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Hyperovals of Hermitian polar spaces

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Abstract The first infinite family of hyperovals of the Hermitian generalized quadrangle arising from $\mathcal{H}(4, q^2)$, q even, is constructed. Alternative geometric descriptions of the known hyperovals of $\mathcal{H}(5, 4)$ are given.

Keywords Hermitian polar space · Symmetric group · Hyperoval · Complete span · Cyclic spread of a unital

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1 Introduction

A connected incidence system is an *extended polar space* if its point residues are finite thick, non degenerate polar spaces. Extensions of polar spaces play an important role as incidence geometries admitting interesting groups, such as sporadic simple, or some classes of (extensions of) classical groups.

A *hyperoval* or a *local subspace* of a polar space $\mathcal{P}$ is a non-empty set of points of $\mathcal{P}$ which intersects every singular line of $\mathcal{P}$ in either 0 or 2 points.

Hyperovals of polar spaces arise in the context of locally polar spaces. Indeed, from a result of Buekenhout and Hubaut [3, Proposition 3] if A is a polar space of polar rank ≥ 3 and order n, and H is a hyperoval of A then H equipped with the graph induced by A

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on H, is the adjacency graph of a locally polar space of polar rank $r-1$ and order n such that the residual space H_P at any point $P \in H$ is isomorphic to $\mathrm{Cone}_P(A)$. This result makes interesting the classification of all local subspaces of polar spaces. As observed in [3, Remark 2, p. 404] when $r = 2$ we can still say that a hyperoval of a generalized quadrangle $\mathcal{S}$ is a graph of degree equal to $|\mathrm{Cone}_P(\mathcal{S})|$ which has the property to be triangle free.

In this article we will focus on hyperovals of the polar spaces arising from the Hermitian varieties $\mathcal{H}(n, q^2)$, $n = 4, 5$ with automorphism groups $\mathrm{P\Gamma U}(n+1, q^2)$. For general information on hyperovals of polar spaces we will refer to [8]. For more results on hyperovals on Hermitian generalized quadrangles see also [5].

Firstly, we construct an infinite family of hyperovals of the generalized quadrangle $\mathcal{H}(4, q^2)$, q even, based on the existence of the cyclic spread of the Hermitian curve $\mathcal{H}(2, q^2)$ [1].

The hyperovals of $\mathcal{H}(5, 4)$ were classified by Pasechnick in [10, Proposition 3.1] with the aid of a computer. He showed that there are, up to isomorphisms, two classes of hyperovals of $\mathcal{H}(5, 4)$: a class of hyperovals consisting of 126 points and a class of hyperovals consisting of 162 points. From [10, Theorem 1.1], if Γ is an extension of $\mathcal{H}(5, 4)$, then Γ is the extended polar space for Fi_{22}. It is also related to near subhexagons of $\mathcal{H}(5, 4)$-dual polar spaces [8].

In a recent article [8] De Bruyn, among other interesting results, gave a computer-free proof for the uniqueness, up to isomorphisms, of the hyperoval of size 126 of $\mathcal{H}(5, 4)$. Also, in the article [7] the authors gave another geometric description of the 126-hyperoval of $\mathcal{H}(5, 4)$ by means of the smallest Split Cayley hexagon $H(2)$ [11].

In the last section of the article, we give an alternative description of both known hyperovals of $\mathcal{H}(5, 4)$ based on the action of the stabilizer in $\mathrm{PSU}_6(4)$ of a self-polar simplex of $\mathrm{PG}(5, 4)$.

2 Hyperovals of $\mathcal{H}(4, q^2)$

We construct the first infinite family of hyperovals of $\mathcal{H}(4, q^2)$, q even.

Proposition 2.1 *There exists an infinite family of hyperovals of* $\mathcal{H}(4, q^2)$, q *even, of size* $q^5 - q^4 + q^3 + q^2 + 2$.

