001110000\\ 111100000000000000001111\\ 110000000000111000101010\\ 000011000000110101001010\\ 000000001100110100101100\\ 101000000000110100011001\\ 000010100000110010101001\\ 000000001010110010011010\\ 100010001000110000001000 \end{pmatrix},$$

respectively.

By investigating all alternatives for the choice of the 3-cycle coordinates we obtain, up to equivalence, all possibilities for the generator matrix of the code F_σ. We constructed 24 inequivalent codes, namely $B_1, \ldots, B_{24}$ listed in Table 1. The generator matrix for a code can be obtained by permuting the corresponding matrix G_t, $t = 1, \ldots, 4$ with $\tau \in S_{24}$ given in the table.

According to Theorem 2 the subcode C_φ is a hermitian self-dual $[14, 7, \geq 5]$ code over $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$. There is a unique such code q_{14} [8] with a generator matrix

$$H_{14,1} = \begin{pmatrix} 0 & \overline{\omega} & \omega & 0 & \overline{\omega} & \omega & \omega & \omega & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & \overline{\omega} & \omega & 1 & 1 & \overline{\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \overline{\omega} & \overline{\omega} & \overline{\omega} & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & \overline{\omega} & 1 & \overline{\omega} & 0 & 1 & 0 & 1 & 0 & 0 & 0\\ 0 & \overline{\omega} & \overline{\omega} & 0 & 0 & \omega & \overline{\omega} & 0 & \overline{\omega} & 0 & 0 & \overline{\omega} & 0 & 0\\ 0 & \overline{\omega} & 0 & \omega & \omega & \overline{\omega} & 0 & 0 & \omega & 0 & 0 & 0 & \omega & 0\\ 0 & 0 & \overline{\omega} & 1 & \overline{\omega} & 0 & 1 & 0 & \omega & 0 & 0 & 0 & 0 & \omega \end{pmatrix}. \tag{5}$$

Let $\tau \in S_{14}$ be a permutation. Denote by C_i^τ the code with generator matrix (3), determined by $C_\pi = B_i$, $i = 1, \ldots, 24$, and C_φ generated by the matrix $H_{14,1}$ with columns permuted by τ. We use the following lemma.

Table 1 Generators of C_π

Code	Matrix	τ
B_1	G_1	()
B_2	G_1	(6,21,7,22,12,14,23,15,9,16,19,10,20)(17,24,18)
B_3	G_2	()
B_4	G_3	()
B_5	G_3	(8,15)(9,16,11)(12,17,19)(13,20,22,14)(23, 24)
B_6	G_3	(4,15,7,6,5)(8,16,12,11)(9,10)(14,22)(17,18)(19,21)(23,24)
B_7	G_3	(4,15,8,16,18,11,7,6,5)(12,17)(14,22,23,24,21,19)
B_8	G_3	(4,15,11,17,9,19,14,12,7,6,5)(8,16,10,18)(20,24)(21,23)
B_9	G_3	(4,15,11,17,9,19,14,12,7,6,5)(8,16,10,18)(13,23,21)(20,24)
B_{10}	G_4	()
B_{11}	G_4	(12,15,13)
B_{12}	G_4	(12,15,16,17,18,19,20,21,14,13)
B_{13}	G_4	(12,15,16,17,18,19,20,21,22,23,24,14,13)
B_{14}	G_4	(5,8,7)(9,17,19,10,18)(14,15,20,16)(21,24,22)
B_{15}	G_4	(5,8,7)(10,16,23,19,11)(12,15,24,13,22,20,18,17,21,14)
B_{16}	G_4	(5,8,7)(9,22,11,14,21,12,24,17,10,23,18,13)(19,20)
B_{17}	G_4	(5,8,7)(9,22,12,14,19,13,10,24,11,23,17)(15,16)(18,20,21)
B_{18}	G_4	(5,8,9,18,24,17,6,10,14,21,11,19,22,12,20,23,13)(7,16,15)
B_{19}	G_4	(5,13)(6,20,17)(7,19,18,11,8,14,12,10,9,16,15)(21,24,22)
B_{20}	G_4	(5,14,16,21,12,15,19,9)(6,24,17,20,8,22,11,7,23,18,10)
B_{21}	G_4	(5,9)(6,17,13,21,14,18,11)(7,16,20,10,15,12)(23,24)
B_{22}	G_4	(5,9,12,21,11,20,23,19,14,24,18,13,7,16,6,17,15)(10,22)
B_{23}	G_4	(6,15,10,12,21,8)(7,16,20,23)(9,22,17)(13,19,14)(18,24)
B_{24}	G_4	(4,15,14,12,17,22,13,23,19,18,11,7,16,24,20,10,9,8,5)

