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The pseudo-hyperplanes and homogeneous pseudo-embeddings of AG$(n, 4)$ and PG$(n, 4)$

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Abstract We determine all homogeneous pseudo-embeddings of the affine space AG$(n, 4)$ and the projective space PG$(n, 4)$. We give a classification of all pseudo-hyperplanes of AG$(n, 4)$. We also prove that the two homogeneous pseudo-embeddings of the generalized quadrangle $Q(4, 3)$ are induced by the two homogeneous pseudo-embeddings of AG$(4, 4)$ into which $Q(4, 3)$ is fully embeddable.

Keywords Homogeneous pseudo-embedding · Pseudo-hyperplane · Projective space · Affine space · Generalized quadrangle

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1 Basic definitions and main results

The aim of this section is to state the main results of this article and to define the basic notions which are necessary to understand these results. Throughout this section, $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry with the property that the number of points on each line is finite and at least three.

Suppose V is a vector space over the field $\mathbb{F}_2$ of order 2. A *pseudo-embedding* of $\mathcal{S}$ into the projective space $\Sigma = \mathrm{PG}(V)$ is a mapping e from $\mathcal{P}$ to the point set of Σ satisfying: (1) $< e(\mathcal{P}) >_\Sigma = \Sigma$; (2) if L is a line of $\mathcal{S}$ with points $x_1, x_2, \ldots, x_k$, then the points $e(x_1), e(x_2), \ldots, e(x_{k-1})$ of Σ are linearly independent and $e(x_k) =< \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_{k-1} >$ where $\bar{v}_i$, $i \in \{1, 2, \ldots, k-1\}$, is the unique vector of V for which $e(x_i) =< \bar{v}_i >_\Sigma$. Two pseudo-embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* ($e_1 \cong e_2$) if

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there exists an isomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that $e_2 = \phi \circ e_1$. The notion pseudo-embedding was introduced in De Bruyn [1].

Suppose $e : \mathcal{S} \to \mathrm{PG}(V)$ is a pseudo-embedding of $\mathcal{S}$ and G is a group of automorphisms of $\mathcal{S}$. We say that e is *G-homogeneous* if for every $\theta \in G$, there exists a (necessarily unique) projectivity η_θ of $\mathrm{PG}(V)$ such that $e(x^\theta) = e(x)^{\eta_\theta}$ for every point x of $\mathcal{S}$. If G is the full automorphism group of $\mathcal{S}$, then e is also called a *homogeneous pseudo-embedding*.

Suppose $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$ and α is a subspace of Σ satisfying the following two properties:

(Q1) if x is a point of $\mathcal{S}$, then $e(x) \notin \alpha$;
(Q2) if L is a line of $\mathcal{S}$ with points $x_1, x_2, \ldots, x_k$, then $\alpha \cap < e(x_1), e(x_2), \ldots, e(x_k) >_\Sigma = \emptyset$.

Then a new pseudo-embedding $e/\alpha : \mathcal{S} \to \Sigma/\alpha$ can be defined which maps each point x of $\mathcal{S}$ to the point $< \alpha, e(x) >$ of the quotient projective space Σ/α. This new pseudo-embedding e/α is called a *quotient* of e. If $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two pseudo-embeddings of $\mathcal{S}$, then we say that $e_1 \geq e_2$ if e_2 is isomorphic to a quotient of e_1. A pseudo-embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ is called *universal* if $\widetilde{e} \geq e$ for any pseudo-embedding e of $\mathcal{S}$. By [1, Theorem 1.2(1)], we know that if $\mathcal{S}$ has a pseudo-embedding, then $\mathcal{S}$ also has a universal pseudo-embedding. This universal pseudo-embedding is unique, up to isomorphism, and is also homogeneous (De Bruyn [2, Theorem 2.4]). If $\widetilde{e} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ is the universal pseudo-embedding of $\mathcal{S}$, where $\widetilde{V}$ is some vector space over $\mathbb{F}_2$, then the dimension of $\widetilde{V}$ is called the *pseudo-embedding rank* of $\mathcal{S}$.

A *pseudo-hyperplane* of $\mathcal{S}$ is a proper subset H of $\mathcal{P}$ such that every line contains an even number of points of $\mathcal{P} \setminus H$. If $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$ and Π is a hyperplane of Σ, then by De Bruyn [1, Theorem 1.1], $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a pseudo-hyperplane of $\mathcal{S}$. Any pseudo-hyperplane of $\mathcal{S}$ which arises from a pseudo-embedding e in the above-described way is said to *arise from e*. If $\mathcal{S}$ has a pseudo-embedding, then by De Bruyn [1, Theorem 1.3], all pseudo-hyperplanes of $\mathcal{S}$ arise from the universal pseudo-embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ of $\mathcal{S}$. More precisely, if H is a pseudo-hyperplane of $\mathcal{S}$, then there exists a unique hyperplane Π of $\widetilde{\Sigma}$ such that $H = \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi)$.

Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0, 1\}$ and $n \geq 0$. The map e_1 which maps every point $(X_0, X_1, \ldots, X_n)$ of $\mathrm{PG}(n, 4)$ to the point $(X_0^3, X_1^3, \ldots, X_n^3, X_i X_j^2 + X_j X_i^2, \delta X_i X_j^2 + \delta^2 X_j X_i^2 \,|\, 0 \leq i < j \leq n)$ of $\mathrm{PG}(n^2 + 2n, 2)$ is called a *Hermitian Veronese embedding* of $\mathrm{PG}(n, 4)$. Observe that the map e_1 depends on the chosen reference systems in $\mathrm{PG}(n, 4)$ and $\mathrm{PG}(n^2 + 2n, 2)$. If e_1 and e_1' are two Hermitian Veronese embeddings of $\mathrm{PG}(n, 4)$ into $\mathrm{PG}(n^2 + 2n, 2)$, then there exists a projectivity η of $\mathrm{PG}(n^2 + 2n, 2)$ such that $e_1' = \eta \circ e_1$. So, up to isomorphism, there exists a unique Hermitian Veronese embedding of $\mathrm{PG}(n, 4)$ into $\mathrm{PG}(n^2 + 2n, 2)$. If α is an m-dimensional subspace ($m \in \{0, 1, \ldots, n\}$) of $\mathrm{PG}(n, 4)$, then the Hermitian Veronese embedding of $\mathrm{PG}(n, 4)$ will induce “an embedding” of α into a subspace of $\mathrm{PG}(n^2 + 2n, 2)$ which is isomorphic to the Hermitian Veronese embedding of $\alpha \cong \mathrm{PG}(m, 4)$. By De Bruyn [1, Proposition 4.2], the Hermitian Veronese embedding e_1 of $\mathrm{PG}(n, 4)$ is a pseudo-embedding and the pseudo-hyperplanes of $\mathrm{PG}(n, 4)$ arising from e_1 are precisely the (possibly degenerate) Hermitian varieties of $\mathrm{PG}(n, 4)$, distinct from the whole point-set.

Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0, 1\}$ and $n \geq 0$. The map e_2 which maps every point $(X_1, X_2, \ldots, X_n)$ of $\mathrm{AG}(n, 4)$ to the point $(1, X_i + X_i^2, \delta X_i + \delta^2 X_i^2 \,|\, 1 \leq i \leq n)$ of $\mathrm{PG}(2n, 2)$ is called a *quadratic embedding* of $\mathrm{AG}(n, 4)$ into $\mathrm{PG}(2n, 2)$. Observe that the map e_2 depends on the chosen reference systems in $\mathrm{AG}(n, 4)$ and $\mathrm{PG}(2n, 2)$. If e_2 and e_2' are two quadratic embeddings of $\mathrm{AG}(n, 4)$ into $\mathrm{PG}(2n, 2)$, then there exists a projectivity η of PG$(2n,2)$ such that $e_2' = \eta \circ e_2$. So, up to isomorphism, there exists a unique quadratic embedding of AG$(n,4)$ into PG$(2n,2)$. If α is an m-dimensional subspace ($m \in \{0,1,\ldots,n\}$) of AG$(n,4)$, then the quadratic embedding of AG$(n,4)$ will induce "an embedding" of α into a subspace of PG$(2n,2)$ which is isomorphic to the quadratic embedding of $\alpha \cong$ AG$(m,4)$. We will prove later (Proposition 3.10(1)) that the quadratic embedding of AG$(n,4)$ is a homogeneous pseudo-embedding.

In De Bruyn [1, Proposition 3.3(1)], we proved that the projective space PG$(n,4)$, $n \geq 0$, has pseudo-embeddings. We used Sherman's classification [9] of the pseudo-hyperplanes of PG$(n,4)$ to prove that the pseudo-embedding rank of PG$(n,4)$ is equal to $\frac{1}{3}(n+1)(n^2+2n+3)$ (see [1, Proposition 4.1]). In [1, Proposition 3.3(2) and Corollary 4.4], we also proved that the affine space AG$(n,4)$, $n \geq 0$, has pseudo-embeddings and that its pseudo-embedding rank is equal to n^2+n+1. In the present article, we will invoke Sherman's classification of the pseudo-hyperplanes of PG$(n,4)$ to give explicit descriptions for the universal pseudo-embeddings of PG$(n,4)$ and AG$(n,4)$.

Theorem 1.1 *Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0,1\}$ and $n \geq 0$. Let $\widetilde{e_1}$ be a map from* PG$(n,4)$ *to* PG$(k,2)$, $k = \frac{n^3+3n^2+5n}{3}$, *mapping the point $p = (X_0, X_1, \ldots, X_n)$ of* PG$(n,4)$ *to the point $\widetilde{e_1}(p) = (Y_0, Y_1, \ldots, Y_k)$ of* PG$(k,2)$*, where*

- *$n+1$ coordinates of $\widetilde{e_1}(p)$ are of the form X_i^3, where $i \in \{0,1,\ldots,n\}$;*
- *$\binom{n+1}{2}$ coordinates of $\widetilde{e_1}(p)$ are of the form $X_iX_j^2 + X_i^2X_j$, where $i,j \in \{0,1,\ldots,n\}$ and $i < j$;*
- *$\binom{n+1}{2}$ coordinates of $\widetilde{e_1}(p)$ are of the form $\delta X_iX_j^2 + \delta^2X_i^2X_j$, where $i,j \in \{0,1,\ldots,n\}$ and $i < j$;*
- *$\binom{n+1}{3}$ coordinates of $\widetilde{e_1}(p)$ are of the form $X_iX_jX_k + X_i^2X_j^2X_k^2$, where $i,j,k \in \{0,1,\ldots,n\}$ and $i < j < k$;*
- *$\binom{n+1}{3}$ coordinates of $\widetilde{e_1}(p)$ are of the form $\delta X_iX_jX_k + \delta^2X_i^2X_j^2X_k^2$, where $i,j,k \in \{0,1,\ldots,n\}$ and $i < j < k$.*

Then $\widetilde{e_1}$ is a pseudo-embedding of PG$(n,4)$ *which is isomorphic to the universal pseudo-embedding of* PG$(n,4)$.

Theorem 1.2 *Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0,1\}$ and $n \geq 0$. Let $\widetilde{e_2}$ be the map from* AG$(n,4)$ *to* PG$(n^2+n,2)$ *mapping the point $p = (X_1, X_2, \ldots, X_n)$ of* AG$(n,4)$ *to the point $\widetilde{e_2}(p) = (Y_0, Y_1, \ldots, Y_{n^2+n})$ of* PG$(n^2+n,2)$*, where*

- *one coordinate of $\widetilde{e_2}(p)$ is equal to 1;*
- *n coordinates of $\widetilde{e_2}(p)$ are of the form $X_i + X_i^2$, where $i \in \{1,2,\ldots,n\}$;*
- *n coordinates of $\widetilde{e_2}(p)$ are of the form $\delta X_i + \delta^2X_i^2$, where $i \in \{1,2,\ldots,n\}$;*
- *$\binom{n}{2}$ coordinates of $\widetilde{e_2}(p)$ are of the form $X_iX_j + X_i^2X_j^2$, where $i,j \in \{1,2,\ldots,n\}$ and $i < j$;*
- *$\binom{n}{2}$ coordinates of $\widetilde{e_2}(p)$ are of the form $\delta X_iX_j + \delta^2X_i^2X_j^2$, where $i,j \in \{1,2,\ldots,n\}$ and $i < j$.*

Then $\widetilde{e_2}$ is a pseudo-embedding of AG$(n,4)$ *which is isomorphic to the universal pseudo-embedding of* AG$(n,4)$.

The following is an immediate consequence of Theorems 1.1 and 1.2 (choose suitable reference systems).

Corollary 1.3 (1) *Suppose $\widetilde{e_1}$ is the universal pseudo-embedding of* PG$(n,4)$, $n \geq 0$, *and π is a nonempty subspace of* PG$(n,4)$. *Then the pseudo-embedding of π induced by $\widetilde{e_1}$ is isomorphic to the universal pseudo-embedding of π.*

(2) *Suppose* $\widetilde{e}_2$ *is the universal pseudo-embedding of* $\mathrm{AG}(n,4)$, $n \geq 0$, *and* π *is a nonempty subspace of* $\mathrm{AG}(n,4)$. *Then the pseudo-embedding of* π *induced by* $\widetilde{e}_2$ *is isomorphic to the universal pseudo-embedding of* π.

In the next two theorems, we determine all homogeneous pseudo-embeddings of $\mathrm{PG}(n,4)$ and $\mathrm{AG}(n,4)$. In fact, we do a little more. We determine all G-homogeneous pseudo-embeddings where $G \in \{PGL(n+1,4), AGL(n,4)\}$ is the group of collineations of $\mathrm{PG}(n,4)$ or $\mathrm{AG}(n,4)$ whose companion automorphism of $\mathbb{F}_4$ is the identity.

Theorem 1.4 *Up to isomorphism, the projective space* $\mathrm{PG}(n,4)$, $n \geq 2$, *has two* $PGL(n+1,4)$*-homogeneous pseudo-embeddings, the universal pseudo-embedding in* $\mathrm{PG}(\frac{1}{3}(n^3+3n^2+5n),2)$ *and the Hermitian Veronese embedding in* $\mathrm{PG}(n^2+2n,2)$.

Theorem 1.5 *Up to isomorphism, the affine space* $\mathrm{AG}(n,4)$, $n \geq 2$, *has two* $AGL(n,4)$*-homogeneous pseudo-embeddings, the universal pseudo-embedding in* $\mathrm{PG}(n^2+n,2)$ *and the quadratic pseudo-embedding in* $\mathrm{PG}(2n,2)$. *There are two types of pseudo-hyperplanes arising from the quadratic pseudo-embedding of* $\mathrm{AG}(n,4)$, $n \geq 1$, *namely the empty set and those pseudo-hyperplanes which are the union of two distinct parallel hyperplanes.*

In Theorem 1.6 below, we give a list of all pseudo-hyperplanes of $\mathrm{AG}(n,4)$, $n \geq 2$. In order to understand that theorem, we need to give some definitions.

Suppose the affine space $\mathrm{AG}(n,4)$, $n \geq 2$, is obtained by removing a hyperplane Π_∞ from the projective space $\mathrm{PG}(n,4)$. Suppose D is a subspace[1] of Π_∞ and X is a nonempty set of points of $\mathrm{AG}(n,4)$ in a subspace of $\mathrm{PG}(n,4)$ which is disjoint from D. If $D=\emptyset$, then we define $\mathcal{C}(D,X) := X$. If $D \neq \emptyset$, then $\mathcal{C}(D,X)$ denotes the set of all points of $\mathrm{AG}(n,4)$ which lie on a line joining a point of D to a point of X. So, if $\mathcal{C}'(D,X)$ denotes the cone of $\mathrm{PG}(n,4)$ with top D and basis X, then $\mathcal{C}(D,X) = \mathcal{C}'(D,X) \setminus \Pi_\infty$. If Π is a subspace of $\mathrm{AG}(n,4)$, then D_Π denotes the set of points of Π_∞ such that $\Pi \cup D_\Pi$ is the subspace of $\mathrm{PG}(n,4)$ generated by Π.

Let Q be a nonsingular parabolic quadric[2] in $\mathrm{PG}(n,4)$, $n \geq 4$ even, let k be the kernel of Q, let $p \neq k$ be a point of $\mathrm{PG}(n,4)$ not contained in Q and let Π be a hyperplane of $\mathrm{PG}(n,4)$ not containing p. The line kp intersects Q in a point p' and the tangent hyperplane $T_{p'}$ at the point p' to the quadric Q intersects Π in a hyperplane Π_∞ of Π. We denote by $\mathrm{AG}(n-1,4)$ the affine space obtained from $\Pi \cong \mathrm{PG}(n-1,4)$ by removing the hyperplane Π_∞ of Π. Now, the projection of Q from the point p onto Π is a set Y of points of Π containing Π_∞. By Hirschfeld and Thas [5, Theorem 13], every line of Π intersects Y in either 1, 3 or 5 points. This implies that the set $X := Y \setminus \Pi_\infty$ is a pseudo-hyperplane of $\mathrm{AG}(n-1,4)$. We call X a *set of parabolic type* of $\mathrm{AG}(n-1,4)$.

Let Q be a nonsingular hyperbolic or elliptic quadric in $\mathrm{PG}(n,4)$, $n \geq 3$ odd, let p be a point of $\mathrm{PG}(n,4)$ not contained in Q and let Π be a hyperplane of $\mathrm{PG}(n,4)$ not containing p. Let ζ be the symplectic polarity of $\mathrm{PG}(n,4)$ associated with Q. Then the hyperplane p^ζ of $\mathrm{PG}(n,4)$ intersects Π in a hyperplane Π_∞ of Π. We denote by $\mathrm{AG}(n-1,4)$ the affine space obtained from $\Pi \cong \mathrm{PG}(n-1,4)$ by removing the hyperplane Π_∞ from Π. Now, the projection of Q from the point p onto Π is a set Y of points of Π containing Π_∞. By Hirschfeld and Thas [5, Theorem 13], every line of Π intersects Y in either 1, 3 or 5 points. This implies that the set $X := Y \setminus \Pi_\infty$ is a pseudo-hyperplane of $\mathrm{AG}(n-1,4)$. We call X

[1] The elements of D correspond to certain directions in the affine space $\mathrm{AG}(n,4)$.

