$\mathbb{Z}_2\mathbb{Z}_4$-linear codes: rank and kernel

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Abstract A code $\mathcal{C}$ is $\mathbb{Z}_2\mathbb{Z}_4$-additive if the set of coordinates can be partitioned into two subsets X and Y such that the punctured code of $\mathcal{C}$ by deleting the coordinates outside X (respectively, Y) is a binary linear code (respectively, a quaternary linear code). The corresponding binary codes of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes under an extended Gray map are called $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. In this paper, the invariants for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, the rank and dimension of the kernel, are studied. Specifically, given the algebraic parameters of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, the possible values of these two invariants, giving lower and upper bounds, are established. For each possible rank r between these bounds, the construction of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with rank r is given. Equivalently, for each possible dimension of the kernel k, the construction of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with dimension of the kernel k is given. Finally, the bounds on the rank, once the kernel dimension is fixed, are established and the construction of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code for each possible pair (r, k) is given.

Keywords Quaternary linear codes · $\mathbb{Z}_4$-linear codes · $\mathbb{Z}_2\mathbb{Z}_4$-additive codes · $\mathbb{Z}_2\mathbb{Z}_4$-linear codes · Kernel · Rank

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1 Introduction

Let $\mathbb{Z}_2$ and $\mathbb{Z}_4$ be the ring of integers modulo 2 and modulo 4, respectively. Let $\mathbb{Z}_2^n$ be the set of all binary vectors of length n and let $\mathbb{Z}_4^n$ be the set of all n-tuples over the ring $\mathbb{Z}_4$. In this paper, the elements of $\mathbb{Z}_4^n$ will also be called quaternary vectors of length n.

Any nonempty subset C of $\mathbb{Z}_2^n$ is a binary code and a subgroup of $\mathbb{Z}_2^n$ is called a *binary linear code* or a *$\mathbb{Z}_2$-linear code*. Equivalently, any nonempty subset $\mathcal{C}$ of $\mathbb{Z}_4^n$ is a quaternary code and a subgroup of $\mathbb{Z}_4^n$ is called a *quaternary linear code*. Quaternary codes can be viewed as binary codes under the usual Gray map defined as $\phi(0) = (0, 0)$, $\phi(1) = (0, 1)$, $\phi(2) = (1, 1)$, $\phi(3) = (1, 0)$ in each coordinate. If $\mathcal{C}$ is a quaternary linear code, then the binary code $C = \phi(\mathcal{C})$ is called a *$\mathbb{Z}_4$-linear* code. The dual of a quaternary linear code $\mathcal{C}$, denoted by $\mathcal{C}^\perp$, is called the *quaternary dual code* and is defined in the standard way [19] in terms of the usual inner product for quaternary vectors [15]. The binary code $C_\perp = \phi(\mathcal{C}^\perp)$ is called the *$\mathbb{Z}_4$-dual code* of $C = \phi(\mathcal{C})$.

Since 1994, quaternary linear codes have became significant due to its relationship to some classical well-known binary codes as the Nordstrom-Robinson, Kerdock, Preparata, Goethals or Reed-Muller codes [15]. It was proved that the Kerdock code and some Preparata-like codes are $\mathbb{Z}_4$-linear codes and, moreover, the $\mathbb{Z}_4$-dual code of the Kerdock code is a Preparata-like code. Lately, more families of quaternary linear codes, called QRM, ZRM and $\mathcal{RM}$, related to the Reed-Muller codes have been studied in [3,4,25], respectively.

Additive codes were first defined by Delsarte in 1973 in terms of association schemes [11,12]. In general, an additive code, in a translation association scheme, is defined as a subgroup of the underlying Abelian group. In the special case of a binary Hamming scheme, that is, when the underlying Abelian group is of order 2^n, the only structures for the Abelian group are those of the form $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, with $\alpha + 2\beta = n$. Therefore, the subgroups $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ are the only additive codes in a binary Hamming scheme. In order to distinguish them from additive codes over finite fields [2], we will hereafter call them *$\mathbb{Z}_2\mathbb{Z}_4$-additive codes* [5,9,22]. The $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are also included in other families of codes with an algebraic structure, such as mixed group codes [18] and translation invariant propelinear codes [24].