Lemma 1 [13] *The following transformations preserve the decomposition and send the code C to an equivalent one:*

(a) *the substitution $x \to x^t$ in C_φ, where t is an integer, $1 \le t \le p-1$;*
(b) *multiplication of the jth coordinate of C_φ by x^{t_j} where t_j is an integer, $0 \le t_j \le p-1$, $j = 1, 2, \ldots, c$;*
(c) *permutation of the first c cycles of C;*
(d) *permutation of the last f coordinates of C.*

The permutational part of the transformations from Lemma 1, preserving the hermitian code C_φ, forms a subgroup of the symmetric group S_{14}, denoted by L. We have calculated that L is a group of order 2184 with generators $(1, 2, 5, 10, 4, 14, 11)(3, 7, 12, 9, 8, 13, 6)$ and $(1, 7, 9, 2, 4, 5, 12, 11, 6, 8, 3, 14)$.

Lemma 2 [14] *If τ_1 and τ_2 are in one and the same right coset of L in S_{14}, then C^{τ_1} and C^{τ_2} are equivalent.*

In order to classify all codes we have considered all representatives of the right transversal of S_{14} with respect to L. The number of codes obtained and the type of their weight enumerators are listed in Table 2. Note that the value $\beta = 8$ for $W_{52,2}$ is new. We summarize the results in the following.

Table 2 [52, 26, 10] self-dual codes with an automorphism of type 3 − (14, 10)

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}
#	4005	708	72259	43	8369	93528	72361	183555	150249	8066	159280	71671
β	1	1	1, 4	6	0, 3, 6	3, 6	1, 3, 7	0, 3, 6	2, 5	2	2, 5	1, 4
	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}
#	11130	25730	11324	148174	5980	27802	72361	67299	15216	93067	2068	4005
β	4, 7	3, 6	2, 5	2, 5	$W_{54,1}$	2, 5	1, 4, 7	4, 7	5, 8	1, 4, 7	$W_{54,1}$	1

Proposition 1 *There are exactly 1308250 inequivalent binary* [52, 26, 10] *self-dual codes with an automorphism of type* 3 − (14, 10). *Exactly 640 of these codes have weight enumerator* $W_{52,2}$ *for* $\beta = 8$. *There does not exist a binary self-dual* [52, 26, 10] *code with weight enumerator* $W_{52,2}$ *for* $\beta = 9, 10$, *and 12 possessing an automorphism of type* 3 − (14, 10).

3.2 Codes with an automorphism of type 3 − (16, 4)

Let C be a binary self-dual code of length 52 with an automorphism σ of type 3 − (16, 4). Then C_φ is a hermitian [16, 8, $\geq$ 5] code over $\mathbb{F}_4$. There are exactly 4 inequivalent such codes $2f_8$, $1_6 + 2f_5$, 1_{16}, $4f_4$ [3] with generator matrices

$$H_{16,1} = \begin{pmatrix} 1000000001111111 \\ 01000000101\omega\overline{\omega}\overline{\omega}\omega 1 \\ 001000001101\omega\overline{\omega}\overline{\omega}\omega \\ 000100001\omega 101\omega\overline{\omega}\overline{\omega} \\ 000010001\overline{\omega}\omega 101\omega\overline{\omega} \\ 000001001\overline{\omega}\overline{\omega}\omega 101\omega \\ 000000101\omega\overline{\omega}\overline{\omega}\omega 101 \\ 0000000111\omega\overline{\omega}\overline{\omega}\omega 10 \end{pmatrix}, \ H_{16,2} = \begin{pmatrix} 1100000\omega 00\omega 00\overline{\omega}\overline{\omega}0 \\ 101000\omega 0\omega 00000\overline{\omega}\overline{\omega} \\ 1001000\omega 0\omega 0\overline{\omega}000\overline{\omega} \\ 10001000\omega 0\omega\overline{\omega}\overline{\omega}000 \\ 100001\omega 00\omega 00\overline{\omega}\overline{\omega}00 \\ 00\overline{\omega}00\overline{\omega}0\omega 0\overline{\omega}000\overline{\omega}0\omega \\ 0\overline{\omega}0\overline{\omega}0000\omega 0\overline{\omega}\omega 00\overline{\omega}0 \\ 0000001\omega 00\omega 1\omega 00\omega \end{pmatrix},$$