[2] For the basic notions of properties regarding quadrics of finite projective spaces which we will use in this article, see Hirschfeld and Thas [7, Chapter 22].

Table 1 The pseudo-hyperplanes of AG$(n, 4)$, $n \geq 2$

Type	# Pseudo-hyperplanes	# Points	Complement
(1)	1	0	AG$(n, 4)$
(2)	$2^{2n+1} - 2$	2^{2n-1}	(2)
(3)	$6 \cdot 4^{m(m-1)} \cdot \begin{bmatrix} n \\ 2m-1 \end{bmatrix}_4 \cdot \prod_{i=1}^{m-1} (4^{2i+1} - 1)$	2^{2n-1}	(3)
(4)	$3 \cdot 4^{m(m+1)} \cdot \begin{bmatrix} n \\ 2m \end{bmatrix}_4 \cdot \prod_{i=1}^{m-1} (4^{2i+1} - 1)$	$2^{2n-1} + 2^{2n-2m-1}$	(5)
(5)	$3 \cdot 4^{m(m+1)} \cdot \begin{bmatrix} n \\ 2m \end{bmatrix}_4 \cdot \prod_{i=1}^{m-1} (4^{2i+1} - 1)$	$2^{2n-1} - 2^{2n-2m-1}$	(4)

a *set of hyperbolic* or *elliptic type* of AG$(n-1, 4)$ depending on whether Q is a hyperbolic or elliptic quadric of PG$(n, 4)$.

Theorem 1.6 *Let* AG$(n, 4)$, $n \geq 2$, *be the affine space obtained from* PG$(n, 4)$ *by removing a hyperplane* Π_∞. *A pseudo-hyperplane of* AG$(n, 4)$ *is one of the following sets of points:*

(1) *the empty set;*
(2) *the union of two disjoint parallel hyperplanes;*
(3) *a set* $\mathcal{C}(D, X)$, *where* D *is a subspace of dimension* $(n-2m)$, $m \in \{2, \ldots, \lfloor \frac{n+1}{2} \rfloor\}$, *of* Π_∞ *and* X *is a set of parabolic type of a* $(2m-1)$*-dimensional subspace* Π *of* AG$(n, 4)$ *for which* $D \cap D_\Pi = \emptyset$;
(4) *a set* $\mathcal{C}(D, X)$, *where* D *is a subspace of dimension* $(n-2m-1)$, $m \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, *of* Π_∞ *and* X *is set of hyperbolic type of a* $2m$*-dimensional subspace* Π *of* AG$(n, 4)$ *for which* $D \cap D_\Pi = \emptyset$;
(5) *a set* $\mathcal{C}(D, X)$, *where* D *is a subspace of dimension* $(n-2m-1)$, $m \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, *of* Π_∞ *and* X *is set of elliptic type of a* $2m$*-dimensional subspace* Π *of* AG$(n, 4)$ *for which* $D \cap D_\Pi = \emptyset$.

In Table 1, we list a few basic properties of the five classes of pseudo-hyperplanes of AG$(n, 4)$, $n \geq 2$, as they occur in Theorem 1.6. We list how many pseudo-hyperplanes there are of each type, the total number of points in each pseudo-hyperplane and the type of the complement of the pseudo-hyperplane. Notice here that for each of the pseudo-hyperplanes of Type (3), (4) and (5), the pseudo-hyperplane which arises as complement has the same value for the parameter m. Observe also the occurrence of Gaussian binomial coefficients in the formulas for the total number of pseudo-hyperplanes.

The points and lines of the projective space PG$(4, 3)$ that are contained in a given non-singular quadric of PG$(4, 3)$ are the points and lines of a generalized quadrangle which we denote by $Q(4, 3)$. In De Bruyn [2], we used the computer algebra system GAP [16] to show that $Q(4, 3)$ has, up to isomorphism, two homogeneous pseudo-embeddings, the universal pseudo-embedding in PG$(14, 2)$ and a certain homogeneous pseudo-embedding in PG$(8, 2)$. No direct constructions for these two homogeneous embeddings were however given in [2]. Theorem 1.7 below gives direct constructions for these pseudo-embeddings.

Thas [14, Section 5.2] (see also Payne and Thas [8, Theorem 7.4.1]) proved that the generalized quadrangle $Q(4, 3)$ is fully embeddable into AG$(4, 4)$. From Thas and Van Maldeghem [15, Theorem 5.1], we know that every full embedding e of $Q(4, 3)$ into AG$(4, 4)$ is homogeneous, i.e. for every automorphism θ of $Q(4, 3)$, there exists a (necessarily unique) collineation η_θ of AG$(4, 4)$ such that $e(x^\theta) = e(x)^{\eta_\theta}$ for every point x of $Q(4, 3)$.

The fact that every full embedding of $Q(4,3)$ into $\mathrm{AG}(4,4)$ is homogeneous implies that if the generalized quadrangle $Q(4,3)$ is a full subgeometry of $\mathrm{AG}(4,4)$, then every homogeneous pseudo-embedding of $\mathrm{AG}(4,4)$ will induce a homogeneous pseudo-embedding of $Q(4,3)$. We will prove the following.

Theorem 1.7 *Regard $Q(4,3)$ as a full subgeometry of $\mathrm{AG}(4,4)$. Then the following holds.*

(1) *The universal pseudo-embedding of $\mathrm{AG}(4,4)$ will induce a pseudo-embedding of $Q(4,3)$ which is isomorphic to the universal pseudo-embedding of $Q(4,3)$.*
(2) *The quadratic embedding of $\mathrm{AG}(4,4)$ will induce a pseudo-embedding of $Q(4,3)$ which is isomorphic to the homogeneous pseudo-embedding of $Q(4,3)$ into $\mathrm{PG}(8,2)$.*

2 The recognition of G-homogeneous pseudo-embeddings

Let $\mathcal{S}$ be a point-line geometry with the property that the number of points on each line is finite and at least three, and let G be a group of automorphisms of $\mathcal{S}$. In this section, we give a criterion, proved in De Bruyn [2], to decide whether a given pseudo-embedding of $\mathcal{S}$ is G-homogeneous. This criterion was used in [2] to determine all homogeneous pseudo-embeddings of all generalized quadrangles of order $(3,t)$. In the present article, we will use this criterion to determine all homogeneous pseudo-embeddings of $\mathrm{PG}(n,4)$ and $\mathrm{AG}(n,4)$. While the classification of the homogeneous pseudo-embeddings in [2] needed the use of a computer (GAP), the classification of the homogeneous pseudo-embeddings in the present article will be computer free.

Proposition 2.1 ([2, Corollary 2.7]) *Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. Let G be a group of automorphisms of $\mathcal{S}$.*

- *If $e:\mathcal{S}\to\Sigma$ is a G-homogeneous pseudo-embedding of $\mathcal{S}$, then the set $\mathcal{A}_e$ of all pseudo-hyperplanes of $\mathcal{S}$ arising from e satisfies the following properties:*
  (a) *$\mathcal{A}_e$ can be written as a disjoint union $\bigcup_{i\in I}\mathcal{H}_i$, where each $\mathcal{H}_i$, $i\in I$, is a G-orbit of pseudo-hyperplanes of $\mathcal{S}$;*
  (b) *if H_1 and H_2 are two distinct elements of $\mathcal{A}_e$, then also the complement of the symmetric difference of H_1 and H_2 belongs to $\mathcal{A}_e$;*
  (c) *if L is a line of $\mathcal{S}$ containing an odd number of points, then for every point x of L there exists a pseudo-hyperplane of $\mathcal{A}_e$ which has only the point x in common with L;*
  (d) *if L is a line of $\mathcal{S}$ containing an even number of points, then for any two distinct points x_1 and x_2 of L, there exists a pseudo-hyperplane of $\mathcal{A}_e$ having only the points x_1 and x_2 in common with L;*
  (e) *for every point x of $\mathcal{S}$, there exists a pseudo-hyperplane of $\mathcal{A}_e$ not containing x.*
- *Conversely, suppose that $\mathcal{A}$ is a finite set of pseudo-hyperplanes of $\mathcal{S}$ satisfying the conditions* (a), (b), (c), (d) *and* (e) *above. Then there exists a pseudo-embedding e of $\mathcal{S}$ such that the pseudo-hyperplanes of $\mathcal{S}$ arising from e are precisely the elements of $\mathcal{A}$. This pseudo-embedding e is uniquely determined, up to isomorphism, and is G-homogeneous.*

Observe that condition (e) in Proposition 2.1 follows from conditions (c) and (d) if there is at least one line incident with x.

3 The homogeneous pseudo-embeddings of PG$(n, 4)$ and AG$(n, 4)$

3.1 The universal pseudo-embeddings of PG$(n, 4)$ and AG$(n, 4)$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three, and let e be a map from $\mathcal{P}$ to the point set of a projective space. The following theorem can be useful to decide whether the map e is a pseudo-embedding of $\mathcal{S}$.

Theorem 3.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. Let V_1 and V_2 be two vector spaces over $\mathbb{F}_2$. For every $i \in \{1, 2\}$, let e_i be a map from the point set $\mathcal{P}$ of $\mathcal{S}$ to the point set of $\mathrm{PG}(V_i)$ and let $\mathcal{H}_i$ be the set of all sets of the form $e_i^{-1}(e_i(\mathcal{P}) \cap \Pi)$, where Π is some hyperplane of $\mathrm{PG}(V_i)$. If e_1 is a pseudo-embedding of $\mathcal{S}$ and $\mathcal{H}_1 = \mathcal{H}_2$, then also e_2 is a pseudo-embedding of $\mathcal{S}$. Moreover, e_2 is isomorphic to e_1.*

Proof (1) By definition, the set $\mathcal{H}_1$ is the set of pseudo-hyperplanes of $\mathcal{S}$ arising from e_1. By De Bruyn [1, Lemma 2.2], we know that $\mathcal{H}_1$ satisfies the following property:

($*$) For every line L of $\mathcal{S}$ and every set X of points of L for which $|L| - |X| \neq 0$ is even, there exists a pseudo-hyperplane of $\mathcal{H}_1$ intersecting L in X.

(2) Suppose $< e_2(\mathcal{P}) >$ is a proper subspace of $\mathrm{PG}(V_2)$. Then there exists a hyperplane Π of $\mathrm{PG}(V_2)$ through $< e_2(\mathcal{P}) >$ and we have $\mathcal{P} = e_2^{-1}(e_2(\mathcal{P}) \cap \Pi) \in \mathcal{H}_2 = \mathcal{H}_1$. This is however impossible since $\mathcal{P}$ is not a pseudo-hyperplane of $\mathcal{S}$. Hence, $< e_2(\mathcal{P}) >= \mathrm{PG}(V_2)$.

(3) Let L be an arbitrary line of $\mathcal{S}$ with points $x_1, x_2, \ldots, x_k$. If the points $e_2(x_1), e_2(x_2), \ldots, e_2(x_k)$ are linearly independent, then there is a hyperplane Π of $\mathrm{PG}(V_2)$ containing $e_2(x_1), e_2(x_2), \ldots, e_2(x_{k-1})$, but not $e_2(x_k)$. Then $H = e_2^{-1}(e_2(\mathcal{P}) \cap \Pi)$ contains the points $x_1, x_2, \ldots, x_{k-1}$ but not the point x_k and hence cannot be a pseudo-hyperplane of $\mathcal{S}$. But this is impossible. The set H belongs to $\mathcal{H}_2$ and hence also to the set $\mathcal{H}_1 = \mathcal{H}_2$ of pseudo-hyperplanes of $\mathcal{S}$.
Now, let $I = \{i_1, i_2, \ldots, i_l\}$ be a subset of $\{1, 2, \ldots, k\}$ of smallest size l such that $e_2(x_{i_1}), e_2(x_{i_2}), \ldots, e_2(x_{i_l})$ is a linearly dependent collection of points. Without loss of generality, we may suppose that $I = \{1, 2, \ldots, l\}$. We prove that $l = k$. Suppose to the contrary that $l < k$. Every subspace of $\mathrm{PG}(V_2)$ containing $e_2(x_1), e_2(x_2), \ldots, e_2(x_{l-1})$ also contains $e_2(x_l)$. As a consequence, every pseudo-hyperplane of $\mathcal{H}_1 = \mathcal{H}_2$ containing $x_1, x_2, \ldots, x_{l-1}$ also contains x_l. But this is impossible. By Property ($*$), there exists a pseudo-hyperplane of $\mathcal{H}_1$ which intersects L in either $\{x_1, x_2, \ldots, x_{l-1}\}$ or $\{x_1, x_2, \ldots, x_{l-1}, x_{l+1}\}$.

(4) By (2) and (3) above, e_2 is a pseudo-embedding of $\mathcal{S}$. Now, let $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ denote the universal pseudo-embedding of $\mathcal{S}$ and let α_1 and α_2 be subspaces of $\widetilde{\Sigma}$ such that $\widetilde{e}/\alpha_1 \cong e_1$ and $\widetilde{e}/\alpha_2 \cong e_2$. If $\alpha_1 \neq \alpha_2$, then there exists a hyperplane Π of $\widetilde{\Sigma}$ containing precisely one of α_1, α_2. This implies that the pseudo-hyperplane $\widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi)$ belongs to precisely one of $\mathcal{H}_1, \mathcal{H}_2$, clearly impossible since $\mathcal{H}_1 = \mathcal{H}_2$. So, $\alpha_1 = \alpha_2$ and $e_1 \cong e_2$. $\square$

A set X of points of a point-line geometry $\mathcal{S}$ is called a *set of even* [resp. *odd*] *type* if it intersects every line of $\mathcal{S}$ in an even [resp. odd] number of points. In [9], Sherman classified all sets of odd type of $\mathrm{PG}(n, 4)$, $n \geq 0$. The following two propositions summarize his classification.

Proposition 3.2 ([9]) *Let* $(X_0, X_1, \dots, X_n)$ *denote the homogeneous coordinates of the points of* $\mathrm{PG}(n,4)$, $n \geq 0$, *with respect to a certain reference system of* $\mathrm{PG}(n,4)$. *Then the sets of odd type of* $\mathrm{PG}(n,4)$ *are precisely those sets whose equation*[3] *with respect to the reference system of* $\mathrm{PG}(n,4)$ *has the form* $H + E + E^2 = 0$, *where*

(1) $H = \sum_{i=0}^{n} a_i X_i^3 + \sum_{0 \leq i < j \leq n} b_{ij} X_i X_j^2 + b_{ij}^2 X_j X_i^2$,
(2) $E = \sum_{0 \leq i < j < k \leq n} c_{ijk} X_i X_j X_k$,
(3) $a_i \in \{0, 1\}$ *for every* $i \in \{0, 1, \dots, n\}$,
(4) $b_{ij} \in \mathbb{F}_4$ *for all* $i, j \in \{0, 1, \dots, n\}$ *satisfying* $i < j$,
(5) $c_{ijk} \in \mathbb{F}_4$ *for all* $i, j, k \in \{0, 1, \dots, n\}$ *satisfying* $i < j < k$.

Proposition 3.3 ([9]) *Let* A_1 *and* A_2 *be two sets of odd type of* $\mathrm{PG}(n,4)$, $n \geq 0$, *with respective equations* $H_1 + E_1 + E_1^2 = 0$ *and* $H_2 + E_2 + E_2^2 = 0$, *where* H_1, E_1, H_2 *and* E_2 *satisfy the conditions* (1), (2), (3), (4) *and* (5) *of Proposition* 3.2. *Then* $A_1 = A_2$ *if and only if* $(H_1, E_1) = (H_2, E_2)$.

The pseudo-hyperplanes of $\mathrm{PG}(n,4)$, $n \geq 0$, arising from the universal pseudo-embedding of $\mathrm{PG}(n,4)$ are all the sets of odd type of $\mathrm{PG}(n,4)$, distinct from the whole point-set. Theorem 1.1 therefore immediately follows from Theorem 3.1 and Propositions 3.2 and 3.3.

The following theorem easily follows from Propositions 3.2 and 3.3.

Theorem 3.4 *Let* $(X_1, X_2, \dots, X_n)$ *denote the coordinates of the points of* $\mathrm{AG}(n,4)$, $n \geq 0$, *with respect to a certain coordinate system of* $\mathrm{AG}(n,4)$. *Then the sets of even type of* $\mathrm{AG}(n,4)$ *are precisely those sets whose equation with respect to the coordinate system of* $\mathrm{AG}(n,4)$ *has the form* $H + E + E^2 = 0$, *where*

(1) $H = a + \sum_{1 \leq i \leq n} b_i X_i + b_i^2 X_i^2$,
(2) $E = \sum_{1 \leq i < j \leq n} c_{ij} X_i X_j$,
(3) $a \in \{0, 1\}$,
(4) $b_i \in \mathbb{F}_4$ *for every* $i \in \{1, 2, \dots, n\}$,
(5) $c_{ij} \in \mathbb{F}_4$ *for all* $i, j \in \{1, 2, \dots, n\}$ *satisfying* $i < j$.

If A_1 *and* A_2 *are two sets of even type of* $\mathrm{AG}(n,4)$ *with respective equations* $H_1 + E_1 + E_1^2 = 0$ *and* $H_2 + E_2 + E_2^2 = 0$, *where* H_1, E_1, H_2 *and* E_2 *satisfy the conditions* (1), (2), (3), (4) *and* (5) *above, then* $A_1 = A_2$ *if and only if* $(H_1, E_1) = (H_2, E_2)$.

Proof Suppose $\mathrm{AG}(n,4)$ is obtained from $\mathrm{PG}(n,4)$ by removing a hyperplane Π_∞ from $\mathrm{PG}(n,4)$. Choose a reference system in $\mathrm{PG}(n,4)$ with coordinates $(X_0, X_1, \dots, X_n)$ such that Π_∞ has equation $X_0 = 0$. We denote the point $(1, X_1, X_2, \dots, X_n)$ of $\mathrm{PG}(n,4)$ also by $(X_1, X_2, \dots, X_n)$.