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, which is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Let $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$, where $n = \alpha + 2\beta$, be an extension of the usual Gray map given by

$$\Phi(x, y) = (x, \phi(y_1), \ldots, \phi(y_\beta))$$

$$\text{for any } x \in \mathbb{Z}_2^\alpha, \quad \text{and any } y = (y_1, \ldots, y_\beta) \in \mathbb{Z}_4^\beta.$$

This Gray map is an isometry which transforms Lee distances defined in a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ to Hamming distances defined in the corresponding binary code $C = \Phi(\mathcal{C})$. The binary code $C = \Phi(\mathcal{C})$ is called *$\mathbb{Z}_2\mathbb{Z}_4$-linear*. Note that $\mathbb{Z}_2\mathbb{Z}_4$-linear codes include binary linear codes and $\mathbb{Z}_4$-linear codes.

Two binary codes C_1 and C_2 of length n are said to be *isomorphic* if there exists a coordinate permutation π such that $C_2 = \{\pi(c) \mid c \in C_1\}$. Two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be *monomially equivalent*, if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain $\mathbb{Z}_4$ coordinates. They are said to be *permutation equivalent* if they differ only by a permutation of coordinates [16]. Note that if two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are monomially equivalent, then, after the Gray map,

the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes $C_1 = \Phi(\mathcal{C}_1)$ and $C_2 = \Phi(\mathcal{C}_2)$ are isomorphic as binary codes.

Two structural properties of nonlinear binary codes are the rank and dimension of the kernel. The *rank* of a binary code C, $rank(C)$, is simply the dimension of $\langle C \rangle$, which is the linear span of the codewords of C. The *kernel* of a binary code C, $K(C)$, is the set of vectors that leave C invariant under translation, i.e. $K(C) = \{x \in \mathbb{Z}_2^n \mid C + x = C\}$. If C contains the all-zero vector, then $K(C)$ is a binary linear subcode of C. In general, C can be written as the union of cosets of $K(C)$, and $K(C)$ is the largest such linear code for which this is true [1]. We will denote the dimension of the kernel of C by $ker(C)$.

The rank and dimension of the kernel have been studied for some families of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes [3,8,7,17,20,21,23]. These two parameters do not always give a full classification of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, since two nonisomorphic $\mathbb{Z}_2\mathbb{Z}_4$-linear codes could have the same rank and dimension of the kernel. In spite of that, they can help in classification, since if two $\mathbb{Z}_2\mathbb{Z}_4$-linear codes have different ranks or dimensions of the kernel, they are nonisomorphic. Moreover, in this case the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are not monomially equivalent, so these two parameters can also help to distinguish between $\mathbb{Z}_2\mathbb{Z}_4$-additive codes that are not monomially equivalent.

Most of the concepts on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes have been implemented recently as a new package in MAGMA, including the computation of the rank and kernel, and the construction of some families of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes [6,10].

The aim of this paper is the study of the rank and dimension of the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, given their algebraic parameters. The paper is organized as follows. In Sect. 2, we give some properties related to both $\mathbb{Z}_2\mathbb{Z}_4$-additive and $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, including the linearity of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. In Sect. 3, we determine all possible values of the rank for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and we prove the existence of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with rank r for all possible values of r. Equivalently, in Sect. 4, we establish all possible values of the dimension of the kernel for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and we prove the existence of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with dimension of the kernel k for all possible values of k. In Sect. 5, we determine all possible pairs of values (r, k) for which there exist a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with rank r and dimension of the kernel k and we construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code for any of these possible pairs. Finally, the conclusions are given in Sect. 6.

2 Preliminaries

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, it is also isomorphic to an Abelian structure $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Therefore, $\mathcal{C}$ is of type $2^\gamma 4^\delta$ as a group, it has $|\mathcal{C}| = 2^{\gamma+2\delta}$ codewords and the number of order two codewords in $\mathcal{C}$ is $2^{\gamma+\delta}$. Let X (respectively Y) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$) coordinate positions, so $|X| = \alpha$ and $|Y| = \beta$. Unless otherwise stated, the set X corresponds to the first α coordinates and Y corresponds to the last β coordinates. Call $\mathcal{C}_X$ (respectively $\mathcal{C}_Y$) the punctured code of $\mathcal{C}$ by deleting the coordinates outside X (respectively Y). Let $\mathcal{C}_b$ be the subcode of $\mathcal{C}$ which contains all order two codewords and let κ be the dimension of $(\mathcal{C}_b)_X$, which is a binary linear code. For the case $\alpha = 0$, we will write $\kappa = 0$. Considering all these parameters, we will say that $\mathcal{C}$ (or equivalently $C = \Phi(\mathcal{C})$) is of type $(\alpha, \beta; \gamma, \delta; \kappa)$.