Proof Let $\mathcal{H}(4, q^2)$ be a Hermitian variety of $\mathrm{PG}(4, q^2)$, q even. Let π be a secant plane to $\mathcal{H}(4, q^2)$ and let $\ell = \pi^\perp$, where $\perp$ is the polarity induced by $\mathcal{H}(4, q^2)$ in $\mathrm{PG}(4, q^2)$. The stabilizer of π in $\mathrm{PGU}(5, q^2)$ is the quotient $\mathrm{G} = X/Z(X)$ of the group $X = \mathrm{GU}_2(q^2) \times \mathrm{GU}_3(q^2)$ by its center $Z(X) = C_{q+1}$. The group G has four orbits on singular points of $\mathcal{H}(4, q^2)$: apart from the orbits of size $q^3 + 1$ and $q + 1$, it has an orbit, say O_1, of size $(q^3 - q)(q^4 - q^3 + q^2)$ consisting of points whose conjugate meets π at a secant line to the Hermitian curve $\mathcal{U} = \mathcal{H}(4, q^2) \cap \pi$ and an orbit O_2 of size $(q^2 - 1)(q + 1)(q^3 + 1)$ consisting of points whose conjugate meets π at a line that is tangent to $\mathcal{U}$. There are $(q + 1)(q^3 + 1)$ generators meeting $\mathcal{U}$ and ℓ. If two of them have non trivial intersection then they meet either in a point of $\mathcal{U}$ or in a point of ℓ. The points of O_2 are those on such generators ($q^2 - 1$ each).

Let $1 \times \mathrm{S}$ be a Singer cyclic group of G of order $q^2 - q + 1$. From [1, Theorem 3.1], since q is even, there exists a unique cyclic spread $\mathcal{F} = \{\ell_1, \ldots, \ell_{q^2-q+1}\}$ of the Hermitian curve $\mathcal{U}$ invariant under $1 \times \mathrm{S}$. Recall that a spread of the Hermitian curve $\mathcal{H}(2, q^2)$ is a family of $q^2 - q + 1$ secant lines of $\mathcal{H}(2, q^2)$ no two of them intersecting in a singular point. Notice that $\mathcal{F}$ is also a dual arc: no three lines of $\mathcal{F}$ are in a pencil.

Consider now the subgroup $\mathrm{H} = (\mathrm{PGU}_2(q^2) \times \mathrm{S})/\mathrm{C}_{q+1}$ of G. The orbit O_1 splits into H-orbits one of which has size $(q^2 - q + 1)(q^3 - q)$, say O' consisting of points of $\mathcal{H}(4, q^2)$

whose polar space intersects π in a line r of $\mathcal{F}$. Indeed, the unital $r^\perp \cap \mathcal{H}(4, q^2)$ contains the chord $\ell \cap \mathcal{H}(4, q^2)$ and hence there are $q^3 - q$ points on $\mathcal{H}(4, q^2)$ whose conjugate is a given line of $\mathcal{F}$. In other words, O' is the union of $q^2 - q + 1$ unitals $\{\mathcal{U}_1, \dots, \mathcal{U}_{q^2-q+1}\}$ sharing the chord $\ell \cap \mathcal{H}(4, q^2)$. Moreover, no three points of O' lie on the same generator, otherwise there should exist three concurrent lines of $\mathcal{F}$. We show that the union $\mathcal{H} = \mathcal{U} \cup (\ell \cap \mathcal{H}(4, q^2)) \cup O'$ is a hyperoval of $\mathcal{H}(4, q^2)$. If P is a point of ℓ then a generator on P meets $\mathcal{U}$ and skips O'. Assume that P is a point of $\mathcal{U}$ and let ℓ_P the unique line of $\mathcal{F}$ on P. A generator on P meets the unital of $(\ell_P)^\perp \cap \mathcal{H}(4, q^2)$ in a point Q and hence either $Q \in \ell$ or $Q \in O'$. Let $P \in O'$, let $\mathcal{U}_i$ be the unital of O' on P and let $\ell_P = P^\perp \cap \pi \in \mathcal{F}$. It follows that $P^\perp$ meets all the other unitals of O' at a chord distinct from ℓ. A counting argument shows that $q + 1$ of the $q^3 + 1$ generators on P meet ℓ_P and the remaining $q^3 - q$ are partitioned into $q^2 - q$ subsets each of size $q + 1$ forming a pencil. Since $\mathcal{F}$ is a dual arc, generators in a subset meet one and only one unital $\mathcal{U}_j$, $j \neq i$. The proof is now complete. □

Remark 2.2 Notice that the above construction also applies to the case of the Hermitian surface giving rise to a 2-ovoid. On the other hand it is clear that it cannot work on $\mathcal{H}(n, q^2)$, $n \geq 5$.