Now, a set A of points of $\mathrm{AG}(n,4)$ is a set of even type of $\mathrm{AG}(n,4)$ if and only if $A \cup \Pi_\infty$ is a set of odd type of $\mathrm{PG}(n,4)$. If $H + E + E^2 = 0$ is the equation of $A \cup \Pi_\infty$, where H and E are as in Proposition 3.2, then the fact that $\Pi_\infty \subseteq A \cup \Pi_\infty$ implies by Proposition 3.3 that $a_i = 0$ for all $i \in \{1, 2, \dots, n\}$, $b_{ij} = 0$ for all $i, j \in \{1, 2, \dots, n\}$ with $i < j$ and $c_{ijk} = 0$ for all $i, j, k \in \{1, 2, \dots, n\}$ satisfying $i < j < k$.

So, if we put $a := a_0, b_i := b_{0i}^2$ for every $i \in \{1, 2, \dots, n\}$ and $c_{ij} = c_{0ij}$ for all $i, j \in \{1, 2, \dots, n\}$ satisfying $i < j$, we readily see that the theorem holds. □

The pseudo-hyperplanes of $\mathrm{AG}(n,4)$, $n \geq 0$, arising from the universal pseudo-embedding of $\mathrm{AG}(n,4)$ are all the sets of even type of $\mathrm{AG}(n,4)$ distinct from the whole set of points. Theorem 1.2 therefore immediately follows from Theorems 3.1 and 3.4.

[3] The homogeneous coordinates of a point are only determined up to a nonzero factor. However, since $\lambda^3 = 1$ for every $\lambda \in \mathbb{F}_4 \setminus \{0\}$, these equations are well-defined.

3.2 The homogeneous pseudo-embeddings of PG$(n, 4)$, $n \geq 2$

Consider the projective space PG$(n, 4), n \geq 2$. The universal pseudo-embedding of PG$(n, 4)$ is homogeneous. The pseudo-hyperplanes of PG$(n, 4)$ arising from the Hermitian Veronese embedding of PG$(n, 4)$ are precisely the (possibly degenerate) Hermitian varieties distinct from the whole point set. So, by Proposition 2.1, also the Hermitian Veronese embedding of PG$(n, 4)$ is a homogeneous pseudo-embedding (off course, one can also verify this in a more direct way). We now prove that the universal pseudo-embedding of PG$(n, 4)$ and the Hermitian Veronese embedding of PG$(n, 4)$ are the only $PGL(n+1, 4)$-homogeneous pseudo-embeddings of PG$(n, 4), n \geq 2$ (and hence also the only homogeneous pseudo-embeddings of PG$(n, 4), n \geq 2$).

Fix a certain reference system in PG$(n, 4)$ and let $(X_0, X_1, \ldots, X_n)$ denote the coordinates of a general point of PG$(n, 4)$ with respect to that reference system. We denote by $\mathcal{H}$ the set of all polynomials of the form $\sum_{i=0}^{n} a_i X_i^3 + \sum_{0 \leq i < j \leq n} b_{ij} X_i X_j^2 + b_{ij}^2 X_j X_i^2$, where $a_i \in \{0, 1\}$ for every $i \in \{0, 1, \ldots, n\}$ and $b_{ij} \in \mathbb{F}_4$ for all $i, j \in \{0, 1, \ldots, n\}$ satisfying $i < j$. We denote by $\mathcal{E}$ the set of all polynomials of the form $\sum_{0 \leq i < j < k \leq n} c_{ijk} X_i X_j X_k$, where $c_{ijk} \in \mathbb{F}_4$ for all $i, j, k \in \{0, 1, \ldots, n\}$ satisfying $i < j < k$. If $H \in \mathcal{H}$ and $E \in \mathcal{E}$, then $\Omega(H, E)$ denotes the set of odd type of PG$(n, 4)$ whose equation with respect to the fixed reference system is given by $H + E + E^2 = 0$. We denote by $\mathcal{I}$ the ideal of the polynomial ring $\mathbb{F}_4[X_0, X_1, \ldots, X_n]$ generated by the polynomials $X_0^4 - X_0, X_1^4 - X_1, \ldots, X_n^4 - X_n$.

Suppose e is a $PGL(n+1, 4)$-homogeneous pseudo-embedding of PG$(n, 4)$ and let $\mathcal{A}_e$ denote the set of all pseudo-hyperplanes of PG$(n, 4)$ arising from e. The condition mentioned in Proposition 2.1(b) translates to:

- (P1) Let $H_1, H_2 \in \mathcal{H}$ and $E_1, E_2 \in \mathcal{E}$ such that $(H_1, E_1) \neq (H_2, E_2)$. If $\Omega(H_1, E_1)$ and $\Omega(H_2, E_2)$ belong to $\mathcal{A}_e$, then also $\Omega(H_1 + H_2, E_1 + E_2)$ belongs to $\mathcal{A}_e$.

The condition mentioned in Proposition 2.1(a) and the fact that e is $PGL(n+1, 4)$-homogeneous implies that the properties (P2), (P3) and (P4) below hold.

- (P2) Let σ be a permutation of $\{0, 1, \ldots, n\}$ and let $(H_1, E_1) \in \mathcal{H} \times \mathcal{E}$. Let H_2 and E_2 be derived from H_1 and E_1, respectively, by applying the following substitutions: $X_i \mapsto X_{\sigma(i)}, \forall i \in \{0, 1, \ldots, n\}$. Then $\Omega(H_1, E_1) \in \mathcal{A}_e$ if and only if $\Omega(H_2, E_2) \in \mathcal{A}_e$.
- (P3) Let $i \in \{0, 1, \ldots, n\}, \lambda \in \mathbb{F}_4 \setminus \{0\}$ and $(H_1, E_1) \in \mathcal{H} \times \mathcal{E}$. Let H_2 and E_2 be derived from H_1 and E_1, respectively, by applying the following substitutions: $X_j \mapsto X_j, \forall j \in \{0, 1, \ldots, n\} \setminus \{i\}$, and $X_i \mapsto \lambda \cdot X_i$. Then $\Omega(H_1, E_1) \in \mathcal{A}_e$ if and only if $\Omega(H_2, E_2) \in \mathcal{A}_e$.
- (P4) Let $i_1, i_2 \in \{0, 1, \ldots, n\}$ with $i_1 \neq i_2$ and let $(H_1, E_1) \in \mathcal{H} \times \mathcal{E}$. Let $H_2, H_2' \in \mathcal{H}$, $E_2 \in \mathcal{E}$ and $I \in \mathcal{I}$ such that H_2 and $H_2' + E_2 + E_2^2 + I$ are derived from respectively H_1 and $E_1 + E_1^2$ by applying the following substitutions: $X_j \mapsto X_j, \forall j \in \{0, 1, \ldots, n\} \setminus \{i_1\}$, and $X_{i_1} \mapsto X_{i_1} + X_{i_2}$. Then $\Omega(H_1, E_1) \in \mathcal{A}_e$ if and only if $\Omega(H_2 + H_2', E_2) \in \mathcal{A}_e$.

Lemma 3.5 *If $\Omega(X_0 X_1^2 + X_1 X_0^2, 0) \in \mathcal{A}_e$, then $\Omega(H, 0) \in \mathcal{A}_e$ for all $H \in \mathcal{H} \setminus \{0\}$.*

Proof
- By Properties (P2) and (P3), we have $\Omega(b_{ij} X_i X_j^2 + b_{ij}^2 X_j X_i^2, 0) \in \mathcal{A}_e$ for all $i, j \in \{0, 1, \ldots, n\}$ with $i < j$ and all $b_{ij} \in \mathbb{F}_4 \setminus \{0\}$.
- Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0, 1\}$ and consider the substitutions $X_0 \mapsto X_0 + \delta X_1, X_i \mapsto X_i, \forall i \in \{1, 2, \ldots, n\}$. By Properties (P3) and (P4), $\Omega(X_0 X_1^2 + X_1 X_0^2 + X_1^3, 0) \in \mathcal{A}_e$. Hence, also $\Omega(X_1^3, 0) = \Omega(X_0 X_1^2 + X_1 X_0^2 + X_1^3 + X_0 X_1^2 + X_1 X_0^2, 0) \in \mathcal{A}_e$ by Property (P1). Property (P2) then implies that $\Omega(X_i^3, 0) \in \mathcal{A}_e$ for all $i \in \{0, 1, \ldots, n\}$.
- The two previous paragraphs and Property (P1) imply that $\Omega(H, 0) \in \mathcal{A}_e$ for all $H \in \mathcal{H} \setminus \{0\}$. □

Lemma 3.6 *If $\Omega(X_0^3, 0) \in \mathcal{A}_e$, then $\Omega(H, 0) \in \mathcal{A}_e$ for all $H \in \mathcal{H} \setminus \{0\}$.*

Proof By Property (P2), we also have $\Omega(X_1^3, 0) \in \mathcal{A}_e$. Now, consider the substitution $X_0 \mapsto X_0 + X_1, X_i \mapsto X_i, \forall i \in \{1, 2, \ldots, n\}$. Then Property (P4) implies that $\Omega(X_0^3 + X_1^3 + X_0X_1^2 + X_1X_0^2, 0) \in \mathcal{A}_e$. By Property (P1), we have $\Omega(X_0X_1^2 + X_1X_0^2, 0) = \Omega(X_0^3 + X_1^3 + X_0^3 + X_1^3 + X_0X_1^2 + X_1X_0^2, 0) \in \mathcal{A}_e$. By Lemma 3.5, $\Omega(H, 0) \in \mathcal{A}_e$ for all $H \in \mathcal{H} \setminus \{0\}$. □

Lemma 3.7 *If $\Omega(0, X_0X_1X_2) \in \mathcal{A}_e$, then $\Omega(H, E) \in \mathcal{A}_e$ for all $(H, E) \in \mathcal{H} \times \mathcal{E} \setminus \{(0, 0)\}$.*

Proof
- By Properties (P2) and (P3), we have $\Omega(0, c_{ijk}X_iX_jX_k) \in \mathcal{A}_e$ for all $i, j, k \in \{0, 1, \ldots, n\}$ with $i < j < k$ and all $c_{ijk} \in \mathbb{F}_4 \setminus \{0\}$. By Property (P1), it then follows that $\Omega(0, E) \in \mathcal{A}_e$ for all $E \in \mathcal{E} \setminus \{0\}$.
- Consider the substitution $X_0 \mapsto X_0 + X_1, X_i \mapsto X_i, \forall i \in \{1, 2, \ldots, n\}$. By Property (P4), $\Omega(X_1X_2^2 + X_2X_1^2, X_0X_1X_2) \in \mathcal{A}_e$. Hence, by Property (P1), $\Omega(X_1X_2^2 + X_2X_1^2, 0) = \Omega(X_1X_2^2 + X_2X_1^2 + 0, X_0X_1X_2 + X_0X_1X_2) \in \mathcal{A}_e$. By Lemma 3.5 and Property (P2), we have $\Omega(H, 0) \in \mathcal{A}_e$ for all $H \in \mathcal{H} \setminus \{0\}$.
- By the previous two paragraphs and Property (P1), we have $\Omega(H, E) \in \mathcal{A}_e$ for all $(H, E) \in \mathcal{H} \times \mathcal{E} \setminus \{(0, 0)\}$. □

Proposition 3.8 *If each element of $\mathcal{A}_e$ is a (possibly degenerate) Hermitian variety of* PG$(n, 4)$*, then e is isomorphic to the Hermitian Veronese embedding of* PG$(n, 4)$.

Proof In this case, there exists an $H \in \mathcal{H} \setminus \{0\}$ such that $\Omega(H, 0) \in \mathcal{A}_e$.

Suppose first that there exist $i, j \in \{0, 1, \ldots, n\}$ with $i < j$ and a $b_{ij} \in \mathbb{F}_4 \setminus \{0\}$ such that the sum $b_{ij}X_iX_j^2 + b_{ij}^2X_jX_i^2$ occurs in H. Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0, 1\}$. Let $H_1 \in \mathcal{H}$ be derived from H by applying the following substitutions: $X_i \mapsto \delta \cdot X_i, X_k \mapsto X_k, \forall k \in \{0, 1, \ldots, n\} \setminus \{i\}$. Then $\Omega(H_1, 0) \in \mathcal{A}_e$ and hence also $\Omega(H_2, 0) \in \mathcal{A}_e$ where $H_2 = H + H_1$. Observe that H_2 only contains terms which involve X_i. Let $H_3 \in \mathcal{H}$ be derived from H_2 by applying the following substitutions: $X_j \mapsto \delta \cdot X_j, X_k \mapsto X_k, \forall k \in \{0, 1, \ldots, n\} \setminus \{j\}$. Then $\Omega(H_3, 0) \in \mathcal{A}_e$ and hence $\Omega(H_4, 0) \in \mathcal{A}_e$ where $H_4 = H_2 + H_3$. Observe that H_4 only contains terms which involve X_i and X_j. We have $H_4 = b_{ij}X_iX_j^2 + b_{ij}^2X_jX_i^2$. By Properties (P2) and (P3), also $\Omega(X_0X_1^2 + X_1X_0^2, 0) \in \mathcal{A}_e$. Lemma 3.5 now implies that $\mathcal{A}_e$ consists of all (possibly degenerate) Hermitian varieties of PG$(n, 4)$. By Theorem 3.1 it then follows that e is isomorphic to the Hermitian Veronese embedding of PG$(n, 4)$.

Suppose next that H has the form $\sum_{i=0}^n a_iX_i^3$ where $a_i \in \{0, 1\}$ for every $i \in \{0, 1, \ldots, n\}$. Without loss of generality, we may suppose that $a_0 = 1$. Let H_1 be derived from H by applying the following substitutions: $X_0 \mapsto X_0 + X_1, X_i \mapsto X_i, \forall i \in \{1, 2, \ldots, n\}$. Then $\Omega(H_1, 0) \in \mathcal{A}_e$. Since H_1 contains $X_0X_1^2 + X_1X_0^2$, we know by the discussion in the previous paragraph that e must be isomorphic to the Hermitian Veronese embedding of PG$(n, 4)$. □

Proposition 3.9 *If there exists an element of $\mathcal{A}_e$ which is not a Hermitian variety of* PG$(n, 4)$*, then e is isomorphic to the universal pseudo-embedding of* PG$(n, 4)$.

Proof In this case, there exists an $H \in \mathcal{H}$ and an $E \in \mathcal{E} \setminus \{0\}$ such that $\Omega(H, E) \in \mathcal{A}_e$. Then there exist $i, j, k \in \{0, 1, \ldots, n\}$ with $i < j < k$ and $c_{ijk} \in \mathbb{F}_4 \setminus \{0\}$ such that $c_{ijk}X_iX_jX_k$ is a term of E. Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0, 1\}$. Let $H_1 \in \mathcal{H}$ and $E_1 \in \mathcal{E}$ be derived from respectively H and E by applying the following substitutions: $X_i \mapsto \delta \cdot X_i, X_l \mapsto X_l, \forall l \in \{0, 1, \ldots, n\} \setminus \{i\}$. Then $\Omega(H_1, E_1) \in \mathcal{A}_e$ and hence also $\Omega(H_2, E_2) \in \mathcal{A}_e$ where $H_2 = H + H_1$ and $E_2 = E + E_1$. Observe that H_2 and E_2 only contains terms which involve X_i. Let $H_3 \in \mathcal{H}$ and $E_3 \in \mathcal{E}$ be derived from respectively H_2 and

E_2 by applying the following substitutions: $X_j \mapsto \delta X_j$, $X_l \mapsto X_l, \forall l \in \{0, 1, \ldots, n\} \setminus \{j\}$. Then $\Omega(H_3, E_3) \in \mathcal{A}_e$ and hence $\Omega(H_4, E_4) \in \mathcal{A}_e$ where $H_4 = H_2 + H_3$ and $E_4 = E_2 + E_3$. Observe that H_4 and E_4 only contains terms which involve X_i and X_j. Let $H_5 \in \mathcal{H}$ and $E_5 \in \mathcal{E}$ be derived from respectively H_4 and E_4 by applying the following substitutions: $X_k \mapsto \lambda \cdot X_k$, $X_l \mapsto X_l, \forall l \in \{0, 1, , \ldots, n\} \setminus \{k\}$. Then $\Omega(H_5, E_5) \in \mathcal{A}_e$ and hence also $\Omega(H_6, E_6) \in \mathcal{A}_e$ where $H_6 = H_4 + H_5$ and $E_6 = E_4 + E_5$. Observe that H_6 and E_6 only contains terms which involve X_i, X_j and X_k. Now, $H_6 = 0$ and $E_6 = c_{ijk} X_i X_j X_k$. By Properties (P2) and (P3), also $\Omega(0, X_0 X_1 X_2) \in \mathcal{A}_e$. Lemma 3.7 then implies that all pseudo-hyperplanes of PG$(n, 4)$, distinct from the whole point set, arise from e. This implies by Theorem 3.1, that e is isomorphic to the universal pseudo-embedding of PG$(n, 4)$. □

Theorem 1.4 is a consequence of Propositions 3.8 and 3.9.

3.3 The homogeneous pseudo-embeddings of AG$(n, 4)$

Consider the affine space AG$(n, 4)$, $n \geq 2$. The universal pseudo-embedding of AG$(n, 4)$ is universal. There is at least one other homogeneous pseudo-embedding.

Proposition 3.10 (1) *The quadratic embedding of* AG$(n, 4)$, $n \geq 0$*, is a homogeneous pseudo-embedding.*

(2) *There are two types of pseudo-hyperplanes arising from the quadratic pseudo-embedding of* AG$(n, 4)$, $n \geq 1$*, namely the empty set and those pseudo-hyperplanes which can be written as the union of two distinct parallel hyperplanes of* AG$(n, 4)$.

Proof We may suppose that $n \geq 2$.

(1) Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0, 1\}$. Choose reference systems in AG$(n, 4)$ and PG$(2n, 2)$ and let e_2 be the map which maps the point $(X_1, X_2, \ldots, X_n)$ of AG$(n, 4)$ to the point $(1, X_i + X_i^2, \delta X_i + \delta^2 X_i^2 \mid 1 \leq i \leq n)$ of PG$(2n, 2)$.