$$H_{16,3} = \begin{pmatrix} 1100000\omega\overline{\omega}\omega 00000\overline{\omega} \\ 10100000\omega\overline{\omega}\omega\overline{\omega}0000 \\ 100100\omega 00\omega\overline{\omega}0\overline{\omega}000 \\ 100010\overline{\omega}\omega 00\omega 00\overline{\omega}00 \\ 100001\omega\overline{\omega}\omega 00000\overline{\omega}0 \\ 010\overline{\omega}0\omega 1\overline{\omega}0000000\omega \\ 0\omega 10\overline{\omega}001\overline{\omega}00\omega 0000 \\ 00\omega 10\overline{\omega}001\overline{\omega}00\omega 000 \end{pmatrix}, \ H_{16,4} = \begin{pmatrix} 1000000001111111 \\ 010000001000\omega\omega\overline{\omega}\overline{\omega} \\ 00100000111\overline{\omega}000\overline{\omega} \\ 0001000011\overline{\omega}1\overline{\omega}\overline{\omega}0\overline{\omega} \\ 000010001\overline{\omega}011\overline{\omega}00 \\ 000001001\overline{\omega}0\omega 1\omega 1\overline{\omega} \\ 000000101\overline{\omega}1\overline{\omega}\overline{\omega}0\overline{\omega}1 \\ 000000011\overline{\omega}\overline{\omega}\omega\omega 011 \end{pmatrix},$$

respectively.

The minimum weight of the code C is 10 hence, by Theorem 2, we can conclude that C_π is a [20, 10, $\geq$ 4] binary self-dual code. There are exactly 7 inequivalent such codes [9]: $d_{12} + d_8$, $d_{12} + e_8$, d_{20}, d_4^5, $d_6^3 + f_2$, $d_8^2 + d_4$, and $c_7^2 + d_6$.

In these seven codes we have to arrange 16 of the coordinate positions $\{1, \ldots, 20\}$ to be the cycle positions X_c and 4 to be the fixed points X_f, in such a way, that the minimum distance of $F_\sigma = \pi^{-1}(C_\pi)$ is at least 10. After calculating all $\binom{20}{4}$ possible subcodes for each of the seven codes we obtain three matrices which lead to different codes F_σ. Denote

$$G_5 = \begin{pmatrix} 11110000000000000000 \\ 00001110000000001000 \\ 00000001110000000100 \\ 00000000001110000010 \\ 00000000000001110001 \\ 00000001100101100010 \\ 11000000001100100001 \\ 01011100000001100000 \\ 11000101100000001000 \\ 00001100101100000100 \end{pmatrix}, G_6 = \begin{pmatrix} 11110000000000000000 \\ 00111100000000000000 \\ 00000011110000000000 \\ 00000000111000001000 \\ 00000000000111100000 \\ 00000000000001110100 \\ 10101010101101010010 \\ 10100110100101001101 \\ 10101000001101001100 \\ 00001110100101011000 \end{pmatrix},$$

$$G_7 = \begin{pmatrix} 11110000000000000000 \\ 00111000000000001000 \\ 00000111100000000000 \\ 00000001110000000100 \\ 00000000001111000000 \\ 00000000000011100010 \\ 10101101011010110000 \\ 10100101001010001111 \\ 10101000011010000110 \\ 00001101001010101100 \end{pmatrix}.$$

Reducing the weight enumerators $W_{52,1}$ and $W_{52,2}$ modulo 3 we have

$$W_{52,1}(y) \equiv 1 + 1.y^{10} + 0.y^{12} + 2y^{14} + \dots, \quad (6)$$

$$W_{52,2}(y) \equiv 1 + (1 + 2\beta)y^{10} + (2 + \beta)y^{12} + 0.y^{14} + \dots. \quad (7)$$

The matrix G_5 generates the code d_4^5, and $\pi^{-1}(G_5)$ generates a code F_σ with weight enumerator

$$1 + 4y^{10} + 5y^{12} + 24y^{14} + \dots \equiv 1 + y^{10} + 2y^{12} + 0.y^{14} + \dots \pmod 3. \quad (8)$$

According the Theorem 3 this code can lead to [52, 26, 10] codes with $W_{52,2}$ for $\beta \equiv 0 \pmod 3$. We have constructed codes with $\beta = 0, 3, 6, 9$, and 12. The values $\beta = 9$ and $\beta = 12$[1] are new.