Although a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is not a free module, every codeword is uniquely expressible in the form

$$\sum_{i=1}^{\gamma} \lambda_i u_i + \sum_{j=1}^{\delta} \mu_j v_j,$$

where $\lambda_i \in \mathbb{Z}_2$ for $1 \le i \le \gamma$, $\mu_j \in \mathbb{Z}_4$ for $1 \le j \le \delta$ and u_i, v_j are vectors in $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ of order two and four, respectively. The vectors u_i, v_j give us a generator matrix $\mathcal{G}$ of size $(\gamma + \delta) \times (\alpha + \beta)$ for the code $\mathcal{C}$. Moreover, we can write $\mathcal{G}$ as

$$\mathcal{G} = \left(\begin{array}{c|c} B_1 & 2B_3 \\ \hline B_2 & Q \end{array} \right), \tag{1}$$

where B_1, B_2 are matrices over $\mathbb{Z}_2$ of size $\gamma \times \alpha$ and $\delta \times \alpha$, respectively; B_3 is a matrix over $\mathbb{Z}_4$ of size $\gamma \times \beta$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and Q is a matrix over $\mathbb{Z}_4$ of size $\delta \times \beta$ with quaternary row vectors of order four.

Let I_n be the identity matrix of size $n \times n$. In [15], it was shown that any quaternary linear code of type $2^{\gamma} 4^{\delta}$ is permutation equivalent to a quaternary linear code with a generator matrix of the form

$$\mathcal{G}_S = \left(\begin{array}{ccc} 2T & 2I_{\gamma} & \mathbf{0} \\ S & R & I_{\delta} \end{array} \right), \tag{2}$$

where R, T are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$, and of size $\delta \times \gamma$ and $\gamma \times (\beta - \gamma - \delta)$, respectively; and S is a matrix over $\mathbb{Z}_4$ of size $\delta \times (\beta - \gamma - \delta)$. The following theorem is a generalization of this result for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, so it gives a canonical generator matrix for these codes.

Theorem 1 *[5] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathcal{C}$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with canonical generator matrix of the form*

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_{\kappa} & T' & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S' & S & R & I_{\delta} \end{array} \right), \tag{3}$$

where T', S' are matrices over $\mathbb{Z}_2$; T_1, T_2, R are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and S is a matrix over $\mathbb{Z}_4$.

The concept of duality for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes was also studied in [5], where the appropriate inner product for any two vectors $u, v \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ was defined. Actually, in [5] it was shown that, given a finite Abelian group, the inner product is uniquely defined after fixing the generators in each one of the Abelian elementary groups in its decomposition. In our case, the inner product in $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ is defined over $\mathbb{Z}_4$ as

$$u \cdot v = 2\left(\sum_{i=1}^{\alpha} u_i v_i \right) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4,$$

where $u, v \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ and the computations are made taking the zeros and ones in the first α coordinates as quaternary zeros and ones, respectively. If $\alpha = 0$, the inner product is the usual one for quaternary vectors, and if $\beta = 0$, it is twice the usual one for binary vectors. Then, the *additive dual code* of $\mathcal{C}$, denoted by $\mathcal{C}^{\perp}$, is defined in the standard way

$$\mathcal{C}^{\perp} = \{ v \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta} \mid u \cdot v = 0 \text{ for all } u \in \mathcal{C} \}.$$

The corresponding binary code $\Phi(\mathcal{C}^\perp)$ is denoted by $C_\perp$ and called the $\mathbb{Z}_2\mathbb{Z}_4$-*dual code* of C. Moreover, in [5] it was proved that the additive dual code $\mathcal{C}^\perp$, which is also a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, is of type $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$, where

$$
\begin{aligned}
\bar{\gamma} &= \alpha + \gamma - 2\kappa, \\
\bar{\delta} &= \beta - \gamma - \delta + \kappa, \\
\bar{\kappa} &= \alpha - \kappa.
\end{aligned}
\tag{4}
$$

The following two lemmas are a generalization of the same results proved for quaternary vectors and quaternary linear codes, respectively, in [15]. Let $u * v$ denote the component-wise product for any $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$.