- By considering the points $(0, 0, 0, \ldots, 0)$, $(1, 0, 0, \ldots, 0)$, $(\delta, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $(0, \delta, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 0, 1)$, $(0, 0, \ldots, 0, \delta)$ of AG$(n, 4)$, we see that the image of e_2 generates PG$(2n, 2)$.
- The group of affine collineations of AG$(n, 4)$ is generated by the following maps: (i) $(X_1, X_2, \ldots, X_n) \mapsto (X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)})$ for some permutation σ of $\{1, 2, \ldots, n\}$; (ii) $(X_1, X_2, \ldots, X_n) \mapsto (X_1 + a, X_2, \ldots, X_n)$ for some $a \in \mathbb{F}_4$; (iii) $(X_1, X_2, \ldots, X_n) \mapsto (\lambda \cdot X_1, X_2, \ldots, X_n)$ for some $\lambda \in \mathbb{F}_4 \setminus \{0\}$; (iv) $(X_1, X_2, X_3, \ldots, X_n) \mapsto (X_1 + X_2, X_2, X_3, \ldots, X_n)$; (v) $(X_1, X_2, \ldots, X_n) \mapsto (X_1^2, X_2^2, \ldots, X_n^2)$. We need to prove that for every collineation θ of AG$(n, 4)$, there exists a projectivity η_θ of PG$(2n, 2)$ such that $e(p^\theta) = e(p)^{\eta_\theta}$ for every point p of AG$(n, 4)$. One can easily verify that this property holds for each of the above generators. Hence, it also holds for any collineation of AG$(n, 4)$.
- Let $L = \{p_1, p_2, p_3, p_4\}$ be an arbitrary line of AG$(n, 4)$. We need to prove that $e_2(p_1), e_2(p_2), e_2(p_3)$ are linearly independent and $e_2(p_1) + e_2(p_2) + e_2(p_3) + e_2(p_4) = 0$. This is easily verified. Observe that by the previous paragraph, we may suppose that $L = \{(\lambda, 0, 0, \ldots, 0) \mid \lambda \in \mathbb{F}_4\}$.

(2) If Π_0 is the hyperplane $Y_0 = 0$ of PG$(2n, 2)$, then $e_2^{-1}(e_2(\text{AG}(n, 4)) \cap \Pi_0) = \emptyset$. If Π_1 is the hyperplane $Y_1 = 0$ of PG$(2n, 2)$, then $e_2^{-1}(e_2(\text{AG}(n, 4)) \cap \Pi_1)$ is the union of the two distinct parallel hyperplanes $X_1 = 0$ and $X_1 = 1$ of AG$(n, 4)$. Since e_2 is homogeneous, all $2^{2n+1} - 2$ pseudo-hyperplanes of AG$(n, 4)$ which are the union of

two distinct parallel hyperplanes arise from e_2. (Off course, it is also possible to prove this directly.) So, we have localized all $2^{2n+1} - 1$ pseudo-hyperplanes of $\mathrm{AG}(n, 4)$ which arise from e_2. □

Now, fix a certain reference system in $\mathrm{AG}(n, 4)$, $n \geq 2$, and let $(X_1, X_2, \dots, X_n)$ denote the coordinates of a general point of $\mathrm{AG}(n, 4)$ with respect to that reference system. We denote by $\mathcal{H}$ the set of all polynomials of the form $a + \sum_{1 \leq i \leq n}(b_i X_i + b_i^2 X_i^2)$, where $a \in \{0, 1\}$ and $b_i \in \mathbb{F}_4$ for all $i \in \{1, 2, \dots, n\}$. We denote by $\mathcal{E}$ the set of all polynomials of the form $\sum_{1 \leq i < j \leq n} c_{ij} X_i X_j$, where $c_{ij} \in \mathbb{F}_4$ for all $i, j \in \{1, 2, \dots, n\}$ with $i < j$. If $H \in \mathcal{H}$ and $E \in \mathcal{E}$, then $\Omega(H, E)$ denotes the set of even type of $\mathrm{AG}(n, 4)$ whose equation with respect to the fixed reference system is given by $H + E + E^2 = 0$. We denote by $\mathcal{I}$ the ideal of the polynomial ring $\mathbb{F}_4[X_1, X_2, \dots, X_n]$ generated by the polynomials $X_1^4 - X_1, X_2^4 - X_2, \dots, X_n^4 - X_n$.

Suppose e is an $AGL(n, 4)$-homogeneous pseudo-embedding of $\mathrm{AG}(n, 4)$ and let $\mathcal{A}_e$ denote the set of all pseudo-hyperplanes of $\mathrm{AG}(n, 4)$ arising from e. The condition mentioned in Proposition 2.1(b) translates to

(P1) Let $H_1, H_2 \in \mathcal{H}$ and $E_1, E_2 \in \mathcal{E}$ such that $(H_1, E_1) \neq (H_2, E_2)$. If $\Omega(H_1, E_1)$ and $\Omega(H_2, E_2)$ belong to $\mathcal{A}_e$, then also $\Omega(H_1 + H_2, E_1 + E_2)$ belongs to $\mathcal{A}_e$.

The condition mentioned in Proposition 2.1(a) and the fact that e is $AGL(n, 4)$-homogeneous implies that the properties (P2), (P3), (P4) and (P5) below hold.

(P2) Let σ be a permutation of $\{1, 2, \dots, n\}$ and let $(H_1, E_1) \in \mathcal{H} \times \mathcal{E}$. Let H_2 and E_2 be derived from H_1 and E_1, respectively, by applying the following substitutions: $X_i \mapsto X_{\sigma(i)}, \forall i \in \{1, 2, \dots, n\}$. Then $\Omega(H_1, E_1) \in \mathcal{A}_e$ if and only if $\Omega(H_2, E_2) \in \mathcal{A}_e$.

(P3) Let $i \in \{1, 2, \dots, n\}, \lambda \in \mathbb{F}_4 \setminus \{0\}$ and $(H_1, E_1) \in \mathcal{H} \times \mathcal{E}$. Let H_2 and E_2 be derived from H_1 and E_1, respectively, by applying the following substitutions: $X_j \mapsto X_j, \forall j \in \{1, 2, \dots, n\} \setminus \{i\}$ and $X_i \mapsto \lambda \cdot X_i$. Then $\Omega(H_1, E_1) \in \mathcal{A}_e$ if and only if $\Omega(H_2, E_2) \in \mathcal{A}_e$.

(P4) Let $i \in \{1, 2, \dots, n\}, \lambda \in \mathbb{F}_4$ and let $(H_1, E_1) \in \mathcal{H} \times \mathcal{E}$. Let $H_2, H_2' \in \mathcal{H}$ such that H_2 and $H_2' + E_1 + E_1^2$ are derived from respectively H_1 and $E_1 + E_1^2$ by applying the following substitutions: $X_j \mapsto X_j, \forall j \in \{1, 2, \dots, n\} \setminus \{i\}$, and $X_i \mapsto X_i + \lambda$. Then $\Omega(H_1, E_1) \in \mathcal{A}_e$ if and only if $\Omega(H_2 + H_2', E_1) \in \mathcal{A}_e$.

(P5) Let $i_1, i_2 \in \{1, 2, \dots, n\}$ with $i_1 \neq i_2$ and let $(H_1, E_1) \in \mathcal{H} \times \mathcal{E}$. Let $H_2, H_2' \in \mathcal{H}$, $E_2 \in \mathcal{E}$ and $I \in \mathcal{I}$ such that H_2 and $H_2' + E_2 + E_2^2 + I$ are derived from respectively H_1 and $E_1 + E_1^2$ by applying the following substitutions: $X_j \mapsto X_j, \forall j \in \{1, 2, \dots, n\} \setminus \{i_1\}$, and $X_{i_1} \mapsto X_{i_1} + X_{i_2}$. Then $\Omega(H_1, E_1) \in \mathcal{A}_e$ if and only if $\Omega(H_2 + H_2', E_2) \in \mathcal{A}_e$.

Lemma 3.11 *If $\Omega(X_1 + X_1^2, 0) \in \mathcal{A}_e$, then $\Omega(H, 0) \in \mathcal{A}_e$ for all $H \in \mathcal{H} \setminus \{0\}$.*

Proof
- By Properties (P2) and (P3), we have $\Omega(b_i X_i + b_i^2 X_i^2, 0) \in \mathcal{A}_e$ for all $i \in \{1, 2, \dots, n\}$ and all $b_i \in \mathbb{F}_4 \setminus \{0\}$.
- Let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0, 1\}$ and consider the substitutions $X_1 \mapsto X_1 + \delta, X_i \mapsto X_i, \forall i \in \{2, 3, \dots, n\}$. By Property (P4), $\Omega(X_1 + X_1^2 + 1, 0) \in \mathcal{A}_e$. By Property (P1), $\Omega(1, 0) = \Omega(X_1 + X_1^2 + X_1 + X_1^2 + 1, 0) \in \mathcal{A}_e$.
- By Property (P1) and the previous two paragraphs, we have $\Omega(H, 0) \in \mathcal{A}_e$ for all $H \in \mathcal{H} \setminus \{0\}$. □

Lemma 3.12 *If $\Omega(0, X_1 X_2) \in \mathcal{A}_e$, then $\Omega(H, E) \in \mathcal{A}_e$ for all $(H, E) \in \mathcal{H} \times \mathcal{E} \setminus \{(0, 0)\}$.*

Proof • By Properties (P2) and (P3), we have $\Omega(0, c_{ij}X_iX_j) \in \mathcal{A}_e$ for all $i, j \in \{1, 2, \dots, n\}$ with $i < j$ and all $c_{ij} \in \mathbb{F}_4 \setminus \{0\}$. By Property (P1), it then follows that $\Omega(0, E) \in \mathcal{A}_e$ for all $E \in \mathcal{E} \setminus \{0\}$.

- Consider the substitution $X_1 \mapsto X_1 + X_2, X_i \mapsto X_i, \forall i \in \{2, 3, \dots, n\}$. By Property (P5), $\Omega(X_2 + X_2^2, X_1X_2) \in \mathcal{A}_e$. Hence, by Property (P1), we also have $\Omega(X_2 + X_2^2, 0) = \Omega(X_2 + X_2^2 + 0, X_1X_2 + X_1X_2) \in \mathcal{A}_e$. By Lemma 3.11 and Property (P2), we have $\Omega(H, 0) \in \mathcal{A}_e$ for all $H \in \mathcal{H} \setminus \{0\}$.
- By the previous two paragraphs and Property (P1), we have $\Omega(H, E) \in \mathcal{A}_e$ for all $(H, E) \in \mathcal{H} \times \mathcal{E} \setminus \{(0, 0)\}$. □

Observe that $|\mathcal{A}_e| \geq 2$. So, there exists an element in $\mathcal{A}_e \setminus \{\emptyset\}$.

Proposition 3.13 *If each element of $\mathcal{A}_e \setminus \{\emptyset\}$ is the union of two distinct parallel hyperplanes, then e is isomorphic to the quadratic embedding of* AG$(n, 4)$.

Proof In this case, there exists an $H \in \mathcal{H} \setminus \{0, 1\}$ such that $\Omega(H, 0) \in \mathcal{A}_e$. So, there exists an $i \in \{1, 2, \dots, n\}$ and a $b_i \in \mathbb{F}_4 \setminus \{0\}$ such that $b_iX_i + b_i^2X_i^2$ occurs in H. As before, let δ be an arbitrary element of $\mathbb{F}_4 \setminus \{0, 1\}$ and let $H_1 \in \mathcal{H}$ be derived from H by applying the following substitutions: $X_i \mapsto \delta \cdot X_i, X_j \mapsto X_j, \forall j \in \{1, 2, \dots, n\} \setminus \{i\}$. Then $\Omega(H_1, 0) \in \mathcal{A}_e$ and hence also $\Omega(H_2, 0) \in \mathcal{A}_e$ where $H_2 = H + H_1$. We have $H_2 = \delta^2 b_i X_i + \delta b_i^2 X_i^2$. By Properties (P2) and (P3), we have $\Omega(X_1 + X_1^2, 0) \in \mathcal{A}_e$. By Lemma 3.11, we now readily see that $\mathcal{A}_e$ consists of the following pseudo-hyperplanes: (i) the empty set; (ii) the union of two distinct parallel hyperplanes. By Theorem 3.1, e is isomorphic to the quadratic embedding of AG$(n, 4)$. □

Proposition 3.14 *If $\mathcal{A}_e$ has a pseudo-hyperplane which is neither empty, nor the union of two distinct parallel hyperplanes, then e is isomorphic to the universal pseudo-embedding of* AG$(n, 4)$.

Proof There exists an $H \in \mathcal{H}$ and an $E \in \mathcal{E} \setminus \{0\}$ such that $\Omega(H, E) \in \mathcal{A}_e$. Then there exist $i, j \in \{1, 2, \dots, n\}$ with $i < j$ and a $c_{ij} \in \mathbb{F}_4 \setminus \{0\}$ such that $c_{ij}X_iX_j$ is a term of E. With a similar reasoning as in the proof of Proposition 3.9, one can prove that $\Omega(0, X_1X_2) \in \mathcal{A}_e$. Lemma 3.12 then implies that all pseudo-hyperplanes of AG$(n, 4)$ distinct from the whole set of points arise from e. This implies by Theorem 3.1 that e is isomorphic to the universal pseudo-embedding of AG$(n, 4)$. □

Theorem 1.4 is an immediate consequence of Propositions 3.10, 3.13 and 3.14.

4 The pseudo-hyperplanes of AG$(n, 4)$

In this section, we classify all pseudo-hyperplanes of AG$(n, 4)$, $n \geq 2$. The proof highly depends on some results of Hirschfeld and Thas [6], who characterized those sets of points of finite projective spaces which arise as projections of nonsingular quadrics. Supposing the affine space AG$(n, 4)$ arises from PG$(n, 4)$ by removing a hyperplane Π_∞, then for every pseudo-hyperplane X of AG$(n, 4)$, the set $\Pi_\infty \cup X$ is a set of odd type of PG$(n, 4)$. Before we discuss the actual classification of the pseudo-hyperplanes of AG$(n, 4)$, we have to do some preparatory work by discussing and proving some properties of sets of odd type of PG$(n, 4)$.

The sets of odd type of PG$(2, 4)$ can easily be determined by hand and are listed in the following proposition.

Proposition 4.1 *Let X be a set of odd type of* PG(2, 4), *then X is one of the following:*

(I) *a unital of* PG(2, 4)*;*
(II) *a Baer subplane of* PG(2, 4)*;*
(III) *a hyperoval of* PG(2, 4), *plus an external line;*
(IV) *the complement of a hyperoval of* PG(2, 4)*;*
(V) *the union of three distinct lines through a given point;*
(VI) *a line;*
(VII) *the whole set of points of* PG(2, 4).

The result stated in Proposition 4.1 can be found at several places in the literature, like Hirschfeld [3, Theorem 19.6.2] and Hirschfeld and Hubaut [4, Theorem 4]. The discussion in [3] and [4] is based on results of Tallini Scafati who studied more general problems in her papers [10–12].

If X is a set of odd type of PG$(n, 4)$, $n \geq 2$, and α is a plane of PG$(n, 4)$, then $\alpha \cap X$ is a set of odd type of $\alpha \cong$ PG(2, 4) and hence one of the seven possibilities of Proposition 4.1 occurs. If case (Y) of Proposition 4.1 occurs, then we say that $\alpha \cap X$ is a *plane section of Type (Y)*.

Suppose Π is a hyperplane of the projective space PG$(n, 4)$, $n \geq 2$, p is a point of PG$(n, 4)$ not contained in Π and X is a set of odd type of Π. Then the cone pX with top p and basis X is a set of odd type of PG$(n, 4)$. Any set of odd type of PG$(n, 4)$ which arises in this way is called *singular*; otherwise it is called *non-singular*.

We now define two classes of nonsingular sets of odd type of PG$(n, 4)$, $n \geq 2$, which will play a crucial role later.

Construction 1 Consider in PG$(2n + 1, 4)$, $n \geq 1$, a nonsingular quadric Q and a point $p \notin Q$. Let ζ be the symplectic polarity of PG$(2n + 1, 4)$ associated with Q. There are two possibilities for Q. Either Q is a hyperbolic quadric $Q^+(2n + 1, 4)$ or an elliptic quadric $Q^-(2n + 1, 4)$. The number of points of Q is equal to $\frac{4^{2n+1}-1}{3} + \epsilon \cdot 4^n$, where $\epsilon = +1$ in case Q is a hyperbolic quadric and $\epsilon = -1$ in case Q is an elliptic quadric.

There are three types of lines through p: lines which are disjoint from Q (exterior lines), lines which meet Q in precisely one point (tangent lines) and lines which meet Q in precisely two points (secant lines). The tangent lines through p are precisely the lines through p contained in p^ζ. There are $\frac{4^{2n}-1}{3}$ such lines. As a consequence, there are

$$\frac{1}{2}\Big(\frac{4^{2n+1} - 1}{3} + \epsilon \cdot 4^n - \frac{4^{2n} - 1}{3}\Big) = 2^{2n-1}(4^n + \epsilon)$$

secant lines.

Now, consider a hyperplane PG$(2n, 4)$ of PG$(2n + 1, 4)$ not containing p and let X be the projection of Q from the point p onto PG$(2n, 4)$. By the above, we know that the total number of points in X is equal to

$$\frac{4^{2n} - 1}{3} + 2^{2n-1}(4^n + \epsilon). \tag{1}$$

By Hirschfeld and Thas [5, Theorem 13], we know that X is a nonsingular set of odd type of PG$(2n, 4)$. Since X contains the hyperplane $p^\zeta \cap$ PG$(2n, 4)$ of PG$(2n, 4)$, there are no plane sections of Type (I), nor of type (II).

Now, consider the case $n = 1$. If Q is a hyperbolic quadric $Q^+(3, 4)$ of PG(3, 4), then we have $|X| = 15$ and hence, after consulting Proposition 4.1, we see that X is the complement of a hyperoval of PG(2, 4). If Q is an elliptic quadric $Q^-(3, 4)$ of PG(3, 4), then we have $|X| = 11$ and hence, after consulting Proposition 4.1, we see that X is a hyperoval of PG$(2, 4)$, plus a line disjoint from that hyperoval. These observations can be used to prove the following lemma.

Lemma 4.2 *If $n \geq 2$, then X has plane sections of Type* (III) *and plane sections of Type* (IV).