3 Hyperovals of $\mathcal{H}(5, 4)$

Let $\mathcal{H}(5, 4)$ be the Hermitian variety of PG(5, 4) with equation $X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 + X_6^3 = 0$, where X_i, $i = 1, \dots, 6$ are projective homogeneous coordinates in PG(5, 4). Let Δ be the 6-simplex of PG(5, 4) with vertices $E_i = (0, \dots, 1, \dots, 0)$. Then Δ is a self-polar simplex with respect to $\mathcal{H}(5, 4)$. Let $\Omega_r = \{(x_1, \dots, x_6) : x_i \in \mathrm{GF}(4)$ and $x_i \neq 0$ for exactly r values of $i\}$. So, in particular Ω_1 is the set of vertices of Δ. Let $\mathrm{G} = \mathrm{PSL}_6(4)$. Then G_Δ, the stabilizer of Δ in G, is a group of order $6!3^4$, with structure $3^4 \cdot S_6$ and $|\Omega_r| = \binom{6}{r} 3^{r-1}$. It turns out that each Ω_r, $1 \leq r \leq 5$ is an orbit for G_Δ. It follows that $|\Omega_1| = 6$, $|\Omega_2| = 45$, $|\Omega_3| = 180$, $|\Omega_4| = 405$, $|\Omega_5| = 486$. The set Ω_6 of size 243 is the union of three G_Δ-orbits, denoted by $\Omega_6(1)$, $\Omega_6(a)$, $\Omega_6(a^2)$, where $a \in \mathrm{GF}(4)$ such that $a^2 + a + 1 = 0$, and defined as follows. For a point $(x_1, x_2, x_3, x_4, x_5, x_6) \in \Omega_6$, put $x = x_1 x_2 x_3 x_4 x_5 x_6$. We have:

$$\Omega_6(1) = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \Omega_6 : x = 1\},$$
$$\Omega_6(a) = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \Omega_6 : x = a\},$$
$$\Omega_6(a^2) = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \Omega_6 : x = a^2\}.$$

Since G_Δ consists of the monomial matrices in G and is generated by elations with centres in Ω_2, the sets $\Omega_6(1)$, $\Omega_6(a)$ and $\Omega_6(a^2)$, are actually G_Δ-orbits and they have all size 81.

The orbits of G_Δ on $\mathcal{H}(5, 4)$ are the orbits Ω_r, for r even.

Proposition 3.1 *The union* $\Omega_2 \cup \Omega_6(1)$ *is a* 126-*hyperoval of* $\mathcal{H}(5, 4)$.

Proof Points of Ω_2 are singular points on edges of Δ which is a self-polar simplex. Let $P = (1, 1, 0, 0, 0, 0) \in \Omega_2$. Then $P^\perp$ is the hyperplane $X_1 = X_2$. Since G_Δ is two-transitive on Δ, we have that on P there are 18 lines of $\mathcal{H}(5, 4)$ meeting Ω_2 in at least 2 points and each such line skips $\Omega_6(1)$. However, it is easy to see, by direct computations, that the above 18 lines are exactly 2-secant to Ω_2. The stabilizer of P in G_Δ is a group K of order 1296 generated by the matrices [4].

$$\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{bmatrix},$$

$$\begin{bmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},$$

$$\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&a\\ 0&0&a^2&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0 \end{bmatrix}.$$

and

$$\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&a^2&0&0&0\\ 0&0&0&a&0&0\\ 0&0&0&0&a&0\\ 0&0&0&0&0&a^2 \end{bmatrix}.$$

The group K permutes such 18 lines and has an orbit of size 27 on the remaining lines on P that are 1-secant to Ω_2 and 1-secant to $\Omega_6(1)$. The 27 points are the images of the point with coordinates $(1, 1, 1, 1, 1, 1)$ under K.

On the other hand, if $P = (1, 1, 1, 1, 1, 1) \in \Omega_6(1)$, then $P^{\perp}$ is the hyperplane $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 0$. The stabilizer M of P in G_{Δ} is the symmetric group S_6 which is generated by the monomial matrices

$$\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},$$

$$\begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0 \end{bmatrix}.$$

The 45 lines of $\mathcal{H}(5, 4)$ on P in $P^{\perp}$ are partitioned under M into two orbits: an orbit of size 15 containing the point $(1, 1, 0, 0, 0, 0)$ and its images under M (it consists of sextuples such that four entries are equal to zero and two entries are equal to one), and an orbit of size 30 containing the point $(1, a, a^2, 1, a, a^2) \in \Omega_6(1)$ and its images under M that are exactly 2-secants to $\Omega_6(1)$ (direct computations). Since G_Δ is transitive on Ω_2 and on $\Omega_6(1)$, we have proved that $\Omega_2 \cup \Omega_6(1)$ is a 126-hyperoval of $\mathcal{H}(5, 4)$. $\square$

In a similar way we can prove the following corollary.