Lemma 1 *For all $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, we have*

$$
\Phi(u + v) = \Phi(u) + \Phi(v) + \Phi(2u * v).
$$

Proof Straightforward using the same arguments as for quaternary vectors to prove that for all $u, v \in \mathbb{Z}_4^\beta$, $\Phi(u + v) = \Phi(u) + \Phi(v) + \Phi(2u * v)$, [15], [26, p. 45]. □

Note that if u or v are vectors in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ of order two, then $\Phi(u + v) = \Phi(u) + \Phi(v)$.

Lemma 2 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. The $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ is a binary linear code if and only if $2u * v \in \mathcal{C}$ for all $u, v \in \mathcal{C}$.*

Proof Straightforward by Lemma 1 and using the same arguments as for quaternary linear codes [15,26]. □

Note that if $\mathcal{G}$ is a generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ as in (1) and $\{u_i\}_{i=1}^\gamma$ and $\{v_j\}_{j=1}^\delta$ are the row vectors of order two and four in $\mathcal{G}$, respectively, then the $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ is a binary linear code if and only if $2v_j * v_k \in \mathcal{C}$, for all j, k satisfying $1 \leq j < k \leq \delta$, since the component-wise product is bilinear.

3 Rank of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code of binary length $n = \alpha + 2\beta$. In this section, we will study the rank of these $\mathbb{Z}_2\mathbb{Z}_4$-linear codes C. We will show that there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $r = rank(C)$ for any possible value of r.

Lemma 3 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. Let $\mathcal{G}$ be a generator matrix of $\mathcal{C}$ as in (1) and let $\{u_i\}_{i=1}^\gamma$ be the rows of order two and $\{v_j\}_{j=1}^\delta$ the rows of order four in $\mathcal{G}$. Then, $\langle C \rangle$ is generated by $\{\Phi(u_i)\}_{i=1}^\gamma$, $\{\Phi(v_j), \Phi(2v_j)\}_{j=1}^\delta$ and $\{\Phi(2v_j * v_k)\}_{1 \leq j < k \leq \delta}$.*

Proof If $x \in \mathcal{C}$, then x can be expressed as $x = v_{j_1} + \cdots + v_{j_m} + w$, where $\{j_1, \ldots, j_m\} \subseteq \{1, \ldots, \delta\}$ and w is a codeword of order two. By Lemma 1, $\Phi(x) = \Phi(v_{j_1} + \cdots + v_{j_m}) + \Phi(w)$, where $\Phi(w)$ is a linear combination of $\{\Phi(u_i)\}_{i=1}^\gamma$ and $\{\Phi(2v_j)\}_{j=1}^\delta$, and $\Phi(v_{j_1} + \cdots + v_{j_m}) = \Phi(v_{j_1}) + \cdots + \Phi(v_{j_m}) + \sum_{1 \leq k < l \leq m} \Phi(2v_{j_k} * v_{j_l})$. Therefore, $\Phi(x)$ is generated by $\{\Phi(u_i)\}_{i=1}^\gamma$, $\{\Phi(v_j), \Phi(2v_j)\}_{j=1}^\delta$ and $\{\Phi(2v_j * v_k)\}_{1 \leq j < k \leq \delta}$. □

Proposition 1 *Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of binary length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then,*

$$rank(C) \in \left\{\gamma + 2\delta, \ldots, \min\left(\beta + \delta + \kappa, \; \gamma + 2\delta + \binom{\delta}{2}\right)\right\}.$$

Proof Let $\mathcal{G}_S$ be a canonical generator matrix of $\mathcal{C} = \Phi^{-1}(C)$ as in (3). In the generator matrix $\mathcal{G}_S$ there are γ rows of order two, $\{u_i\}_{i=1}^{\gamma}$, and δ rows of order four, $\{v_j\}_{j=1}^{\delta}$. Then, by Lemma 3, we can take the matrix G whose row vectors are $\{\Phi(u_i)\}_{i=1}^{\gamma}$, $\{\Phi(v_j), \Phi(2v_j)\}_{j=1}^{\delta}$ and $\{\Phi(2v_j * v_k)\}_{1 \leq j < k \leq \delta}$, as a generator matrix of $\langle C \rangle$.