Proof The hyperplane p^ζ of PG$(2n + 1, 4)$ intersects Q in a nonsingular quadric of Type $Q(2n, 4)$ and p is the kernel of this quadric. Let p_1 and p_2 be two points of $p^\zeta \cap Q$ such that $p_1 p_2$ is not contained in Q. Then the plane $< p, p_1, p_2 >$ intersects Q in a nonsingular conic of $< p, p_1, p_2 >$. Through $< p, p_1, p_2 >$, there exists a 3-space α_1 which intersects Q in a nonsingular elliptic quadric of α_1 and a 3-space α_2 which intersects Q in a nonsingular hyperbolic quadric of α_2. If we project $\alpha_1 \cap Q$ from the point p onto PG$(2n, 4)$, then we get a plane section of Type (III) and if we project $\alpha_2 \cap Q$ from the point p onto PG$(2n, 4)$, then we get a plane section of Type (IV). □

The following proposition is a special case of Hirschfeld and Thas [6, Theorem 6].

Proposition 4.3 ([6]) *Let X be a nonsingular set of odd type of* PG$(2n, 4)$, $n \geq 2$, *such that there exist plane sections of Type* (IV)*, but no plane sections of Type* (I)*, nor of type* (II). *Then X is a projection of a nonsingular hyperbolic or elliptic quadric of a projective space* PG$(2n + 1, 4)$ *which contains* PG$(2n, 4)$ *as a hyperplane. The point from which one projects does not belong to the quadric, nor to the hyperplane* PG$(2n, 4)$.

Construction 2 Consider in PG$(2n, 4)$, $n \geq 2$, a nonsingular parabolic quadric Q and a point $p \notin Q \cup \{k\}$, where k is the kernel of Q. The number of points of Q is equal to $\frac{4^{2n}-1}{3}$. Every line through k is a tangent line. We denote by p' the unique point of Q on the line kp and by $T_{p'}$ the hyperplane of PG$(2n, 4)$ which is tangent to Q at the point p'. The tangent hyperplane $T_{p'}$ contains the line kp and intersects Q in a cone $p'Q(2n-2, 4)$, where $Q(2n-2, 4)$ is a nonsingular parabolic quadric of a hyperplane of $T_{p'}$ which contains p, but not p'. Observe that p is the kernel of $Q(2n-2, 4)$. The tangent lines through p are precisely the lines through p contained in $T_{p'}$. There are $\frac{4^{2n-1}-1}{3}$ such lines. As a consequence, there are

$$\frac{1}{2}\Big(\frac{4^{2n}-1}{3} - \frac{4^{2n-1}-1}{3}\Big) = 2^{4n-3}$$

secant lines.

Now, consider a hyperplane PG$(2n - 1, 4)$ of PG$(2n, 4)$ not containing p and let X be the projection of Q from the point p onto PG$(2n - 1, 4)$. By the above, we know that the total number of points in X is equal to

$$\frac{4^{2n-1}-1}{3} + 2^{4n-3}. \tag{2}$$

By Hirschfeld and Thas [5, Theorem 13], we know that X is a nonsingular set of odd type of PG$(2n - 1, 4)$. Since X contains the hyperplane $T_{p'} \cap$ PG$(2n - 1, 4)$ of PG$(2n - 1, 4)$, there are no plane sections of Type (I), nor of Type (II).

Lemma 4.4 *The set X of odd type has plane sections of Type* (III) *and plane sections of Type* (IV).

Proof Let p_1 and p_2 be two points of $Q(2n-2,4)$ such that p_1p_2 is not contained in $Q(2n-2,4)$. Then the plane $< p, p_1, p_2 >$ intersects $Q(2n-2,4)$ in a nonsingular conic of $< p, p_1, p_2 >$. Through $< p, p_1, p_2 >$, there exists a 3-space α_1 which intersects Q in a nonsingular elliptic quadric of α_1 and a 3-space α_2 which intersects Q in a nonsingular hyperbolic quadric of α_2. If we project $\alpha_1 \cap Q$ from the point p onto $\mathrm{PG}(2n-1,4)$, then we get a plane section of Type (III) and if we project $\alpha_2 \cap Q$ from the point p onto $\mathrm{PG}(2n-1,4)$, then we get a plane section of Type (IV). □

The following proposition is a special case of Hirschfeld and Thas [6, Theorem 5].

Proposition 4.5 ([6]) *Let X be a nonsingular set of odd type of* $\mathrm{PG}(2n-1,4)$, $n \geq 2$*, such that there exist plane sections of Type* (IV)*, but no plane sections of Type* (I)*, nor of Type* (II)*. Then X is a projection of a nonsingular parabolic quadric Q of a projective space* $\mathrm{PG}(2n,4)$ *which contains* $\mathrm{PG}(2n-1,4)$ *as a hyperplane. The point from which one projects does not belong to* $\mathrm{PG}(2n-1,4)$ *nor to Q and is distinct from the kernel of Q.*

In the following three lemmas, we prove some properties regarding the sets of odd type constructed above.

Lemma 4.6 *Let X be a set of odd type of* $\mathrm{PG}(2n,4)$, $n \geq 2$*, which is the projection of a nonsingular hyperbolic or elliptic quadric Q* (*see Construction* 1). *Then there are precisely* $4^{2n}-1$ *hyperplanes* Π *of* $\mathrm{PG}(2n,4)$ *which intersect X in a set Y which is the projection of a nonsingular parabolic quadric* (*see Construction* 2).

Proof The quadric Q belongs to a projective space $\mathrm{PG}(2n+1,4)$ which contains $\mathrm{PG}(2n,4)$ as a hyperplane. Suppose X is the projection of Q from the point p of $\mathrm{PG}(2n+1,4)$ onto the hyperplane $\mathrm{PG}(2n,4)$ of $\mathrm{PG}(2n+1,4)$. Let ζ be the symplectic polarity of $\mathrm{PG}(2n+1,4)$ associated with Q. There are three possibilities for a hyperplane Π of $\mathrm{PG}(2n,4)$.

(1) $< p, \Pi >$ is a hyperplane of $\mathrm{PG}(2n+1,4)$ tangent to Q at some point p'. Then $\Pi \cap X$ is a singular set of odd type of Π. If this case occurs, then p' necessarily belongs to the nonsingular parabolic quadric $p^\zeta \cap Q$ of p^ζ. Conversely, if $p' \in p^\zeta \cap Q$ then the tangent hyperplane $T_{p'}$ at the point p' is of the form $< p, \Pi >$ for some hyperplane Π of $\mathrm{PG}(2n,4)$. So, there are $|p^\zeta \cap Q| = \frac{4^{2n}-1}{3}$ hyperplanes Π of $\mathrm{PG}(2n,4)$ for which this case occurs.
(2) $< p, \Pi >$ is a hyperplane of $\mathrm{PG}(2n+1,4)$ which is not tangent to Q such that the point p is the kernel of the parabolic quadric $< p, \Pi > \cap Q$ of $< p, \Pi >$. Then $\Pi \subseteq X$. This case occurs precisely when $< p, \Pi >= p^\zeta$, i.e. when $\Pi = p^\zeta \cap \mathrm{PG}(2n,4)$.
(3) $< p, \Pi >$ is a hyperplane of $\mathrm{PG}(2n+1,4)$ which is not tangent to Q such that the point p is not the kernel of the parabolic quadric $< p, \Pi > \cap Q$ of $< p, \Pi >$. If this case occurs, then $\Pi \cap X$ is the projection of the nonsingular parabolic quadric $< p, \Pi > \cap Q$ of the subspace $< p, \Pi >$.

Since the total number of hyperplanes of $\mathrm{PG}(2n,4)$ is equal to $\frac{4^{2n+1}-1}{3}$, the required number of hyperplanes is equal to $\frac{4^{2n+1}-1}{3} - \frac{4^{2n}-1}{3} - 1 = 4^{2n}-1$. □

Lemma 4.7 *Let X be a set of odd type of* $\mathrm{PG}(2n-1,4)$, $n \geq 2$*, which is the projection of a nonsingular parabolic quadric Q. Then there are precisely* 4^{2n-1} *hyperplanes* Π *of* $\mathrm{PG}(2n-1,4)$ *which intersect X in a set Y which is the projection of a nonsingular hyperbolic or elliptic quadric.*

Proof The quadric Q belongs to a projective space PG$(2n, 4)$ which contains PG$(2n-1, 4)$ as a hyperplane. Suppose X is the projection of Q from the point p onto the hyperplane PG$(2n-1, 4)$ of PG$(2n, 4)$. The point p is distinct from the kernel k of Q and the line kp intersects Q in a point p'. There are two possibilities for a hyperplane Π of PG$(2n-1, 4)$.

(1) $< p, \Pi >$ is a hyperplane of PG$(2n, 4)$ tangent to Q at some point p''. Then $\Pi \cap X$ is a singular set of odd type of Π. The point p'' necessarily belongs to the tangent hyperplane $T_{p'}$ at the point p'. Conversely, if $p'' \in T_{p'}$, then the tangent hyperplane $T_{p''}$ at the point p'' is of the form $< p, \Pi >$ for some hyperplane Π of PG$(2n-1, 4)$. So, there are $|T_{p'} \cap Q| = \frac{4^{2n-1}-1}{3}$ hyperplanes Π of PG$(2n-1, 4)$ for which this case occurs.

(2) $< p, \Pi >$ is a hyperplane of PG$(2n, 4)$ which is not tangent to Q. If this case occurs, then $\Pi \cap X$ is the projection of the nonsingular (hyperbolic or elliptic) quadric $< p, \Pi > \cap Q$ of the subspace $< p, \Pi >$.

Since the total number of hyperplanes of PG$(2n-1, 4)$ is equal to $\frac{4^{2n}-1}{3}$, the required number of hyperplanes is equal to $\frac{4^{2n}-1}{3} - \frac{4^{2n-1}-1}{3} = 4^{2n-1}$. □

Lemma 4.8 *Let* Π *be a hyperplane of* PG$(n, 4)$, $n \geq 3$. *Let* p *be a point of* PG$(n, 4)$ *not contained in* Π *and let* X *be a set of odd type of* Π *which is the projection of a nonsingular quadric. Then there are precisely* 4^n *hyperplanes* Π' *of* PG$(n, 4)$ *which intersect the cone* pX *in a set* Y *which is the projection of a nonsingular quadric.*

Proof If Π' contains p, then $\Pi' \cap pX$ is a singular set of odd type of Π' (with top p) and hence cannot be the projection of a nonsingular quadric. If Π' is one of the 4^n hyperplanes of PG$(n, 4)$ not containing p, then $\Pi' \cap pX$ is a set of odd type of Π' which is isomorphic to the set X of odd type of Π. □

Lemma 4.9 *Let* X *be a set of odd type of* PG$(n, 4)$, $n \geq 2$, *such that there are no plane sections of Type* (I), (II), (III), *nor* (IV). *Then* X *is either a hyperplane, the union of three distinct hyperplanes through a given* $(n-2)$*-dimensional subspace of* PG$(n, 4)$ *or the whole point set of* PG$(n, 4)$.

Proof If every line of PG$(n, 4)$ intersects X in either 1 or 5 points, then X is either a hyperplane of PG$(n, 4)$ or the whole set of points of PG$(n, 4)$. In the sequel, we will suppose that there exists a line L which intersects X in three points x_1, x_2 and x_3. By Proposition 4.1, every plane α through L intersects X in the union of three lines through a given point k_α. Let K denote the set of all points k_α where α is some plane through L.

We prove that K is a subspace. Suppose α_1 and α_2 are two distinct planes through L. Put $M = k_{\alpha_1}k_{\alpha_2}$. We prove that every $k \in M \cap X$ is of the form k_α for some plane α through L. We may suppose that $k \notin \{k_{\alpha_1}, k_{\alpha_2}\}$. The plane $< x_i k_{\alpha_1}, x_i k_{\alpha_2} >$, $i \in \{1, 2, 3\}$, contains the two lines $x_i k_{\alpha_1}, x_i k_{\alpha_2}$ through x_i which are contained in X, plus the extra point k which is also contained in X. It follows that the line kx_i is contained in X. So, $k = k_\alpha$ where $\alpha =< k, L >$. Now, since the line M contains two points of X, namely k_{α_1} and k_{α_2}, it contains a third point of X. This point is equal to k_{α_3} for some plane α_3 through L. Now, the plane $< x_1 k_{\alpha_1}, x_1 k_{\alpha_2} >$ contains at least three lines through x_1 which are contained in X, namely the lines $x_1 k_{\alpha_1}, x_1 k_{\alpha_2}$ and $x_1 k_{\alpha_3}$. Let α' be a plane of $< L, M >$ through L distinct from α_1, α_2 and α_3. The unique line through x_3 contained in $\alpha' \cap X$ intersects $< x_1 k_{\alpha_1}, x_1 k_{\alpha_2} >$ in a point of X which is not contained in $x_1 k_{\alpha_1} \cup x_1 k_{\alpha_2} \cup x_1 k_{\alpha_3}$. This implies that the plane $< x_1 k_{\alpha_1}, x_1 k_{\alpha_2} >$ is completely contained in X. In particular, $M \subseteq X$. By the above, we then know that each point of M is of the form k_α for some plane α through L. This indeed proves that K is a subspace.

Now, since K is disjoint from L, we have $\dim(K) \leq n-2$. Since every plane α through L meets K, we have $\dim(K) = n-2$. By considering all planes through L, we immediately see that X must be a cone with top K and basis $\{x_1, x_2, x_3\}$, i.e. X is the union of the three hyperplanes $< K, x_1 >$, $< K, x_2 >$ and $< K, x_3 >$. □

Lemma 4.10 *Let X be a set of odd type of* $\mathrm{PG}(n,4)$, $n \geq 2$, *containing a hyperplane* Π_∞ *of* $\mathrm{PG}(n,4)$. *Put* $X' = \Pi_\infty \cup (\mathrm{PG}(n,4) \setminus X)$. *Then X' is a set of odd type of* $\mathrm{PG}(n,4)$. *The set X' is singular if and only if X is singular.*

Proof Let L be a line of $\mathrm{PG}(n,4)$. If $L \subseteq \Pi_\infty$, then $L \subseteq X'$. If L is a line of $\mathrm{PG}(n,4)$ not contained in Π_∞ which intersects X in $i \in \{1,3,5\}$ points, then L intersects X' in $6-i \in \{1,3,5\}$ points. So, X' is a set of odd type of $\mathrm{PG}(n,4)$.

Suppose X is singular. Then X is a cone pY where p is some point of $\mathrm{PG}(n,4)$ and Y is a set of odd type of a hyperplane Π of $\mathrm{PG}(n,4)$ not containing p. If $p \notin \Pi_\infty$, then since $\Pi_\infty \subseteq X$, we have $X = \mathrm{PG}(n,4)$ and hence $X' = \Pi_\infty$ is singular. We may therefore suppose that $p \in \Pi_\infty$. Put $Y' = (\Pi_\infty \cap \Pi) \cup (\Pi \setminus Y)$. By the first paragraph, Y' is a set of odd type of Π. We clearly have $X' = pY'$. So, X' is also singular.

By symmetry, if X' is singular then also X is singular. □

Proposition 4.11 *Let X be a set of odd type of* $\mathrm{PG}(n,4)$, $n \geq 2$, *containing a hyperplane* Π_∞ *of* $\mathrm{PG}(n,4)$. *Then X is either a singular set of odd type or the projection of a nonsingular quadric of a projective space* $\mathrm{PG}(n+1,4)$ *which contains* $\mathrm{PG}(n,4)$ *as a hyperplane.*

Proof By Proposition 4.1, the result holds if $n = 2$. So, we may suppose that $n \geq 3$.

Since X contains a hyperplane, every plane section contains a line. So, there are no plane sections of Type (I) nor of Type (II). If there are no plane sections of Type (III), nor of Type (IV), then X is a singular set of odd type by Lemma 4.9. So, in the sequel, we may suppose that there exist plane sections of Type (III) or (IV). We may also suppose that X is not singular.

Suppose there are plane sections of Type (IV). Then Propositions 4.3 and 4.5 imply that X is the projection of a nonsingular quadric of a projective space $\mathrm{PG}(n+1,4)$ which contains $\mathrm{PG}(n,4)$ as a hyperplane.

Suppose there are plane sections of Type (III), i.e. there exists a plane α of $\mathrm{PG}(n,4)$ which intersects X in a hyperoval of α, plus a line of α which is disjoint from that hyperoval. Put $X' = \Pi_\infty \cup (\mathrm{PG}(n,4) \setminus X)$. Then by Lemma 4.10, X' is a nonsingular set of odd type of $\mathrm{PG}(n,4)$. Moreover, since $\Pi_\infty \subseteq X'$ there are no plane sections of Type (I), nor of Type (II). Now, the plane α intersects X' in the complement of a hyperoval of α. So, X' has plane sections of Type (IV). By Propositions 4.3 and 4.5, X' is the projection of a nonsingular quadric of a projective space $\mathrm{PG}(n+1,4)$ which contains $\mathrm{PG}(n,4)$ as a hyperplane. By Lemmas 4.2 and 4.4, X' also has plane sections of Type (III), or equivalently, X has plane sections of Type (IV). So, we are again in the situation of the previous paragraph. By Propositions 4.3 and 4.5, we conclude again that X is the projection of a nonsingular quadric of a projective space $\mathrm{PG}(n+1,4)$ which contains $\mathrm{PG}(n,4)$ as a hyperplane. □

Corollary 4.12 *Let X be a set of odd type of* $\mathrm{PG}(n,4)$, $n \geq 2$, *containing a hyperplane* Π *of* $\mathrm{PG}(n,4)$. *Then X is one of the following:*

(1) *the hyperplane* Π*;*
(2) *the union of three mutually distinct hyperplanes* Π, Π', Π' *through a hyperplane of* Π*;*

(3) *the whole point set of* PG$(n, 4)$*;*
(4) *a cone* $\pi_1 Y$*, where:* (i) π_1 *is an* m*-dimensional subspace*[4] *of* Π *for some* $m \in \{-1, 0, \ldots, n-3\}$*;* (ii) π_2 *is an* $(n-m-1)$*-dimensional subspace of* PG$(n, 4)$ *which is complementary to* π_1*;* (iii) $Y \subseteq \pi_2$ *is the projection of a nonsingular quadric of a projective space which contains* π_2 *as a hyperplane.*

Proof The corollary follows by induction from Proposition 4.11. Notice that the corollary is valid for $n = 2$ by Proposition 4.1. □

Theorem 1.6 is now an immediate consequence of Corollary 4.12. Indeed, suppose that the affine space AG$(n, 4)$ is obtained from PG$(n, 4)$ by removing a hyperplane Π_∞ from PG$(n, 4)$. If X is a pseudo-hyperplane of AG$(n, 4)$, then $X \cup \Pi_\infty$ is a set of odd type of PG$(n, 4)$ which contains Π_∞, and hence must correspond to one of the cases (1), (2) or (4) of Corollary 4.12.