Corollary 3.2 *The unions* $\Omega_2 \cup \Omega_6(a)$ *and* $\Omega_2 \cup \Omega_6(a^2)$ *are* 126*–hyperovals of* $\mathcal{H}(5, 4)$.

Proof The orbits $\Omega_6(1)$, $\Omega_6(a)$, $\Omega_6(a^2)$ are projectively equivalent. Indeed, the matrix $\begin{bmatrix} 0&0&a&0&0&0\\ 1&0&0&0&0&0\\ 0&a&0&0&0&0\\ 0&0&0&0&0&a^2\\ 0&0&0&a&0&0\\ 0&0&0&0&1&0 \end{bmatrix}$ is an element of $PGU_6(4) \setminus PSU_6(4)$ stabilizing Δ and permuting the three orbits $\Omega_6(a^i)$, $i = 0, 1, 2$. $\square$

Corollary 3.3 *The full stabilizer in* $PSU_6(4)$ *of the* 126*-hyperoval of* $\mathcal{H}(5, 4)$ *is isomorphic to* $PSU_4(9) \cdot 2$ *acting transitively on it.*

Proof It follows from [9, Lemma 7] that there exists an elation $\alpha \in PSU_6(4)$ such that $\langle \alpha, G_\Delta \rangle$ has two orbits on points of $\mathcal{H}(5, 4)$, i.e. $\Omega_2 \cup \Omega_6(1)$ of length 126 and $\Omega_4 \cup \Omega_6(a) \cup \Omega_6(a^2)$ of length 567. It turns out that $\langle \alpha, G_\Delta \rangle$ is the group $PSU_4(9) \cdot 2$ which is maximal in $PSU_6(4)$. $\square$

Proposition 3.4 *The union* $\Omega_6(a^i) \cup \Omega_6(a^j)$, $i, j \in \{0, 1, 2\}$ *is a* 162*-hyperoval of* $\mathcal{H}(5, 4)$.

Proof This is [10, Lemma 5.1]. A direct proof goes as follows. First of all we notice that Ω_6 consists of points $(X_1, X_2, X_3, X_4, X_5, X_6)$ such that $X_1^3 = X_2^3 = X_3^3 = X_4^3 = X_5^3 = X_6^3$ and hence is the base locus of the 4-dimensional linear system

$$\lambda_1(X_2^3 - X_1^3) + \lambda_2(X_3^3 - X_1^3) + \cdots + \lambda_5(X_6^3 - X_1^3) = 0.$$

Let $P = (1, 1, 1, 1, 1, 1) \in \Omega_6(1)$. On P there are 45 lines of $\mathcal{H}(5, 4)$ lying on the 15 planes with equations $X_i = X_j$, $X_k = X_l$, $X_m = X_n$ and each such plane contains 2 lines that are secant to $\Omega_6(1)$ and 3 lines that are tangent to $\Omega_6(1)$ and also to $\Omega_6(a)$ and to $\Omega_6(a^2)$. It is easy to see that the two lines that are secant to $\Omega_6(1)$ skip $\Omega_6(a)$ and $\Omega_6(a^2)$. Since G_Δ acts transitively on $\Omega_6(a^i)$ it follows that $\Omega_6(a^i) \cup \Omega_6(a^j)$, $i, j \in \{0, 1, 2\}$ is a 162-hyperoval of $\mathcal{H}(5, 4)$. $\square$

Proposition 3.5 *The stabilizer of the hyperoval* $O' = \Omega_6(a^i) \cup \Omega_6(a^j)$ *in* $PSU_6(4)$ *is* $3^4 : S_6$.

Proof The group $3^4 : S_6$ has two orbits on O', namely $\Omega_6(a^i)$ and $\Omega_6(a^j)$. In order to get transitivity on O' we need to pass to $P\Sigma U_6(4)$. In this case there exists an involution switching $\Omega_6(a^i)$ and $\cup\Omega_6(a^j)$. $\square$

Remark 3.6 Unfortunately, the above constructions do not generalize to higher values of q.