The binary vectors $\{\Phi(u_i)\}_{i=1}^{\gamma}$ and $\{\Phi(v_j), \Phi(2v_j)\}_{j=1}^{\delta}$ are linearly independent over $\mathbb{Z}_2$. Thus, $rank(C) = \gamma + 2\delta + \bar{r}$, where $\bar{r}$ is the number of additional independent vectors taken from $\{\Phi(2v_j * v_k)\}_{1 \leq j < k \leq \delta}$. Note that there are at most $\binom{\delta}{2}$ of such vectors. Using row reduction in $\Phi^{-1}(G)$, the $\binom{\delta}{2}$ vectors $\{2v_j * v_k\}_{1 \leq j < k \leq \delta}$ can be transformed into vectors with zeroes in the last $\gamma - \kappa + \delta$ coordinates. Therefore, there are at most $\min\left(\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2}\right)$ of such additional independent vectors, so the upper bound of the rank is $\min\left(\beta + \delta + \kappa, \; \gamma + 2\delta + \binom{\delta}{2}\right)$.

The lower bound follows from the case where the code C is both binary linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear.   □

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and $C = \Phi(\mathcal{C})$ with $rank(C) = \gamma + 2\delta + \bar{r}$, where $\bar{r} \in \left\{0, \ldots, \min\left(\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2}\right)\right\}$. Let $\mathcal{G}$ be a generator matrix of $\mathcal{C}$ as in (1), where $\{u_i\}_{i=1}^{\gamma}$ are the rows of order two and $\{v_j\}_{j=1}^{\delta}$ the rows of order four. By the proof of Proposition 1, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{S}_\mathcal{C}$ generated by $\{u_i\}_{i=1}^{\gamma}$, $\{v_j\}_{j=1}^{\delta}$ and $\{2v_j * v_k\}_{1 \leq j < k \leq \delta}$ is of type $(\alpha, \beta; \gamma + \bar{r}, \delta; \kappa)$ and it is easy to check that $\Phi(\mathcal{S}_\mathcal{C}) = \langle C \rangle$, by Lemma 3. The following theorem follows.

Theorem 2 *Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code. Then, $\langle C \rangle$ is both binary linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear.*

For the parameters $\alpha, \beta, \gamma, \delta, \kappa$ given by some families of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes such as, for example, extended 1-perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes ([7,22] or Example 2), the upper bound above is tight. We also know $\mathbb{Z}_2\mathbb{Z}_4$-linear codes such that the rank is in between these two bounds such as, for example, the Hadamard $\mathbb{Z}_4$-linear codes ([23] or Example 1).

Example 1 For any integer $t \geq 3$ and each $\delta \in \{1, \ldots, \lfloor(t+1)/2\rfloor\}$ there exists a unique (up to isomorphism) extended 1-perfect $\mathbb{Z}_4$-linear code C of binary length $n = 2^t$, such that the $\mathbb{Z}_4$-dual code of C is of type $(0, \beta; \gamma, \delta)$, where $\beta = 2^{t-1}$ and $\gamma = t + 1 - 2\delta$ [17]. The Hadamard $\mathbb{Z}_4$-linear codes H are the $\mathbb{Z}_4$-dual of the extended 1-perfect $\mathbb{Z}_4$-linear codes.

The rank of Hadamard $\mathbb{Z}_4$-linear codes was computed in [23] and the rank of extended 1-perfect $\mathbb{Z}_4$-linear codes in [7,17]. Specifically,

$$rank(H) = \begin{cases} \gamma + 2\delta + \binom{\delta-1}{2} & \text{if} \quad \delta \geq 3 \\ \gamma + 2\delta & \text{if} \quad \delta = 1, 2 \end{cases}$$

and $rank(C) = \bar{\gamma} + 2\bar{\delta} + \delta = \beta + \bar{\delta}$ (except when $t = 4$ and $\delta = 1$), where $\bar{\gamma} = \gamma$ and $\bar{\delta} = \beta - \gamma - \delta$ by (4) taking $\alpha = 0 = \kappa$. Note that the rank of the extended 1-perfect $\mathbb{Z}_4$-linear codes satisfies the upper bound.