Proposition 4.13 *Let* X *be a set of odd type of* PG$(n, 4)$, $n \geq 2$*, containing a hyperplane* Π_∞ *of* PG$(n, 4)$*. Put* $X' = \Pi_\infty \cup (\mathrm{PG}(n, 4) \setminus X)$*. Then the following holds.*

(1) *If* n *is odd and* X *is the projection of a nonsingular parabolic quadric* Q*, then also* X' *is the projection of a nonsingular parabolic quadric.*
(2) *If* n *is even and* X *is the projection of a nonsingular hyperbolic [resp. elliptic] quadric* Q*, then* X' *is the projection of a nonsingular elliptic [resp. hyperbolic] quadric.*

Proof By Lemma 4.10 and Proposition 4.11, X' is the projection of a nonsingular quadric Q'. This proves already (1). Suppose now that n is even. Then $|X| = \frac{4^n-1}{3} + 2^{n-1}(2^n + \epsilon)$ with $\epsilon = 1$ if Q is a hyperbolic quadric and $\epsilon = -1$ if Q is an elliptic quadric. It is straightforward to calculate $|X'|$. We find

$$|X'| = \frac{4^n - 1}{3} + 4^n - 2^{n-1}(2^n + \epsilon) = \frac{4^n - 1}{3} + 2^{n-1}(2^n - \epsilon).$$

So, Q' is an elliptic quadric if Q is a hyperbolic quadric and Q' is a hyperbolic quadric if Q is an elliptic quadric. □

The following is a rephrasing of Proposition 4.13.

Corollary 4.14 (1) *Let* X *be a set of parabolic type of* AG$(n-1, 4)$, $n \geq 4$ *even. Then the complement of* X *is also a set of parabolic type of* AG$(n-1, 4)$.
(2) *Let* X *be a set of hyperbolic [resp. elliptic] type of* AG$(n-1, 4)$, $n \geq 3$ *odd. Then the complement of* X *is a set of elliptic [resp. hyperbolic] type of* AG$(n-1, 4)$.

Definition A set X of even type of the affine space AG$(n-1, 4)$ is said to be *reduced* if one of the following cases occurs.

(1) $n \geq 4$ is even and X is a set of parabolic type of AG$(n-1, 4)$;
(2) $n \geq 3$ is odd and X is a set of hyperbolic or elliptic type of AG$(n-1, 4)$.

Lemma 4.15 *Suppose* AG$(n, 4)$, $n \geq 3$*, denotes the affine space which is obtained from* PG$(n, 4)$ *by removing a hyperplane* Π_∞*. Let* X *be a set of even type of* AG$(n, 4)$ *and* Π *a hyperplane of* AG$(n, 4)$ *intersecting* X *in a reduced set of even type of* Π*. Then precisely one of the following two cases occurs:*

[4] If $m = -1$, then $\pi_1 Y = Y$.

(1) *X is a reduced set of even type of* $\mathrm{AG}(n,4)$*;*
(2) *$X=\mathcal{C}(D,Y)$ where D is some singleton of Π_∞ and Y is a reduced set of even type of a hyperplane Π_1 of* $\mathrm{AG}(n,4)$ *for which $D\cap D_{\Pi_1}=\emptyset$.*

Proof Suppose that this is not the case. Then by Theorem 1.6, $X=\mathcal{C}(D,Y)$ where D is some subspace of dimension at least 1 of Π_∞ and Y is a set of even type of an $(n-1-\dim(D))$-dimensional subspace Π_1 of $\mathrm{AG}(n,4)$ for which $D\cap D_{\Pi_1}=\emptyset$. Since $\dim(D)\geq 1$, we have $D\cap D_\Pi\neq\emptyset$. Then $X\cap\Pi=\mathcal{C}(D\cap D_\Pi,Y')$ where Y' is a set of even type of an $(n-2-\dim(D\cap D_\Pi))$-dimensional subspace Π_2 of Π for which $(D\cap D_\Pi)\cap D_{\Pi_2}=\emptyset$. So, $X\cap\Pi$ cannot be a reduced set of even type of Π, a contradiction. □

For every $n\geq 2$, let $N(n)$ denote the total number of reduced sets of $\mathrm{AG}(n,4)$. From Proposition 4.1, one easily deduces that $N(2)=96$.

Lemma 4.16 *We have $N(2n+1)=(4^{2n+1}-1)\cdot N(2n)$ for every $n\geq 1$ and $N(2n)=4^{2n}\cdot N(2n-1)$ for every $n\geq 2$.*

Proof Consider the affine space $\mathrm{AG}(m,4)$, $m\geq 3$, obtained from $\mathrm{PG}(m,4)$ by removing a hyperplane Π_∞. We count in two different ways the number of triples (Π,X,Y), where Y is a pseudo-hyperplane of $\mathrm{AG}(m,4)$, Π is a hyperplane of $\mathrm{AG}(m,4)$ and X is a reduced pseudo-hyperplane of Π such that $X=Y\cap\Pi$.

- There are $\frac{4^{m+1}-4}{3}$ possibilities for Π, and for given Π there are $N(m-1)$ possibilities for X. Now, fix Π and X. Denote by $\widetilde{e_2}:\ \mathrm{AG}(m,4)\to\widetilde{\Sigma}$ the universal pseudo-embedding of $\mathrm{AG}(m,4)$. Then $\dim(\widetilde{\Sigma})=m^2+m$. By Corollary 1.3(2), the pseudo-embedding of Π induced by $\widetilde{e_2}$ is isomorphic to the universal pseudo-embedding of Π. So, $\dim(<\widetilde{e_2}(\Pi)>)=m^2-m$. There exists a unique hyperplane U of $<e_2(\Pi)>$ such that $X=\widetilde{e_2}^{-1}(\widetilde{e_2}(\Pi)\cap U)$. Since every pseudo-hyperplane of $\mathrm{AG}(m,4)$ arises from $\widetilde{e_2}$ (and the corresponding hyperplane of $\widetilde{\Sigma}$ is unique), the number of possibilities for Y is equal to the number of hyperplanes of $\widetilde{\Sigma}$ which intersects $<e_2(\Pi)>$ in U. The set of such subspaces is equal to $2^{2m+1}-2^{2m}=4^m$.
- By Lemma 4.15, there are two possibilities for Y. Either, the set Y is a reduced set of $\mathrm{AG}(m,4)$, or $Y=\mathcal{C}(D,Y')$ where D is some singleton of Π_∞ and Y' is a reduced set of a hyperplane Π_1 of $\mathrm{AG}(m,4)$ for which $D\cap D_{\Pi_1}=\emptyset$. In the former case, there are $N(m)$ possibilities for Y. In the latter case, there are $\frac{4^m-1}{3}\cdot N(m-1)$ possibilities for Y.
Suppose $m=2n+1$ for some $n\geq 1$. Then by Lemmas 4.7 and 4.8, we have
$$4^{2n+1}\cdot\frac{4^{2n+2}-4}{3}\cdot N(2n)=N(2n+1)\cdot 4^{2n+1}+\frac{4^{2n+1}-1}{3}\cdot N(2n)\cdot 4^{2n+1},$$
i.e. $N(2n+1)=(4^{2n+1}-1)\cdot N(2n)$.
Suppose $m=2n$ for some $n\geq 2$. Then by Lemmas 4.6 and 4.8, we have
$$4^{2n}\cdot\frac{4^{2n+1}-4}{3}\cdot N(2n-1)=N(2n)\cdot(4^{2n}-1)+\frac{4^{2n}-1}{3}\cdot N(2n-1)\cdot 4^{2n},$$
i.e. $N(2n)=4^{2n}\cdot N(2n-1)$. □

Corollary 4.17 (1) *The number of sets of parabolic type in* $\mathrm{AG}(2n-1,4)$, $n\geq 2$*, is equal to* $6\cdot 4^{n(n-1)}\cdot\prod_{i=1}^{n-1}(4^{2i+1}-1)$.
(2) *The number of sets of hyperbolic type in* $\mathrm{AG}(2n,4)$, $n\geq 1$*, is equal to* $3\cdot 4^{n(n+1)}\cdot\prod_{i=1}^{n-1}(4^{2i+1}-1)$.

(3) *The number of sets of elliptic type in* AG$(2n, 4), n \geq 1$, *is equal to* $3 \cdot 4^{n(n+1)} \cdot \prod_{i=1}^{n-1}(4^{2i+1} - 1)$.

Proof By Proposition 4.13(2), the number of sets of hyperbolic type of AG$(2n, 4), n \geq 1$, is equal to the number of sets of elliptic type of AG$(2n, 4)$. Taking this fact into account, the corollary is now an immediate consequence of Lemma 4.16 and the fact that $N(2) = 96$. □

The basic properties of the five classes of pseudo-hyperplanes of AG$(n, 4), n \geq 2$, as they occur in Theorem 1.6 have been listed in Table 1. These properties are easily derived from Eqs. 1, 2 and Corollaries 4.14, 4.17.

5 The pseudo-embeddings of $Q(4, 3)$ induced by homogeneous pseudo-embeddings of AG(4, 4)

5.1 The generalized quadrangle $W(3)$

A point-line geometry $\mathcal{Q}$ is called a *generalized quadrangle* if it satisfies the following three properties.

(1) Every two distinct points are incident with at most one line.
(2) There exist two disjoint lines.
(3) For every line L and every point x not incident with L, there exists a unique point on L collinear with x.

The points and lines of PG(3, 3) which are totally isotropic with respect to a given symplectic polarity of PG(3, 3) are the points and lines of a (symplectic) generalized quadrangle which we denote by $W(3)$. The generalized quadrangle $Q(4, 3)$, defined in Sect. 1, is isomorphic to the point-line dual of $W(3)$, see e.g. Payne and Thas [8, Theorem 3.2.1]. The following proposition, which we take from Taylor [13, Theorem 10.18], gives an alternative construction of the generalized quadrangle $W(3)$ which will be useful later.

Proposition 5.1 ([13]) *Let* $H(3, 4)$ *be a nonsingular Hermitian variety of* PG(3, 4) *and let* ζ *be the Hermitian polarity of* PG(3, 4) *associated with* $H(3, 4)$. *Put* $\mathcal{P} :=$ PG$(3, 4) \setminus H(3, 4)$ *and let* $\mathcal{L}$ *denote the set of all subsets* $\{x_1, x_2, x_3, x_4\}$ *of size* 4 *of* $\mathcal{P}$ *such that* $x_i \in x_j^\zeta$ *for all* $i, j \in \{1, 2, 3, 4\}$ *with* $i \neq j$. *Then the point-line geometry* $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ *with point set* $\mathcal{P}$, *line set* $\mathcal{L}$ *and natural incidence relation* I *is isomorphic to* $W(3)$.

Let $G \cong P\Gamma U(4, 2)$ denote the group of collineations of PG(3, 4) fixing $H(3, 4)$ setwise. Then every $\theta \in G$ induces an automorphism $\widetilde{\theta}$ of $(\mathcal{P}, \mathcal{L}, \mathrm{I}) \cong W(3)$. Put $\widetilde{G} := \{\widetilde{\theta} \mid \theta \in G\}$. Then $\widetilde{G} \cong P\Gamma U(4, 2)$. Since $P\Gamma U(4, 2)$ and the automorphism group of $W(3)$ ($\cong PSp(4, 3).2$) have the same order, namely 51840, $\widetilde{G}$ is the full group of automorphisms of $(\mathcal{P}, \mathcal{L}, \mathrm{I}) \cong W(3)$. (Observe also that $PSU(4, 2) \cong PSp(4, 3)$, see e.g. Taylor [13, Corollary 10.19].)

5.2 Construction and properties of the full embeddings of $Q(4, 3)$ into AG(4, 4)

In this subsection, we discuss the classification of the full embeddings of the generalized quadrangle $Q(4, 3)$ into the affine space AG(4, 4). This classification is essentially due to Thas [14, Section 5.2], see also Payne and Thas [8, Theorem 7.4.1]. Another approach to the classification can be found in Sect. 5 of Thas and Van Maldeghem [15]. We follow here the original approach of Thas [14].

Consider in the projective space PG(4, 4) a hyperplane Π_∞ and let AG(4, 4) denote the affine space obtained from PG(4, 4) by removing Π_∞.

Let ω_∞ be a plane of Π_∞, let $\mathcal{U}$ be a unital of ω_∞ and let m be a point of $\Pi_\infty \setminus \omega_\infty$. If $\mathcal{L}_\mathcal{U}$ is the set of twelve secant lines of ω_∞ (i.e. lines intersecting $\mathcal{U}$ in precisely three points), then $(\mathcal{U}, \mathcal{L}_\mathcal{U})$ defines an affine plane $\mathcal{A}_\mathcal{U}$ of order 3. In ω_∞ there are exactly four triangles $m_1^i m_2^i m_3^i$, $i \in \{1, 2, 3, 4\}$, whose vertices are exterior points of $\mathcal{U}$ and whose sides are secants of $\mathcal{U}$. The three secants lines corresponding to any such triangle define a parallel class of lines of the affine plane $\mathcal{A}_\mathcal{U}$. Any line $m_a^1 m_b^2$, $a, b \in \{1, 2, 3\}$, is tangent to $\mathcal{U}$ and contains exactly one vertex $m^3_{c(a,b)} \in \{m_1^3, m_2^3, m_3^3\}$ and one vertex $m^4_{d(a,b)} \in \{m_1^4, m_2^4, m_3^4\}$.

We show that the cross-ratio $(m_a^1, m_b^2; m^3_{c(a,b)}, m^4_{d(a,b)})$ is independent of the choice of $a, b \in \{1, 2, 3\}$. Suppose K and K' are two arbitrary lines of ω_∞ which are tangent to $\mathcal{U}$, and denote by k and k' the respective tangent points. Then $K = \{k, m_a^1, m_b^2, m^3_{c(a,b)}, m^4_{d(a,b)}\}$ and $K' = \{k', m^1_{a'}, m^2_{b'}, m^3_{c(a',b')}, m^4_{d(a',b')}\}$ for certain $a, b, a', b' \in \{1, 2, 3\}$. Let k'' be the third point of $\mathcal{U}$ on the line kk'. Now, there exist a projectivity η of ω_∞ (induced by a unitary transvection) which interchanges the two points of $\mathcal{U} \setminus \{k''\}$ on each secant line of ω_∞ through k'', and interchanges the two points off $\mathcal{U}$ on each secant line of ω_∞ through k''. In particular, η interchanges[5] the points m_a^1 and $m^1_{a'}$, the points m_b^2 and $m^2_{b'}$, the points $m^3_{c(a,b)}$ and $m^3_{c(a',b')}$ and the points $m^4_{d(a,b)}$ and $m^4_{d(a',b')}$. This implies that $(m_a^1, m_b^2; m^3_{c(a,b)}, m^4_{d(a,b)}) = (m^1_{a'}, m^2_{b'}; m^3_{c(a',b')}, m^4_{d(a',b')})$.

Any three mutually disjoint lines of a projective space PG(3, 4) are contained in a unique nonsingular hyperbolic quadric of PG(3, 4). Such a hyperbolic quadric has the structure of a (5×5)-grid. If Q is a nonsingular hyperbolic quadric of PG(4, 4) with points x_{ij} and lines $L_i := \{x_{ij'} \mid 1 \le j' \le 5\}$, $M_j := \{x_{i'j} \mid 1 \le i' \le 5\}$ $(i, j \in \{1, 2, \ldots, 5\})$, then after giving explicit coordinates to the points of Q, one can readily verify that $(x_{11}, x_{12}; x_{13}, x_{14}) = (x_{21}, x_{22}; x_{23}, x_{24})$.

Now, let L be a line of AG(4, 4) which has m as point at infinity and let p_1, p_2, p_3, p_4 be the affine points of L, where notation is chosen in such a way that $(p_1, p_2; p_3, p_4) = (m_a^1, m_b^2; m^3_{c(a,b)}, m^4_{d(a,b)})$ for all $a, b \in \{1, 2, 3\}$. For all $a, b \in \{1, 2, 3\}$, let Q_{ab} be the nonsingular hyperbolic quadric in the hyperplane $< L, m_a^1 m_b^2 >$ of PG(4, 4) which contains the three mutually disjoint lines $p_1 m_a^1$, $p_2 m_b^2$ and $p_3 m^3_{c(a,b)}$. Since $(p_1, p_2; p_3, p_4) = (m_a^1, m_b^2; m^3_{c(a,b)}, m^4_{d(a,b)})$, Q_{ab} also contains the line $p_4 m^4_{d(a,b)}$ by the previous paragraph.

Let $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following point-line geometry. The elements of $\mathcal{P}$ are the 40 affine points on the lines $p_i m_j^i$, $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$, the elements of $\mathcal{L}$ are the affine lines which are contained in one of the nine hyperbolic quadrics Q_{ab}, $a, b \in \{1, 2, 3\}$, and the incidence relation I is containment.

In Thas [14, Section 5.2] (see also Payne and Thas [8, Theorem 7.4.1]), the following was proved.

Proposition 5.2 ([14]) *If* $(\mathcal{P}', \mathcal{L}', \mathrm{I}') \cong Q(4, 3)$ *is a full subgeometry of* AG(4, 4)*, then there exists an affine collineation of* AG(4, 4) *(whose companion automorphism of* $\mathbb{F}_4$ *is the identity) which maps* $\mathcal{P}'$ *to* $\mathcal{P}$ *and* $\mathcal{L}'$ *to* $\mathcal{L}$.

