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Two classes of optimal two-dimensional OOCs

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Abstract Let $\Phi(u \times v, k, \lambda_a, \lambda_c)$ denote the largest possible size among all 2-D $(u \times v, k, \lambda_a, \lambda_c)$-OOCs. In this paper, the exact value of $\Phi(u \times v, k, \lambda_a, k-1)$ for $\lambda_a = k-1$ and k is determined. The case $\lambda_a = k-1$ is a generalization of a result in Yang (Inform Process Lett 40:85–87, 1991) which deals with one dimensional OOCs namely, $u = 1$.

Keywords Optimal · Two-dimensional optical orthogonal code · Orbit · Stabilizer

Mathematics Subject Classification (2000) 05B30

1 Introduction

An optical orthogonal code is a family of (0, 1)-matrices with good auto- and cross-correlation properties. Its study has been motivated by applications in an optical code-division multiple access (OCDMA) system. For more information, the interested reader may refer to [18,21,22,25,26].

Let u, v, k, λ_a and λ_c be positive integers. A *two-dimensional* $(u \times v, k, \lambda_a, \lambda_c)$ *optical orthogonal code* (briefly, 2-D $(u \times v, k, \lambda_a, \lambda_c)$-OOC), $\mathcal{C}$, is a family of $u \times v$ (0, 1)-matrices (called *codewords*) of Hamming weight k satisfying the following two correlation properties:

(1) The auto-correlation property: $\sum_{i=0}^{u-1} \sum_{j=0}^{v-1} a_{ij} a_{i,j+\tau} \leq \lambda_a$ for any matrix $\mathbf{A} = (a_{ij})_{u \times v} \in \mathcal{C}$ and any integer $\tau \not\equiv 0 \pmod{v}$;
(2) The cross-correlation property: $\sum_{i=0}^{u-1} \sum_{j=0}^{v-1} a_{ij} b_{i,j+\tau} \leq \lambda_c$ for any matrices $\mathbf{A} = (a_{ij})_{u \times v} \in \mathcal{C}$, $\mathbf{B} = (b_{ij})_{u \times v} \in \mathcal{C}$ with $\mathbf{A} \neq \mathbf{B}$, and any integer τ,

where the integer $j + \tau$ is reduced modulo v. The number of codewords of $\mathcal{C}$ is called the *size* of $\mathcal{C}$. Let $\Phi(u \times v, k, \lambda_a, \lambda_c)$ denote the largest possible size among all 2-D $(u \times v, k, \lambda_a, \lambda_c)$-

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OOCs. A 2-D $(u\times v, k, \lambda_a, \lambda_c)$-OOC with $\Phi(u\times v, k, \lambda_a, \lambda_c)$ codewords is said to be *optimal*. A 2-D $(1 \times v, k, \lambda_a, \lambda_c)$-OOC is usually called one-dimensional $(v, k, \lambda_a, \lambda_c)$-OOC (or 1-D $(v, k, \lambda_a, \lambda_c)$-OOC). When $\lambda_a = \lambda_c = \lambda$, the notations of 2-D $(u \times v, k, \lambda)$-OOC and $\Phi(u \times v, k, \lambda)$ are employed the abbreviated notation. From the definition of optimal OOC, it is straightforward that an optimal 2-D OOC exists for all parameter values: it is simply an OOC of the largest possible size. Determining $\Phi(u \times v, k, \lambda_a, \lambda_c)$ for the parameters u, v, k, λ_a and λ_c is apparently a difficult task.

Throughout the present paper let $\Omega(u \times v, k)$ be the set of all k-subsets of $I_u \times Z_v$, where $I_u = \{0, 1, \ldots, u-1\}$ and Z_v is the residue group of integers modulo v. The notion of 2-D OOCs can be more conveniently reformulated as follows. By identifying codewords in $\mathcal{C}$ with k-subsets of $I_u \times Z_v$ representing the indices of the nonzero positions in a matrix, $\mathcal{C}$ can be viewed as a family $\mathcal{F} \subseteq \Omega(u \times v, k)$ satisfying the following two properties:

(1′) The auto-correlation property: $|X \bigcap (X + \tau)| \le \lambda_a$ for any $X \in \mathcal{F}$ and every $\tau \in Z_v \setminus \{0\}$;
(2′) The cross-correlation property: $|X \bigcap (Y + \tau)| \le \lambda_c$ for any $X, Y \in \mathcal{F}$ with $X \ne Y$ and every $\tau \in Z_v$,

where $X + i = \{(x, y+i) : (x, y) \in X\}$ and all the additive operations are performed in Z_v.