Example 2 For any integer $t \geq 3$ and each $\delta \in \{0, \ldots, \lfloor t/2 \rfloor\}$ there exists a unique (up to isomorphism) extended 1-perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of binary length $n = 2^t$, such that the

$\mathbb{Z}_2\mathbb{Z}_4$-dual code of C is of type $(\alpha, \beta; \gamma, \delta)$ with $\alpha \neq 0$, where $\alpha = 2^{t-\delta}$, $\beta = 2^{t-1} - 2^{t-\delta-1}$ and $\gamma = t + 1 - 2\delta$ [9]. The Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes H are the $\mathbb{Z}_2\mathbb{Z}_4$-dual of the extended 1-perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

The rank of Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes was computed in [23] and the rank of extended 1-perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes in [7]. Specifically,

$$rank(H) = \begin{cases} \gamma + 2\delta + \binom{\delta}{2} & \text{if } \delta \geq 2 \\ \gamma + 2\delta & \text{if } \delta = 0, 1 \end{cases}$$

and $rank(C) = \bar{\gamma} + 2\bar{\delta} + \delta = \beta + \bar{\delta} + \bar{\gamma}$, where $\bar{\gamma} = \alpha - \gamma$ and $\bar{\delta} = \beta - \delta$ by (4) taking $\gamma = \kappa$. Note that the rank of these two families of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes satisfies the upper bound.

Example 3 Let $\overline{QRM}(r, m)$ be the class of $\mathbb{Z}_4$-linear Reed-Muller codes defined in [3]. These are $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of type $(0, 2^m; 0, \delta; 0)$, where $\delta = \sum_{i=0}^{r} \binom{m}{i}$. An important property is that any $\mathbb{Z}_4$-linear Kerdock-like code of binary length 4^m is in the class $\overline{QRM}(1, 2m-1)$ and any extended $\mathbb{Z}_4$-linear Preparata-like code of binary length 4^m is in the class $\overline{QRM}(2m - 3, 2m - 1)$.

The rank of any code $C \in \overline{QRM}(r, m)$ is

$$rank(C) = \sum_{i=0}^{r} \binom{m}{i} + \sum_{i=0}^{t} \binom{m}{i},$$

where $t = \min(2r, m)$, [3]. Hence, if $2r \geq m$, then $rank(C) = \delta + \beta$, i.e., the maximum possible. A $\mathbb{Z}_4$-linear Kerdock-like code K of binary length $4^m \geq 16$ has $rank(P) = 2m^2 + m + 1$ and an extended $\mathbb{Z}_4$-linear Preparata-like code P of binary length $4^m \geq 64$ has $rank(P) = 2^{2m} - 2m$ [8], attaining the upper bound of Proposition 1.

The next point to be solved is how to construct $\mathbb{Z}_2\mathbb{Z}_4$-linear codes with any rank in the range of possibilities given by Proposition 1.

Lemma 4 *There exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ if and only if*

$$\begin{aligned} \alpha, \beta, \gamma, \delta, \kappa \geq 0, \quad \alpha + \beta > 0, \\ 0 < \delta + \gamma \leq \beta + \kappa \quad \text{and} \quad \kappa \leq \min(\alpha, \gamma). \end{aligned} \tag{5}$$

Proof Straightforward from Theorem 1. □

Theorem 3 *Let $\alpha, \beta, \gamma, \delta, \kappa$ be integers satisfying (5). Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $rank(C) = r$ if and only if*

$$r \in \left\{ \gamma + 2\delta, \ldots, \min\left(\beta + \delta + \kappa, \gamma + 2\delta + \binom{\delta}{2}\right) \right\}.$$

Proof Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix

$$\mathcal{G} = \begin{pmatrix} I_\kappa & T' & 0 & 0 & 0 \\ 0 & 0 & 2T_1 & 2I_{\gamma-\kappa} & 0 \\ 0 & S' & S_r & 0 & I_\delta \end{pmatrix},$$

where S_r is a matrix over $\mathbb{Z}_4$ of size $\delta \times (\beta - (\gamma - \kappa) - \delta)$, and let $C = \Phi(\mathcal{C})$ be its corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. Let $\{u_i\}_{i=1}^{\gamma}$ and $\{v_j\}_{j=1}^{\delta}$ be the row vectors of order two and four in $\mathcal{G}$, respectively.