In Thas [14], it was also mentioned (without proof) that the point-line geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a generalized quadrangle isomorphic to $Q(4, 3)$. This fact in combination with

[5] Observe that the two points coincide for exactly one of the four pairs. In this case, η just fixes the point.

Proposition 5.2 then implies that in some sense there is a unique full embedding of $Q(4,3)$ into AG$(4,4)$.

We are now going to establish an explicit isomorphism between $(\mathcal{P},\mathcal{L},\mathrm{I})$ and the dual of the generalized quadrangle $W(3)$ (which is known to be isomorphic to $Q(4,3)$).

Lemma 5.3 *The complement (in Π_∞) of the set $(\omega_\infty\setminus\mathcal{U})\cup\Big(\bigcup_{p\in\mathcal{U}}(mp\setminus\{p\})\Big)$ is a nonsingular Hermitian variety $H(3,4)$ of Π_∞. If ζ is the Hermitian variety of Π_∞ associated with $H(3,4)$, then $\omega_\infty=m^\zeta$.*

Proof Let $H'(3,4)$ denote an arbitrary nonsingular Hermitian variety of Π_∞, let m' be a point of $\Pi_\infty\setminus H'(3,4)$, let ζ' be the Hermitian polarity of Π_∞ associated with $H'(3,4)$ and put $\omega':=(m')^{\zeta'}$. Then ω' intersects $H'(3,4)$ in a unital $\mathcal{U}'$ of ω'. Every line of Π_∞ through m' intersects $H'(3,4)$ in either one point (tangent line) or three points (secant line). The tangent lines through m' are precisely the lines through m' meeting $\mathcal{U}'$. It follows that the complement of $H'(3,4)$ in Π_∞ is equal to $(\omega'\setminus\mathcal{U}')\cup\bigcup_{p\in\mathcal{U}'}(m'p\setminus\{p\})$. The lemma now follows from the fact that there exists a collineation of Π_∞ mapping m' to m, ω' to ω_∞ and $\mathcal{U}'$ to $\mathcal{U}$. □

Let $H(3,4)$ be the Hermitian variety of Π_∞ occurring in the statement of Lemma 5.3 and let ζ be the Hermitian polarity of Π_∞ associated with $H(3,4)$. Let $W'(3)$ denote the symplectic generalized quadrangle on the point set $\Pi_\infty\setminus H(3,4)$ as defined in Proposition 5.1.

For every $L\in\mathcal{L}$, let p_L denote its point at infinity i.e. the point of Π_∞ which belongs to the unique line of PG$(4,4)$ containing L. By the construction of the set $\mathcal{L}$, we see that the correspondence $L\mapsto p_L$ defines a bijection between $\mathcal{L}$ and $\Pi_\infty\setminus H(3,4)=(\omega_\infty\setminus\mathcal{U})\cup\Big(\bigcup_{p\in\mathcal{U}}(mp\setminus\{p\})\Big)$.

Lemma 5.4 *Every point x of $\mathcal{P}$ is contained in precisely four affine lines of $\mathcal{L}$.*

Proof Suppose first that $x=p_i$ for some $i\in\{1,2,3,4\}$. Then the elements of $\mathcal{L}$ containing x are the affine line L and the affine lines defined by $p_im^i_j$, $j\in\{1,2,3\}$. So, x is indeed contained in precisely four affine lines of $\mathcal{L}$.

Suppose next that $x\notin L$. Then x is contained on a line $p_im^i_j$ for some $i\in\{1,2,3,4\}$ and some $j\in\{1,2,3\}$. The plane $<L,x>$ of PG$(4,4)$ intersects ω_∞ in the singleton $\{m^i_j\}$ and hence the affine line determined by $p_im^i_j$ is the unique element of $\mathcal{L}$ through x meeting L. Now, the point m^i_j of ω_∞ is contained in precisely three tangent lines of ω_∞, which we denote by $\{m^1_{j_1},m^2_{j_2},m^3_{j_3},m^4_{j_4},u\}$, $\{m^1_{j_1'},m^2_{j_2'},m^3_{j_3'},m^4_{j_4'},u'\}$ and $\{m^1_{j_1''},m^2_{j_2''},m^3_{j_3''},m^4_{j_4''},u''\}$. Then $Q_{j_1j_2}$, $Q_{j_1'j_2'}$ and $Q_{j_1''j_2''}$ are those hyperbolic quadrics of the set $\{Q_{ab}\,|\,a,b\in\{1,2,3\}\}$ which contain x. The hyperbolic quadrics $Q_{j_1j_2}$, $Q_{j_1'j_2'}$ and $Q_{j_1''j_2''}$ determine three affine lines M, M' and M'' of $\mathcal{L}$ through x distinct from the affine line contained in $p_im^i_j$. Since the points at infinity of the affine lines M, M' and M'' are respectively contained in mu, mu' and mu'', the lines M, M' and M'' are distinct. So, x is contained in precisely four affine lines of $\mathcal{L}$ as we needed to prove. □

For every point x of $\mathcal{P}$, put $A_x:=\{a_1,a_2,a_3,a_4\}$, where a_1,a_2,a_3 and a_4 are the four points at infinity on the four affine lines of $\mathcal{L}$ through x.

Lemma 5.5 *For every point x of $\mathcal{P}$, A_x is a line of $W'(3)$. Conversely, if A is a line of $W'(3)$, then there exists a unique point $x\in\mathcal{P}$ for which $A=A_x$.*

Proof (1) Let y_1, y_2 be two points of $\Pi_\infty \setminus H(3,4)$. Then there are two possibilities. If the line $y_1 y_2$ is a tangent line to $H(3,4)$, then $y_2 \notin y_1^\zeta$. If the line $y_1 y_2$ is a secant line (intersecting $H(3,4)$ in precisely three points), then $y_1 \in y_2^\zeta$.

(2) Suppose $x = p_i$ for some $i \in \{1,2,3,4\}$. Then $A_x = \{m, m_1^i, m_2^i, m_3^i\}$. We have $\{m_1^i, m_2^i, m_3^i\} \subset \omega_\infty = m^\zeta$. Since $m_{j_1}^i m_{j_2}^i$ is a secant line, we have $m_{j_1}^i \in (m_{j_2}^i)^\zeta$ for all $j_1, j_2 \in \{1,2,3\}$ with $j_1 \neq j_2$. So, A_x is indeed a line of $W'(3)$.

(3) Suppose next that $x \in \mathcal{P} \setminus L$. Then x is contained in a line $p_i m_j^i$ for some $i \in \{1,2,3,4\}$ and some $j \in \{1,2,3\}$. The point m_j^i of ω_∞ is contained in precisely three tangent lines of ω_∞, which we denote by $\{m_{j_1}^1, m_{j_2}^2, m_{j_3}^3, m_{j_4}^4, u\}$, $\{m_{j_1'}^1, m_{j_2'}^2, m_{j_3'}^3, m_{j_4'}^4, u'\}$ and $\{m_{j_1''}^1, m_{j_2''}^2, m_{j_3''}^3, m_{j_4''}^4, u''\}$. Notice that the points u, u' and u'' are contained in the line $(m_j^i)^\zeta \cap \omega_\infty$ of ω_∞. Now, $Q_{j_1 j_2}$, $Q_{j_1' j_2'}$ and $Q_{j_1'' j_2''}$ are precisely the three hyperbolic quadrics of the set $\{Q_{ab} \mid a, b \in \{1,2,3\}\}$ through the point x. These three hyperbolic quadrics determine three affine lines M, M' and M'' of $\mathcal{L}$ through x distinct from the affine line contained in $p_i m_j^i$. Let a, a' and a'' denote the respective points at infinity of the affine lines M, M' and M''. Then $a \in mu$, $a' \in mu'$ and $a'' \in mu''$. We have $A_x = \{m_j^i, a, a', a''\}$.

Since $\{u, u', u''\} \subset (m_j^i)^\zeta$ and $m \in (m_j^i)^\zeta$, we have $a, a', a'' \in (m_j^i)^\zeta$.

Now, let Π be the hyperplane $< L, m_j^i u'' >$ of $\mathrm{PG}(4,4)$. Then Π contains the points p_i, m_j^i, x, u'', m and intersects Π_∞ in the plane $< m_j^i, u'', m >= (u'')^\zeta$. Now, let η be the elation of $\mathrm{PG}(4,4)$ fixing each point of Π, fixing each line through u'' and mapping u to u'. If $i = 1$, then $m_j^i = m_{j_1}^1 = m_{j_1'}^1 = m_{j_1''}^1$, $< u'', m_{j_1}^1 > \subseteq (u'')^\zeta$ and hence η maps $m_{j_1}^1$ to $m_{j_1'}^1 = m_{j_1}^1$. If $i \neq 1$, then the line $< u'', m_{j_1}^1 >$ is a secant line and hence intersects $m_j^i u'$ in the point $m_{j_1'}^1$. So, also in this case η maps $m_{j_1}^1$ to $m_{j_1'}^1$. In a similar way, one proves that η maps $m_{j_2}^2$ to $m_{j_2'}^2$, $m_{j_3}^3$ to $m_{j_3'}^3$ and $m_{j_4}^4$ to $m_{j_4'}^4$. This implies that η maps the hyperbolic quadric $Q_{j_1 j_2}$ to the hyperbolic quadric $Q_{j_1' j_2'}$. Since η fixes x, the projectivity η maps a to a'. So, u'', a and a' are contained in the same line. Since $u'' a$ is not contained in $(u'')^\zeta$, the line $u'' a$ is a secant line. Hence, $a' \in a^\zeta$.

In a similar way, one proves that $a'' \in a^\zeta$ and $a'' \in (a')^\zeta$. So, $A_x = \{m_j^i, a, a', a''\}$ is a line of $W'(3)$.

Conversely, suppose that A is a line of $W'(3)$. Let L_1, L_2, L_3 and L_4 denote those lines of $\mathcal{L}$ for which $A = \{p_{L_1}, p_{L_2}, p_{L_3}, p_{L_4}\}$. If x is a point of $\mathcal{P}$ for which $A = A_x$, then x necessarily is contained in the lines L_1, L_2, L_3 and L_4, proving that there is at most one such point. The uniqueness of x follows from the fact that there are as many points in $\mathcal{P}$ as there are lines of $W'(3)$, namely 40. □

Corollary 5.6 *The maps $x \mapsto A_x$ and $L \mapsto p_L$ ($x \in \mathcal{P}$ and $L \in \mathcal{L}$) define an isomorphism between the point-line geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ and the dual of $W'(3)$. As a consequence, $(\mathcal{P}, \mathcal{L}, \mathrm{I}) \cong Q(4,3)$.*

Lemma 5.7 *If $\mathcal{G}$ is a (4×4)-subgrid of $(\mathcal{P}, \mathcal{L}, \mathrm{I}) \cong Q(4,3)$, then there exists a nonsingular hyperbolic quadric Q of $\Pi =< \mathcal{G} >$ tangent to $\Pi \cap \Pi_\infty$ such that $\mathcal{G} = Q \setminus (\Pi \cap \Pi_\infty)$. Moreover, $\Pi \cap \mathcal{P} = \mathcal{G}$.*

Proof The eight points at infinity of the eight lines of $\mathcal{G}$ have distinct points at infinity. This implies that $\mathcal{G}$ is contained in a unique nonsingular hyperbolic quadric Q of the 3-dimensional subspace $\Pi =< \mathcal{G} >$ of $\mathrm{PG}(4,4)$. The two lines of Q which are disjoint from $\mathcal{G}$

are contained in Π_∞. This implies that the plane $\Pi \cap \Pi_\infty$ of Π is tangent to Q and that $\mathcal{G} = Q \setminus (\Pi \cap \Pi_\infty)$.

Since $\Pi \cap \mathcal{P}$ is a proper subquadrangle of $(\mathcal{P}, \mathcal{L}, \mathrm{I}) \cong Q(4, 3)$ containing $\mathcal{G}$ it must coincide with $\mathcal{G}$. □

Lemma 5.8 *The 40 elements of $\mathcal{L}$ are precisely those lines of* AG$(4, 4)$ *which are contained in $\mathcal{P}$.*

Proof Obviously, every element of $\mathcal{L}$ is contained in $\mathcal{P}$. Conversely, suppose that K is a line of AG$(4, 4)$ which is contained in $\mathcal{P}$ and let $\mathcal{G}$ be a (4×4)-grid of $(\mathcal{P}, \mathcal{L}, \mathrm{I}) \cong Q(4, 3)$ containing at least two points of K. Let Q be the unique nonsingular hyperbolic quadric of $< \mathcal{G} >$ containing $\mathcal{G}$. By Lemma 5.7, $K \subseteq < \mathcal{G} > \cap \mathcal{P}$ is completely contained in Q and hence is contained in one of the ten lines of Q, i.e. K is one of the eight lines of $\mathcal{G}$. So, $K \in \mathcal{L}$. □

Lemma 5.9 *Let $\mathcal{G}$ be a (4×4)-subgrid of $(\mathcal{P}, \mathcal{L}, \mathrm{I}) \cong Q(4, 3)$, let x be a point of $\mathcal{P} \setminus \mathcal{G}$ and let x_1, x_2, x_3, x_4 denote the four points of $\mathcal{G}$ which are collinear (in $(\mathcal{P}, \mathcal{L}, \mathrm{I})$) with x. Then $< x_1, x_2, x_3, x_4 >=< \mathcal{G} >$.*

Proof Since $< A_x >= \Pi_\infty$, we have $< xx_1, xx_2, xx_3, xx_4 >=$ PG$(4, 4)$. So, $< x, < x_1, x_2, x_3, x_4 >>=$ PG$(4, 4)$ and $< x_1, x_2, x_3, x_4 >=< \mathcal{G} >$. □

In Lemma 5.9, the points x_1, x_2, x_3 and x_4 of $\mathcal{G}$ form a so-called *ovoid* of $\mathcal{G}$, this is a set of points of $\mathcal{G}$ having a unique point of common with each line. We call $\{x_1, x_2, x_3, x_4\}$ the ovoid of $\mathcal{G}$ *subtended* by x.

In Sect. 5 of [15], Thas and Van Maldeghem classified all affine embeddings of $Q(4, 3)$ into AG$(4, 4)$ by making use of the so-called coordinates of the generalized quadrangle $Q(4, 3)$. From Theorem 5.1 of [15] and the last part of its proof in [15], we know that the following holds.

Proposition 5.10 *Every full embedding e of $Q(4, 3)$ into* AG$(4, 4)$ *is homogeneous, i.e. for every automorphism θ of $Q(4, 3)$, there exists a (necessarily unique) collineation ϕ_θ of* AG$(4, 4)$ *such that $e(p^\theta) = e(p)^{\phi_\theta}$ for every point p of $Q(4, 3)$.*

The following also holds.

Proposition 5.11 *Up to isomorphism, there is a unique full embedding of $Q(4, 3)$ into* AG$(4, 4)$*, i.e. if e_1 and e_2 are two full embeddings of $Q(4, 3)$ into* AG$(4, 4)$*, then there exists a collineation ϕ of* AG$(4, 4)$ *such that $e_1 = \phi \circ e_2$.*

Proof This is a consequence of Propositions 5.2 and 5.10. Observe that by Lemma 5.8 the image of the point set of $Q(4, 3)$ under the embedding e_i, $i \in \{1, 2\}$, not only determines the embedded points but also the embedded lines. □

The original version of this article also contained a proof of Proposition 5.10. It was however pointed out by the referee that Proposition 5.10 is also implied by Theorem 5.1 of [15]. In the original approach of the author, Proposition 5.10 was derived from Proposition 5.11, while Proposition 5.11 was proved in another way. Indeed, by relying on Propositions 5.1 and 5.2, Lemmas 5.3, 5.4 and 5.5 and Corollary 5.6, it is possible to show that there exists a collineation ϕ of AG$(4, 4)$ such that: (1) for every line L of $Q(4, 3)$, the lines $e_1(L)$ and $\phi \circ e_2(L)$ of AG$(4, 4)$ have the same point at infinity; (2) there exist two distinct collinear points x and y of $Q(4, 3)$ such that $e_1(x) = \phi \circ e_2(x)$ and $e_1(y) = \phi \circ e_2(y)$. It is also possible to show that conditions (1) and (2) imply that $e_1 = \phi \circ e_2$.

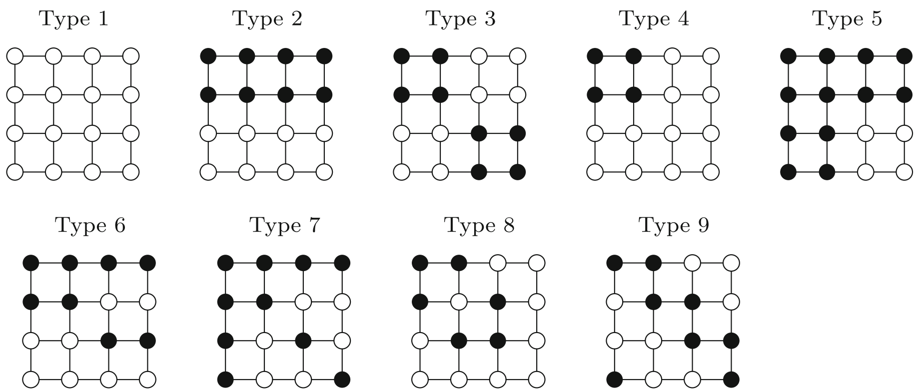
5.3 The pseudo-embeddings of the (4×4)-grid induced by the pseudo-embeddings of $AG(n, 4)$, $n \in \{2, 3\}$

Let $\mathcal{G}$ be a (4×4)-grid. Without loss of generality, we may suppose that the points of $\mathcal{G}$ are the symbols x_{ij}, $1 \leq i, j \leq 4$, where we suppose that two distinct points $x_{i_1 j_1}$ and $x_{i_2 j_2}$ are collinear if and only if either $i_1 = i_2$ or $j_1 = j_2$. We now define a relation R on the set of 24 ovoids of $\mathcal{G}$. If $O = \{x_{1i}, x_{2j}, x_{3k}, x_{4l}\}$ and $O' = \{x_{1i'}, x_{2j'}, x_{3k'}, x_{4l'}\}$ are two ovoids of $\mathcal{G}$, then we say that $(O, O') \in R$ if the permutation

$$\begin{pmatrix} i & j & k & l \\ i' & j' & k' & l' \end{pmatrix}$$

of $\{1, 2, 3, 4\}$ is even. The relation R is an equivalence relation with two classes. We call these two classes the *two families of ovoids* of $\mathcal{G}$. Let G denote the subgroup of $Aut(\mathcal{G})$ consisting of all automorphisms of $\mathcal{G}$ mapping any ovoid of $\mathcal{G}$ to an ovoid of the same family. Clearly, G is a normal subgroup of index 2 of $Aut(\mathcal{G})$.