Both matrices G_6 and G_7 generate codes equivalent to $d_6^3 + f_2$ but $\pi^{-1}(G_6)$ and $\pi^{-1}(G_7)$ are generator matrices of codes F_σ with different weight enumerators:

$$1{+}4y^{10}{+}9y^{12}{+}32y^{14}{+}\dots \equiv 1 + 1.y^{10} + 0.y^{12} + 2.y^{14} + \dots \pmod 3 \quad (9)$$

$$1{+}6y^{10}{+}9y^{12}{+}24y^{14}{+}\dots \equiv 1 + 0.y^{10} + 0.y^{12} + 0.y^{14} + \dots \pmod 3, \quad (10)$$

respectively. The weight function (9) leads to self-dual codes with $W_{52,1}$ and (10) to codes with $W_{52,2}$ for $\beta \equiv 1 \pmod 3$. Using Theorem 3 and (7) we have the following.

Proposition 2 *There does not exist a* [52, 26, 10] *self-dual code with weight enumerator* $W_{52,2}$ *for* $\beta \equiv 2 \pmod 3$ *having an automorphism of type* $3 - (16, 4)$.

Remark We were unable to construct a [52, 26, 10] self-dual code with weight enumerator $W_{52,2}$ for the last unobtained value $\beta = 10$ and its existence is still an open question. We give some examples for codes with new value of β in Table 3.

[1] The code with $\beta = 12$ is equivalent to a code first constructed and communicated to the author by Stefka Bouyuklieva.

Table 3 Some new [52, 26, 10] self-dual codes

Code	C_π	C_φ	τ	β
$C_{14,1}$	B_{21}	$H_{14,1}$	(2,3,9,10,12,14,8,11,13,6,7)	8
$C_{14,2}$	B_{21}	$H_{14,1}$	(6,12,9,13,14,8)	8
$C_{14,3}$	B_{21}	$H_{14,1}$	(3,8,6)(7,14,12)(9,13)	8
$C_{14,4}$	B_{21}	$H_{14,1}$	(3,8,6,7,14,12)(9,13)	8
$C_{14,5}$	B_{21}	$H_{14,1}$	(3,8,6,7,14,9,13,12)	8
$C_{14,6}$	B_{21}	$H_{14,1}$	(3,12)(6,14,13,8)	8
$C_{14,7}$	B_{21}	$H_{14,1}$	(3,12)(6,14,8)(9,13)	8
$C_{14,8}$	B_{21}	$H_{14,1}$	(3,12)(6,9)(7,14,8)	8
$C_{14,9}$	B_{21}	$H_{14,1}$	(2,3,14,8)(6,11,13,7,10,12,9)	8
$C_{14,10}$	B_{21}	$H_{14,1}$	(3,13,9,8,14,12)(4,6)	8
$C_{16,1}$	G_5	$H_{16,4}$	(1,4,10,8,7,3,14,9,12,11,15,16,5)(2,13)	9
$C_{16,2}$	G_5	$H_{16,4}$	(2,8,11,13,4,7,15,6)(3,14,5)(9,12)	9
$C_{16,3}$	G_5	$H_{16,4}$	(2,14,15,10,4,8,3,5,7,16,9,11,12,13,6)	9
$C_{16,4}$	G_5	$H_{16,4}$	(1,12,3,6)(2,14,9,5,8)(4,7)	9
$C_{16,5}$	G_5	$H_{16,4}$	(1,6,8,2,14,12,4,15,13)(3,5,7,10,16)	9
$C_{16,6}$	G_5	$H_{16,4}$	(1,4,13,5,6,2,9,12,8,3,7,10,14)(11,16)	9
$C_{16,7}$	G_5	$H_{16,4}$	(1,9,16,7,8,6,13,3,5)(4,12)(11,15)	9
$C_{16,8}$	G_5	$H_{16,4}$	(1,3,4,2,13,7,14,10,8,12,11,15,16)	9
$C_{16,9}$	G_5	$H_{16,4}$	(1,2,7,14,11,5,6,8,12,4)(3,10,13,9,16)	9
$C_{16,10}$	G_5	$H_{16,4}$	(1,2,7,14,11,5,6,8,12,4)(3,10,13,9,16)	9
$C_{16,11}$	G_5	$H_{16,3}$	(1,11,13,2,10,14,16,4,3,7)(6,8,12,15,9)	12

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