The necessary conditions for the values of r are given in Proposition 1, that is, $rank(C) = r = \gamma + 2\delta + \bar{r}$, where $\bar{r} \in \left\{ 0, \dots, \min\left(\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2} \right) \right\}$. In the generator matrix $\mathcal{G}$, the Gray map image of the γ row vectors $\{u_i\}_{i=1}^{\gamma}$ and the 2δ row vectors $\{v_j\}_{j=1}^{\delta}, \{2v_j\}_{j=1}^{\delta}$ are independent binary vectors over $\mathbb{Z}_2$. For each $\bar{r} \in \left\{ 0, \dots, \min\left(\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2} \right) \right\}$, we will define S_r in an appropriate way such that $rank(C) = r = \gamma + 2\delta + \bar{r}$.

Let e_k, $1 \le k \le \delta$, denote the column vector of length δ, with a one in the kth coordinate and zeroes elsewhere. For each $\bar{r} \in \left\{ 0, \dots, \min\left(\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2} \right) \right\}$, we can construct S_r as a quaternary matrix where in $\bar{r}$ columns there are $\bar{r}$ different column vectors $e_k + e_l$ of length δ, $1 \le k < l \le \delta$, and in the remaining columns there is the all-zero column vector. For each one of the $\bar{r}$ column vectors the rank increases by 1. In fact, if the column vector $e_k + e_l$ is included in S_r, then the quaternary vector $2v_k * v_l$ has only a two in the same coordinate where the column vector $e_k + e_l$ is and $\Phi(2v_k * v_l)$ is independent to the vectors $\{\Phi(u_i)\}_{i=1}^{\gamma}, \{\Phi(v_j)\}_{j=1}^{\delta}, \{\Phi(2v_j)\}_{j=1}^{\delta}$ and $\{\Phi(2v_s * v_t)\}$, $\{s, t\} \ne \{k, l\}$. Since the maximum number of columns of S_r is $\beta - (\gamma - \kappa) - \delta$ and the maximum number of different such columns is $\binom{\delta}{2}$, the result follows. $\qquad\square$

Note that in the proof of the previous theorem the matrices T', T and S' can be chosen arbitrarily.

Example 4 By Theorem 3, we know that the possible ranks for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of type $(\alpha, 9; 2, 5; 1)$ are $r \in \{12, 13, 14, 15\}$. For each possible r, we can construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C with $rank(C) = r$, taking the following generator matrix of $\mathcal{C} = \Phi^{-1}(C)$:

$$
\mathcal{G}_S = \begin{pmatrix} 1 & T' & \mathbf{0} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & 2T_1 & 2 & \mathbf{0} \\ \mathbf{0} & S' & S_r & \mathbf{0} & I_5 \end{pmatrix},
$$

where $S_{12} = (\mathbf{0})$ and S_{13}, S_{14}, and S_{15} are constructed as follows:

$$
S_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{14} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{15} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

4 Kernel dimension of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

In this section, we will study the dimension of the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes $C = \Phi(\mathcal{C})$. We will also show that there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $k = ker(C)$ for any possible value of k.

Lemma 5 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. Then,*

$$
K(C) = \{\Phi(u) \mid u \in \mathcal{C} \text{ and } 2u * v \in \mathcal{C}, \forall v \in \mathcal{C}\}.
$$

Proof By Lemma 2, $\Phi(u) + \Phi(v) \in C$ if and only if $2u * v \in \mathcal{C}$ for all $u, v \in \mathcal{C}$. Thus, the result follows. $\qquad\square$

Note that if $\mathcal{G}$ is a generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ and $C = \Phi(\mathcal{C})$, $\Phi(u) \in K(C)$ if and only if $u \in \mathcal{C}$ and $2u * v \in \mathcal{C}$ for all row vector v in $\mathcal{G}$. Moreover, all codewords of order two in $\mathcal{C}$ belong to $K(C)$.