Up to isomorphism, the (4×4)-grid has nine pseudo-hyperplanes. We list them below.



Now, denote by $\mathcal{F}_a$ and $\mathcal{F}_b$ the two families of ovoids of $\mathcal{G}$. Suppose H is a pseudo-hyperplane of Type 7 of $\mathcal{G}$. Then there are two lines L_1 and L_2 which are contained in H and the set $O_H := (H \setminus (L_1 \cup L_2)) \cup (L_1 \cap L_2)$ is an ovoid of $\mathcal{G}$. We say that H is a pseudo-hyperplane of *Type 7a* if $O_H \in \mathcal{F}_a$ and of *Type 7b* if $O_H \in \mathcal{F}_b$. A pseudo-hyperplane of Type 8 is said to be of *Type 8a* if its complement has Type 7a, and of *Type 8b* if its complement has Type 7b. One can easily verify that G has 11 orbits on the pseudo-hyperplanes of $\mathcal{G}$. The set of pseudo-hyperplanes of Type 7 will split into two orbits (Type 7a and 7b) and also the set of pseudo-hyperplanes of Type 8 will split into two orbits (Type 8a and 8b).

(I) Let $AG(2, 4)$ be the affine plane obtained from $PG(2, 4)$ by removing a line l_∞ and let $\mathcal{G}$ be a (4×4)-subgrid of $AG(2, 4)$. Then there exist two distinct points p_1^* and p_2^* of l_∞ such that the eight lines of $\mathcal{G}$ are the eight lines of $AG(2, 4)$ whose point at infinity is equal to either p_1^* and p_2^*. We will coordinatize $PG(2, 4)$ in such a way that $p_1^* = (0, 1, 0)$ and $p_2^* = (0, 0, 1)$. A point (of $AG(2, 4)$) with coordinates $(1, x, y)$ will also be denoted by (x, y).
If K is a line of $AG(2, 4)$ whose point at infinity is distinct from p_1^* and p_2^*, then K is an ovoid of $\mathcal{G}$. The 12 ovoids of $\mathcal{G}$ which arise in this way form one of the two families of ovoids of $\mathcal{G}$. We denote this family by $\mathcal{F}_a$.

Each automorphism of $G \leq Aut(\mathcal{G})$ is induced by an automorphism of AG(2, 4). So, every homogeneous pseudo-embedding of AG(2, 4) will induce a G-homogeneous pseudo-embedding of $\mathcal{G}$.

(Ia) Let e be the quadratic pseudo-embedding of AG(2, 4). Then e maps the point (x, y) of AG(2, 4) to the point $(X_0, X_1, X_2, X_3, X_4) = (1, x + x^2, \delta x + \delta^2 x^2, y + y^2, \delta y + \delta^2 y^2)$ of PG(4, 2). Since $\mathcal{G}$ and AG(4, 2) have the same point-set, e is also a pseudo-embedding of $\mathcal{G}$. There are $2^5 - 1 = 31$ pseudo-hyperplanes of $\mathcal{G}$ arising from e.

- If Π_0 is the hyperplane $X_0 = 0$ of PG(4, 2), then $e^{-1}(e(\mathcal{G}) \cap \Pi_0) = \emptyset$. So, the unique pseudo-hyperplane of Type 1 arises from e.
- If Π_1 is the hyperplane $X_1 = 0$ of PG(4, 2), then $e^{-1}(e(\mathcal{G}) \cap \Pi_1)$ is the union of the two lines $x = 0$ and $x = 1$ of AG(2, 4) and hence is a pseudo-hyperplane of Type 2 of $\mathcal{G}$. Since e is G-homogeneous, all 12 pseudo-hyperplanes of Type 2 of $\mathcal{G}$ arise from e.
- If Π_2 is the hyperplane $X_1 + X_3 = 0$ of PG(4, 2), then $e^{-1}(e(\mathcal{G}) \cap \Pi_2) = \{(0, 0), (0, 1), (1, 0), (1, 1), (\delta, \delta), (\delta, \delta^2), (\delta^2, \delta), (\delta^2, \delta^2)\}$ is a pseudo-hyperplane of $\mathcal{G}$ of Type 3. Since e is G-homogeneous, all 18 pseudo-hyperplanes of Type 3 of $\mathcal{G}$ arise from e.

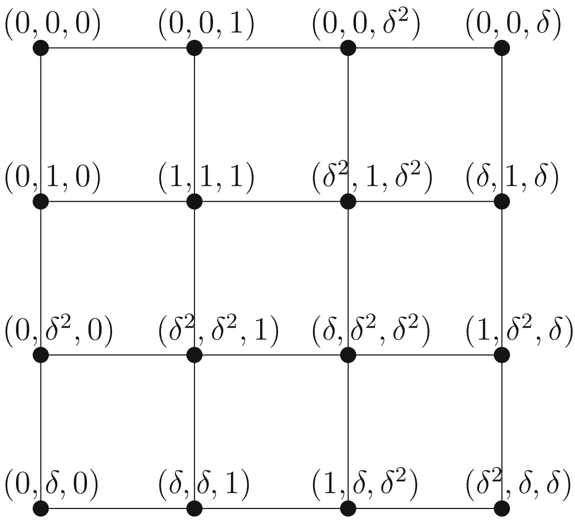
So, we have localized all 31 pseudo-hyperplanes of $\mathcal{G}$ which arise from e. By Proposition 2.1, e is homogeneous. The homogeneous pseudo-embedding e of $\mathcal{G}$ is isomorphic to one of the homogeneous pseudo-embeddings described in De Bruyn [2, Theorem 3.1].

(Ib) Let $\widetilde{e}$ be the universal pseudo-embedding of AG(2, 4). Then $\widetilde{e}$ maps the point (x, y) of AG(2, 4) to the point $(X_0, X_1, X_2, X_3, X_4, X_5, X_6) = (1, x + x^2, \delta x + \delta^2 x^2, y + y^2, \delta y + \delta^2 y^2, xy + x^2 y^2, \delta xy + \delta^2 x^2 y^2)$ of PG(6, 2). Since $\mathcal{G}$ and AG(2, 4) have the same point set, $\widetilde{e}$ is also a pseudo-embedding of $\mathcal{G}$. There are $2^7 - 1 = 127$ pseudo-hyperplanes of $\mathcal{G}$ arising from $\widetilde{e}$.

- As before, by considering the hyperplanes $X_0 = 0$, $X_1 = 0$ and $X_1 + X_3 = 0$, we see that all pseudo-hyperplanes of Type 1, 2 and 3 of $\mathcal{G}$ arise from $\widetilde{e}$.
- If Π_3 is the hyperplane of PG(6, 2) with equation $X_5 = 0$, then $\widetilde{e}^{-1}(\widetilde{e}(\mathcal{G}) \cap \Pi_3) = \{(0, y) \,|\, y \in \mathbb{F}_4\} \cup \{(x, 0) \,|\, x \in \mathbb{F}_4\} \cup \{(1, 1), (\delta, \delta^2), (\delta^2, \delta)\}$ is a pseudo-hyperplane of $\mathcal{G}$ of Type 7b, since the points $(0, 0)$, $(1, 1)$, (δ, δ^2) and (δ^2, δ) are not contained in some line of AG(2, 4). Since $\widetilde{e}$ is a G-homogeneous pseudo-embedding of $\mathcal{G}$, all 48 pseudo-hyperplanes of Type 7b of $\mathcal{G}$ arise from $\widetilde{e}$.
- If Π_4 is the hyperplane of PG(6, 2) with equation $X_0 + X_5 = 0$, then $\widetilde{e}^{-1}(\widetilde{e}(\mathcal{G}) \cap \Pi_4)$ is the complement of the pseudo-hyperplane described in the previous paragraph and hence is a pseudo-hyperplane of Type 8b. Since $\widetilde{e}$ is a G-homogeneous pseudo-embedding of $\mathcal{G}$, all 48 pseudo-hyperplanes of Type 8b of $\mathcal{G}$ will arise from $\widetilde{e}$.

So, we have localized all 127 pseudo-hyperplanes of $\mathcal{G}$ which arise from $\widetilde{e}$. By Proposition 2.1, $\widetilde{e}$ is G-homogeneous, but not homogeneous. In the terminology of De Bruyn [2], $\widetilde{e}$ is the almost-homogeneous pseudo-embedding of $\mathcal{G}$ whose corresponding family of ovoids of $\mathcal{G}$ is equal to $\mathcal{F}_b$.
So, the map $\widetilde{e}$ defined above provides direct constructions for the almost-homogeneous pseudo-embedding of $\mathcal{G}$.

(II) Suppose AG(3, 4) is the affine space obtained from PG(3, 4) by removing a hyperplane Π_∞. Suppose $\mathcal{G}$ is a (4×4)-subgrid of AG(3, 4) such that $<\mathcal{G}>=$ PG(3, 4). Then there exists a unique nonsingular hyperbolic quadric Q of PG(3, 4) such that Π_∞ is tangent to Q and $\mathcal{G} = Q \setminus \Pi_\infty$. We can choose a coordinate system such that the points of $\mathcal{G}$ have the following coordinates.



Let L_1 and L_2 be the two lines of Π_∞ such that $Q \cap \Pi_\infty = L_1 \cup L_2$ and put $\{p^*\} = L_1 \cap L_2$. If Π is one of the twelve planes of PG(3, 4) through p^* not containing L_1, nor L_2, then $\Pi \cap \mathcal{G}$ is an ovoid of $\mathcal{G}$. The set of twelve ovoids of $\mathcal{G}$ arising in this way form one of the two families of ovoids of $\mathcal{G}$. We denote this family by $\mathcal{F}_a$. Each automorphism of $\mathcal{G}$ belonging to G is induced by an automorphism of AG(3, 4) which stabilizes the point-set of $\mathcal{G}$. So, every homogeneous pseudo-embedding of AG(3, 4) will induce a G-homogeneous pseudo-embedding of $\mathcal{G}$.

(IIa) Let e be the quadratic pseudo-embedding of AG(3, 4). Then e maps the point (x, y, z) of AG(3, 4) to the point $(X_0, X_1, X_2, X_3, X_4, X_5, X_6) = (1, x + x^2, y + y^2, z + z^2, \delta x + \delta^2 x^2, \delta y + \delta^2 y^2, \delta z + \delta^2 z^2)$ of PG(6, 2). The pseudo-embedding e will induce a pseudo-embedding e' of $\mathcal{G}$ into a subspace Σ of PG(6, 2). Since $e[(0, 0, 0)] = (1, 0, 0, 0, 0, 0, 0)$, $e[(0, 0, 1)] = (1, 0, 0, 0, 0, 0, 1)$, $e[(0, 0, \delta^2)] = (1, 0, 0, 1, 0, 0, 0)$, $e[(0, 1, 0)] = (1, 0, 0, 0, 0, 1, 0)$, $e[(1, 1, 1)] = (1, 0, 0, 0, 1, 1, 1)$, $e[(\delta^2, 1, \delta^2)] = (1, 1, 0, 1, 0, 1, 0)$ and $e[(0, \delta^2, 0)] = (1, 0, 1, 0, 0, 0, 0)$ generate PG(6, 2), we have $\Sigma =$ PG (6, 2). So, there are $2^7 - 1 = 127$ pseudo-hyperplanes of $\mathcal{G}$ arising from e'.

- If Π_0 is the hyperplane $X_0 = 0$ of PG(6, 2), then $e^{-1}(e(\mathcal{G}) \cap \Pi_0) = \emptyset$. So, the unique pseudo-hyperplane of Type 1 arises from e'.
- If Π_1 is the hyperplane $X_2 = 0$ of PG(6, 2), then $e^{-1}(e(\mathcal{G}) \cap \Pi_1)$ is a pseudo-hyperplane of Type 2 of $\mathcal{G}$. Since e' is G-homogeneous, all 12 pseudo-hyperplanes of Type 2 arise from e'.

- If Π_2 is the hyperplane $X_2 + X_3 = 0$ of PG$(6, 2)$, then $e^{-1}(e(\mathcal{G}) \cap \Pi_2)$ is a pseudo-hyperplane of Type 3 of $\mathcal{G}$. Since e' is G-homogeneous, all 18 pseudo-hyperplanes of Type 3 of $\mathcal{G}$ arise from e'.
- If Π_3 is the hyperplane $X_1 = 0$ of PG$(6, 2)$, then $e^{-1}(e(\mathcal{G}) \cap \Pi_3) = \{(0, 0, 0), (0, 0, 1), (0, 0, \delta^2), (0, 0, \delta), (0, 1, 0), (0, \delta^2, 0), (0, \delta, 0), (1, 1, 1), (1, \delta^2, \delta), (1, \delta, \delta^2)\}$. Since the points $(0, 0, 0)$, $(1, 1, 1)$, $(1, \delta^2, \delta)$ and $(1, \delta, \delta^2)$ are not contained in a plane, $e^{-1}(e(\mathcal{G}) \cap \Pi_3)$ is a pseudo-hyperplane of Type 7b of $\mathcal{G}$. Since e' is G-homogeneous, all 48 pseudo-hyperplanes of Type 7b of $\mathcal{G}$ arise from e'.
- If Π_4 is the hyperplane $X_0 + X_1 = 0$ of PG$(6, 2)$, then $e^{-1}(e(\mathcal{G}) \cap \Pi_4)$ is the complement of the pseudo-hyperplane mentioned in the previous paragraph and hence is a pseudo-hyperplane of Type 8b of $\mathcal{G}$. Since e' is G-homogeneous, all 48 pseudo-hyperplanes of Type 8b arise from e'.

So, we have located all 127 pseudo-hyperplanes of $\mathcal{G}$ which arise from e'. By Proposition 2.1, e' is G-homogeneous, but not homogeneous. In the terminology of De Bruyn [2], we have:

Lemma 5.12 *e' is isomorphic to the almost-homogeneous pseudo-embedding of $\mathcal{G}$ whose corresponding family of ovoids of $\mathcal{G}$ is equal to $\mathcal{F}_b$.*

(IIb) Finally, suppose that $\widetilde{e}$: AG$(3, 4) \to$ PG$(12, 2)$ is the universal pseudo-embedding of AG$(3, 4)$. Then $\widetilde{e}$ will induce a pseudo-embedding $\widetilde{e}'$ of $\mathcal{G}$ into a subspace Σ of PG$(12, 2)$. Using the explicit description of $\widetilde{e}$ given in Theorem 1.2, it is possible to determine Σ. We find that $\dim(\Sigma) = 8$. Since the pseudo-embedding rank of $\mathcal{G}$ is equal to 9, see e.g. De Bruyn [1, Proposition 3.7], we obtain:

Lemma 5.13 *The pseudo-embedding $\widetilde{e}'$ is isomorphic to the universal pseudo-embedding of $\mathcal{G}$.*

5.4 Two homogeneous pseudo-embeddings of $Q(4, 3)$

In De Bruyn [2], we used the computer algebra system GAP [16] to show that the generalized quadrangle $Q(4, 3)$ has up to isomorphism two homogeneous pseudo-embeddings, the universal pseudo-embedding in PG$(14, 2)$ and a certain pseudo-embedding in PG$(8, 2)$. In [2], we did however not give any direct constructions for these two homogeneous pseudo-embeddings. The aim of this subsection is to show that these two homogeneous pseudo-embeddings of $Q(4, 3)$ are induced by the two homogeneous pseudo-embeddings of AG$(4, 4)$ into which $Q(4, 3)$ is fully embeddable.

Proposition 5.14 *Suppose the generalized quadrangle $Q(4, 3)$ is fully embedded into the affine space AG$(4, 4)$ and let $\mathcal{G}$ be a (4×4)-subgrid of $Q(4, 3)$. Let $\widetilde{e}$ be the universal pseudo-embedding of AG$(4, 4)$ and let $\widetilde{e}'$ be the pseudo-embedding of $Q(4, 3)$ induced by e. Let e be the quadratic pseudo-embedding of AG$(4, 4)$ and let e' be the pseudo-embedding of $Q(4, 3)$ induced by e. Then $\widetilde{e}'$ and e' are homogeneous pseudo-embeddings of $Q(4, 3)$, $\widetilde{e}' \geq e'$ and*

(1) *the pseudo-embedding of $\mathcal{G}$ induced by $\widetilde{e}'$ is isomorphic to the universal pseudo-embedding of $\mathcal{G}$,*
(2) *the pseudo-embedding of $\mathcal{G}$ induced by e' is isomorphic to the almost-homogeneous pseudo-embedding of $\mathcal{G}$ whose corresponding family of ovoids equals the set of subtended ovoids of $\mathcal{G}$.*

So, $\widetilde{e}'$ and e' are not isomorphic.

Proof The fact that $\widetilde{e}'$ and e' are homogeneous pseudo-embeddings of $Q(4,3)$ follows from Proposition 5.10 and the fact that $\widetilde{e}$ and e are homogeneous pseudo-embeddings of $\mathrm{AG}(4,4)$. Since $\widetilde{e} \geq e$, we also have $\widetilde{e}' \geq e'$. The claims (1) and (2) of the proposition follow from Lemmas 5.9, 5.12 and 5.13. □

Corollary 5.15 *With the notations of Proposition 5.14, we have that $\widetilde{e}'$ is isomorphic to the universal pseudo-embedding of $Q(4,3)$ and that e' is isomorphic to the homogeneous pseudo-embedding of $Q(4,3)$ into* $\mathrm{PG}(8,2)$.

Remark The claims mentioned in (1) and (2) of Proposition 5.14 were already obtained in De Bruyn [2, Theorem 1.7(b)]. In [2] however these claims were verified with the aid of computer computations in GAP.

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