The research on optimal 1-D OOC has concentrated on the case when $\lambda_a = \lambda_c$ in many papers, see, for example, [1–3,6,11,8–10,12–17,19,20,35]); a little work has been done on the case when $\lambda_a \ne \lambda_c$, the reader refers to [5,23,31]. For the research on optimal 2-D OOC, the reader may refer to [4,7,24,27,29,32,34] and the references therein. It should be mentioned that because of the practical point of view in the definition of OOC λ_a must be less than k. Therefore $\lambda_a = k$ is an extreme case.

For any $g \in Z_v$ and $B \in \Omega(u \times v, k)$, define $B + g = \{(x, y + g) : (x, y) \in B\}$. Then Z_v acts on $\Omega(u \times v, k)$. The orbit generated by B is defined by the set of all distinct $B + g$ where g takes over Z_v. If an orbit has v elements, then the orbit is said to be *full*, otherwise *short*. The subgroup $\{g \in Z_v : B + g = B\}$ is called the *stabilizer* of B in Z_v. We know that $\Omega(u \times v, k)$ can be partitioned into some orbits.

In the present paper we consider two classes of optimal 2-D $(u \times v, k, \lambda_a, k-1)$-OOCs, where $\lambda_a = k-1, k$. In this situation, let $\mathcal{F} \subseteq \Omega(u\times v, k)$ be a 2-D $(u\times v, k, \lambda_a, k-1)$-OOC. By (2′) of the definition, X and Y belong to distinct orbits of $\Omega(u \times v, k)$ if $X, Y \in \mathcal{F}$ with $X \ne Y$. Combined with the property (1′) of the definition, we have the following lemma.

Lemma 1.1 $\Phi(u \times v, k, k-1)$ *is the number of all full orbits in* $\Omega(u \times v, k)$;
$\Phi(u \times v, k, k, k-1)$ *is the number of all orbits in* $\Omega(u \times v, k)$.

In the rest of this paper, we will determine the number of all full orbits (or all orbits, respectively) in $\Omega(u \times v, k)$. We then give two classes of optimal 2-D $(u \times v, k, \lambda_a, k-1)$-OOCs where $\lambda_a = k-1, k$.

2 Exact value of $\Phi(u \times v, k, k-1)$

To determine the exact value of $\Phi(u \times v, k, k-1)$, by Lemma 1.1 we only need to find the number of all full orbits in $\Omega(u \times v, k)$. Before proceeding, we recall that the well-known Möbius function is defined as follows.

$$\mu(n)=\begin{cases}1, & \text{if } n=1,\\ (-1)^r, & \text{if } n \text{ is the product of } r \text{ distinct}\\ & \text{prime numbers,}\\ 0, & \text{if } n \text{ has a square factor.}\end{cases}$$

The following result on Möbius function is well-known, for example, see [30, Theorem 10.3].

Lemma 2.1

$$\sum_{d|n}\mu(d)=\begin{cases}1 & n=1,\\ 0 & n\neq 1,\end{cases}$$

where $d \mid n$ means that d runs over all positive factors of n.

Lemma 2.2 *Let $B\in\Omega(u\times v,k)$. Then the order d of the stabilizer of B in Z_v is a divisor of (k,v). Furthermore, B can be written as $B=\cup_{i=0}^{d-1}(B_0+\frac{vi}{d})$ for some $B_0\in\Omega(u\times\frac{v}{d},\frac{k}{d})$, where $B_0+vi/d=\{(x,y+vi/d):\ (x,y)\in B_0\}$.*

Proof Let $B=\{(x_1,y_1),(x_2,y_2),\ldots,(x_k,y_k)\}$, where $x_i\in I_u$ and $y_i\in Z_v$ for $1\le i\le k$. The stabilizer of B is $G_B=\{\delta\in Z_v:\ B+\delta=B\}$. Since G_B is a subgroup of order d in Z_v, we have $d|v$ and $G_B=\langle v/d\rangle$. Noting that $B+v/d=B$, we then conclude that

$$\sum_{i=1}^{k}y_i=\sum_{i=1}^{k}(y_i+\frac{v}{d}),$$

which implies that $\frac{kv}{d}\equiv 0 \pmod v$, i.e. $k\equiv 0 \pmod d$. Hence, d is a divisor of (k,v).