Another way of constructing hyperovals of $\mathcal{H}(4, q^2)$ is given in the following well-known result [10, Lemma 2.5].

Proposition 3.7 *Let* Π *be a subspace of* $\mathcal{H}(5, q^2)$ *and let* H *be a hyperoval of* $\mathcal{H}(5, q^2)$. *Then* $\Pi \cap H$ *is a hyperoval of* Π.

In [7, Proposition 4.3] we proved that the 126-hyperoval is a 6-tight set of $\mathcal{H}(5,4)$. From [2, Lemma 7] a non-degenerate hyperplane section of $\mathcal{H}(5,4)$ is a 5-ovoid. Then, it follows from [2, Corollary 5] that the 126-hyperoval induces a 30-hyperoval of $\mathcal{H}(4,4)$. Although we can provide the size of the 4-dimensional hyperoval in this case, in general we cannot say anything about its automorphism group apart when the non-degenerate hyperplane is generated by five of the six points of Δ. In that case the group certainly contains S_5, and it is interesting to note that the 30-hyperoval is related to the geometry of the complete span of $\mathcal{H}(4,4)$ admitting PSL(2, 11) studied in [6]. As showed in [6] there exist two complete 11-spans of $\mathcal{H}(4,4)$, L_1 and L_2, both stabilized by the linear group PSL(2, 11) each of them covering the same pointset of $\mathcal{H}(4,4)$ and for this reason we called them *companion spans*. The group PSL(2, 11) has the group $K = A_5$ as a subgroup: it has two orbits X_5 and X_6 on L_1 of sizes 5 and 6, respectively and fixes one line, say r of L_2 transversal to all the lines of X_5. The group K acts transitively on r and so the stabilizer K_Q of a point $Q \in r$ in K has order 12. There are 8 generators of $\mathcal{H}(4,4)$ on Q distinct from r and K_Q has on them two orbits of size 3 and two orbits of size 1. It follows that apart from r, X_5 and X_6, K certainly has another line orbit Y_5 of size 5, and two line orbits of size 15, Y_{15} and Y'_{15}. A counting argument shows that all the orbits described above cover the point set of $\mathcal{H}(4,4)$ and that our 30-hyperoval consists of points on one of the line orbits of size 15, say Y_{15} in such a way that each line of Y_{15} contains exactly two points of it.

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References

1. Baker R.D., Ebert G.L., Korchmàros G., Szönyi T.: Orthogonally divergent spreads of Hermitian curves, Finite geometry and combinatorics (Deinze, 1992), 1730, LMS. Lecture Note Ser., 191, Cambridge Univ. Press, Cambridge (1993).
2. Bamberg J., Kelly S., Law M., Penttila T.: Tight sets and m-ovoids of finite polar spaces. J. Combin. Theory A **114**(7), 1293–1314 (2007).
3. Buekenhout F., Hubaut X.: Locally polar spaces and related rank 3 groups. J. Algebra **45**, 391–434 (1977).
4. Cannon J., Playoust C.: An Introduction to MAGMA.University of Sydney Press, Sydney (1993).
5. Cossidente A.: On hyperovals of $\mathcal{H}(3,q^2)$. J. Combin. Theory A **118**, 1190–1195 (2011).
6. Cossidente A., Ebert G.L., Marino G.: A complete span of $\mathcal{H}(4,4)$ admitting $PSL_2(11)$ and related structures. Contrib. Discrete Math. **3**(1), 52–57 (2008).
7. Cossidente A., Marino G., Penttila T.: Some geometry of $G_2(q) < PSU_6(q^2)$, q even, (under review).
8. De Bruyn B.: On hyperovals of polar spaces, Des. Codes Cryptogr. **56**(2–3), 183–195 (2010).
9. Key J.D.: Some maximal subgroups of $PSL(n,q)$, $n \geq 3$, $q = 2^r$, Geom. Ded. **4**, 377–386 (1975).
10. Pasechnik D.V.: Extending polar spaces of rank at least 3, J. Combin. Theory A **72**, 232–242 (1995).
11. Van Maldeghem H.: Generalized Polygons. Birkhäuser verlag, Basel (1998).