Define

$$B_0=\{(x,y):\ (x,y)\in B, 0\le y\le v/d-1\}.$$

It is easy to see that $(B_0+vi/d)\cap(B_0+vj/d)=\emptyset$ for $i\neq j$ and $0\le i,j\le d-1$. Note that if $(x,y)\in B$ then $(x,y+vi/d)\in B$ for any $0\le i\le d-1$. Then $\bigcup_{i=0}^{d-1}(B_0+vi/d)\subseteq B$. From the definition of B_0, clearly $B\subseteq\bigcup_{i=0}^{d-1}(B_0+vi/d)$. Hence $B=\bigcup_{i=0}^{d-1}(B_0+vi/d)$ and then $|B_0|=k/d$. We know that $B_0\in\Omega(u\times\frac{v}{d},\frac{k}{d})$. □

Lemma 2.3 *Let $g_d(u\times v,k)$ denote the number of k-subsets $B\in\Omega(u\times v,k)$ such that the order of the stabilizer of B is d. Then*

$$|\Omega(u\times v,k)|=\sum_{d|(k,v)}g_d(u\times v,k)=\binom{uv}{k}.$$

Proof From the definition of $\Omega(u\times v,k)$, we have $|\Omega(u\times v,k)|=\binom{uv}{k}$. Let Ω_d denote the set of k-subsets $B\in\Omega(u\times v,k)$ such that the order of stabilizer of B is d. Then $g_d(u\times v,k)=|\Omega_d|$. By Lemma 2.2 it follows that $d|(k,v)$. By the definition of Ω_d, we have $\Omega(u\times v,k)=\cup_{d|(k,v)}\Omega_d$. Therefore, $|\Omega(u\times v,k)|=\sum_{d\mid(k,v)}|\Omega_d|=\sum_{d\mid(k,v)}g_d(u\times v,k)=\binom{uv}{k}$. □

Lemma 2.4 *Let $g_d(u\times v,k)$ denote the number of k-subsets $B\in\Omega(u\times v,k)$ such that the order of the stabilizer of B is d. Then the number of all k-subsets in $\Omega(u\times v,k)$ with trivial stabilizer is*

$$g_1(u\times v,k)=\sum_{d|(k,v)}\mu(d)\binom{\frac{uv}{d}}{\frac{k}{d}}.$$

Proof Let $\alpha = (k, v)$. For given $d \mid \alpha$, we consider the set $\bar{\Omega}(u \times \frac{v}{d}, \frac{k}{d})$ of all (k/d)-subsets of $I_u \times Z_{v/d}$. Denote by $\bar{g}_x(u \times \frac{v}{d}, \frac{k}{d})$ the number of (k/d)-subsets $B \in \bar{\Omega}(u \times \frac{v}{d}, \frac{k}{d})$ such that the order of stabilizer of B is x, where $x | \frac{\alpha}{d}$. Applying Lemma 2.3 gives the following formula,

$$\left|\bar{\Omega}\left(u \times \frac{v}{d}, \frac{k}{d}\right)\right| = \sum_{x|\frac{\alpha}{d}} \bar{g}_x\left(u \times \frac{v}{d}, \frac{k}{d}\right) = \binom{\frac{uv}{d}}{\frac{k}{d}}. \tag{2.1}$$

Consider the 1-1 mapping σ from $\bar{\Omega}(u \times \frac{v}{d}, \frac{k}{d})$ onto $\Omega(u \times v, k)$ given by

$$\sigma(B_0) = \bigcup_{i=0}^{d-1} \left(B_0 + \frac{vi}{d}\right)$$

for any $B_0 \in \bar{\Omega}(u \times \frac{v}{d}, \frac{k}{d})$. By Lemma 2.2, it is not difficult to show that the stabilizer of B_0 in $\bar{\Omega}(u \times \frac{v}{d}, \frac{k}{d})$ is x if and only if the stabilizer of $\sigma(B_0)$ in $\Omega(u \times v, k)$ is dx. We then have $\bar{g}_x(u \times \frac{v}{d}, \frac{k}{d}) = g_{dx}(u \times v, k)$. Hence, by formula (2.1), we have

$$\begin{aligned}\sum_{x|\frac{\alpha}{d}} \bar{g}_x(u \times \frac{v}{d}, \frac{k}{d}) &= \sum_{x|\frac{\alpha}{d}} g_{dx}(u \times v, k)\\ &= \sum_{d|x|\alpha} g_x(u \times v, k) = \binom{\frac{uv}{d}}{\frac{k}{d}}.\end{aligned}$$

Therefore, using Lemma 2.1, we then have

$$\begin{aligned}&\sum_{d|\alpha} \mu(d)\binom{\frac{uv}{d}}{\frac{k}{d}}\\ &= \sum_{d|\alpha}\sum_{d|x|\alpha} g_x(u \times v, k)\mu(d) = \sum_{x|\alpha}\sum_{d|x} g_x(u \times v, k)\mu(d)\\ &= \sum_{x|\alpha} g_x(u \times v, k)\sum_{d|x}\mu(d) = g_1(u \times v, k).\end{aligned}$$

□

Theorem 2.5 *Let u, v and k be positive integers. Then*

$$\Phi(u \times v, k, k-1) = \frac{1}{v}\sum_{d|(k,v)} \mu(d)\binom{\frac{uv}{d}}{\frac{k}{d}}.$$

Proof From the definition of 2-D OOC, $\Phi(u \times v, k, k-1)$ is actually the number of all full orbits of the group action $(Z_v, \Omega(u \times v, k))$. Since each full orbit contains exactly v k-subsets of $\Omega(u \times v, k)$ with trivial stabilizer, by Lemma 2.4, we obtain

$$\begin{aligned}\Phi(u \times v, k, k-1) &= \frac{1}{v} g_1(u \times v, k)\\ &= \frac{1}{v}\sum_{d|(k,v)} \mu(d)\binom{\frac{uv}{d}}{\frac{k}{d}}.\end{aligned}$$

□

Taking $u = 1$ in Theorem 2.5, we have the following corollary, which was first proved by Yang [33].

Corollary 2.6 *Let v and k be positive integers. Then*

$$\Phi(v, k, k-1) = \frac{1}{v} \sum_{d|(k,v)} \mu(d) \binom{\frac{v}{d}}{\frac{k}{d}}.$$

3 Exact value of $\Phi(u \times v, k, k, k-1)$

To determine the exact value of $\Phi(u \times v, k, k, k-1)$, by Lemma 1.1 we only need to find the number of all orbits in $\Omega(u \times v, k)$. Recall that Z_v acts on $\Omega(u \times v, k)$. By the well-known Cauchy-Frobenius-Burnside Lemma (refer to [28, Lemma 1.25]), we have that the number of all orbits in $\Omega(u \times v, k)$ is

$$\frac{1}{v} \sum_{g \in Z_v} |fix(g)|,$$

where $fix(g) = \{B : B + g = B,\ B \in \Omega(u \times v, k)\}$. We state the result as follows.

Lemma 3.1 *Let u, v and k be positive integers. Then*

$$\Phi(u \times v, k, k, k-1) = \frac{1}{v} \sum_{g \in Z_v} |fix(g)|.$$

Theorem 3.2 *Let u, v and k be positive integers. Then*

$$\Phi(u \times v, k, k, k-1) = \frac{1}{v} \sum_{d|(k,v)} \varphi(d) \binom{\frac{uv}{d}}{\frac{k}{d}}.$$

Proof By Lemma 3.1, we only need to compute the size of $fix(g)$ for every $g \in Z_v$.

For any $B \in fix(g)$, then $B + g = B$. It is clear that the spanning group $\langle g \rangle$ is a subgroup of the stabilizer G_B of B in Z_v. From Lemma 2.2 we have $|\langle g \rangle|$ is a divisor of $|G_B|$, and hence $|\langle g \rangle|$ is a divisor of (k, v). By Lemma 2.2, B can be written as $B = \cup_{i=0}^{|G_B|-1}(B_0 + \frac{vi}{|G_B|})$ for some $B_0 \in \Omega(u \times \frac{v}{|G_B|}, \frac{k}{|G_B|})$, where $B_0 + vi/|G_B| = \{(x, y + vi/|G_B|) : (x, y) \in B_0\}$. Let $A = \cup_{i=0}^{x-1}(B_0 + \frac{vi}{|G_B|})$ where $x = |G_B|/|\langle g \rangle|$. Then $A \in \Omega(u \times \frac{v}{|\langle g \rangle|}, \frac{k}{|\langle g \rangle|})$. It is readily checked that $B = \cup_{i=0}^{|\langle g \rangle|-1}(A + \frac{vi}{|\langle g \rangle|})$. Conversely, if $B = \cup_{i=0}^{|\langle g \rangle|-1}(A + \frac{vi}{|\langle g \rangle|})$ for some $A \in \Omega(u \times \frac{v}{|\langle g \rangle|}, \frac{k}{|\langle g \rangle|})$, by noting that $|\langle g \rangle| = v/(g, v)$, we then have $B + g = B$, that is $B \in fix(g)$. That means that $B \in fix(g)$ if and only if B can be written as $B = \cup_{i=0}^{|\langle g \rangle|-1}(A + \frac{vi}{|\langle g \rangle|})$ for some $A \in \Omega(u \times \frac{v}{|\langle g \rangle|}, \frac{k}{|\langle g \rangle|})$. Hence, we have $|fix(g)| = |\Omega(u \times \frac{v}{|\langle g \rangle|}, \frac{k}{|\langle g \rangle|})| = \binom{\frac{uv}{|\langle g \rangle|}}{\frac{k}{|\langle g \rangle|}}$ if $|\langle g \rangle|$ is a divisor of (k, v); or $fix(g) = 0$ otherwise. That is

$$|fix(g)| = \begin{cases} \binom{\frac{uv}{|\langle g \rangle|}}{\frac{k}{|\langle g \rangle|}}, & |\langle g \rangle| \mid (k, v), \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Note that there are $\varphi(d)$ elements g in Z_v such that the order of the spanning group $\langle g\rangle$ is d, that is, $\varphi(d) = \sum_{\substack{g\in Z_v\\ |\langle g\rangle|=d}} 1$. Applying Lemma 3.1 and (3.2), we then have the following

$$\begin{aligned}
\Phi(u\times v,k,k,k-1) &= \frac{1}{v}\sum_{\substack{|\langle g\rangle|\,|(k,v)\\ g\in Z_v}} \binom{\frac{uv}{|\langle g\rangle|}}{\frac{k}{|\langle g\rangle|}} \\
&= \frac{1}{v}\sum_{d|(k,v)}\sum_{\substack{g\in Z_v\\ |\langle g\rangle|=d}} \binom{\frac{uv}{d}}{\frac{k}{d}} \\
&= \frac{1}{v}\sum_{d|(k,v)} \binom{\frac{uv}{d}}{\frac{k}{d}} \sum_{\substack{g\in Z_v\\ |\langle g\rangle|=d}} 1 \\
&= \frac{1}{v}\sum_{d|(k,v)} \varphi(d)\binom{\frac{uv}{d}}{\frac{k}{d}}.
\end{aligned}$$

This completes the proof. □

4 Concluding remarks

In the present paper, we have determined the exact value of $\Phi(u \times v, k, \lambda_a, k-1)$ for $\lambda_a = k-1$ and k. That means that the size of an optimal 2-D $(u\times v, k, \lambda_a, k-1)$-OOC with $\lambda_a = k-1$ and k is characterized. General speaking, the determination of $\Phi(u\times v, k, \lambda_a, \lambda_c)$ is still much open for $\lambda_a < k-1$ or $\lambda_c < k-1$. It is worthy for further investigation.

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