Lemma 6 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. Given $x, y \in \mathcal{C}$, $\Phi(x) + \Phi(y) \in K(C)$ if and only if $\Phi(x + y) \in K(C)$.*

Proof By Lemma 1, $\Phi(x+y+2x*y) = \Phi(x) + \Phi(y)$. Now, by Lemma 5, $\Phi(x+y+2x*y) \in K(C)$ if and only if for all $v \in \mathcal{C}$, $2(x + y + 2x * y) * v = 2(x + y) * v \in \mathcal{C}$; that is, if and only if $\Phi(x + y) \in K(C)$. □

Lemma 7 *Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of binary length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $ker(C) \in \{\gamma + \delta, \gamma + \delta + 1, \ldots, \gamma + 2\delta - 2, \gamma + 2\delta\}$.*

Proof The upper bound $\gamma + 2\delta$ comes from the linear case. The lower bound $\gamma + \delta$ is straightforward, since there are $2^{\gamma+\delta}$ codewords of order two in $\mathcal{C} = \Phi^{-1}(C)$ and, by Lemma 5, the binary images by Φ of all these codewords are in $K(C)$. Also note that if the $\mathbb{Z}_2\mathbb{Z}_4$-linear code C is not linear, then the dimension of the kernel is equal to or less than $\gamma + 2\delta - 2$ [21]. Therefore, $ker(C) \in \{\gamma + \delta, \ldots, \gamma + 2\delta - 2, \gamma + 2\delta\}$. □

Given an integer $m > 0$, a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and a subset $I = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, m\}$, we denote by v_I the vector $v_{i_1} + \cdots + v_{i_l}$. If $I = \emptyset$, then $v_I = \mathbf{0}$. Note that given $I, J \subseteq \{1, \ldots, m\}$, $v_I + v_J = v_{I \triangle J} + 2v_{I \cap J}$, where $I \triangle J = (I \cup J) - (I \cap J)$.

Proposition 2 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, with generator matrix $\mathcal{G}$, and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code with $ker(C) = \gamma + 2\delta - \bar{k}$, where $\bar{k} \in \{2, \ldots, \delta\}$. Then, there exists a set $\{v_1, v_2, \ldots, v_{\bar{k}}\}$ of row vectors of order four in $\mathcal{G}$, such that*

$$C = \bigcup_{I \subseteq \{1, \ldots, \bar{k}\}} (K(C) + \Phi(v_I))$$

Proof We know that C can be written as the union of cosets of $K(C)$ [1]. Since $|K(C)| = 2^{\gamma+2\delta-\bar{k}}$ and $|C| = 2^{\gamma+2\delta}$, there are exactly $2^{\bar{k}}$ cosets.

Let $u_1, \ldots, u_\gamma, v_1, \ldots, v_\delta$ be the γ and δ row vectors in $\mathcal{G}$ of order two and four, respectively. By Lemma 5, the binary images by Φ of all codewords of order two are in $K(C)$. There are $2^{\gamma+\delta}$ codewords of order two generated by $\gamma + \delta$ codewords. Moreover, there are $\delta - \bar{k}$ codewords w_i of order four such that $\Phi(w_i) \in K(C)$ for all $i \in \{1, \ldots, \delta - \bar{k}\}$, and $\Phi(u_1), \ldots, \Phi(u_\gamma)$, $\Phi(2v_1), \ldots, \Phi(2v_\delta)$, $\Phi(w_1), \ldots, \Phi(w_{\delta-\bar{k}})$ are linearly independent vectors over $\mathbb{Z}_2$. The code $\mathcal{C}$ can also be generated by $u_1, \ldots, u_\gamma, w_1, \ldots, w_{\delta-\bar{k}}, v_{i_1}, \ldots, v_{i_{\bar{k}}}$, where $\{i_1, i_2, \ldots, i_{\bar{k}}\} \subseteq \{1, \ldots, \delta\}$. We can assume that $v_{i_1}, \ldots, v_{i_{\bar{k}}}$ are the $\bar{k}$ row vectors $v_1, \ldots, v_{\bar{k}}$ in $\mathcal{G}$. Note that $\Phi(v_I) \notin K(C)$, for any $I \subseteq \{1, \ldots, \bar{k}\}$ such that $I \neq \emptyset$. In fact, if $\Phi(v_I) \in K(C)$, then the set of vectors $\Phi(u_1), \ldots, \Phi(u_\gamma), \Phi(2v_1), \ldots, \Phi(2v_\delta)$, $\Phi(w_1), \ldots, \Phi(w_{\delta-\bar{k}}), \Phi(v_I)$ would be linearly independent.

Finally, we show that the $2^{\bar{k}} - 1$ binary vectors $\Phi(v_I)$, $I \subseteq \{1, \ldots, \bar{k}\}$ and $I \neq \emptyset$, are in different cosets. Let $\Phi(v_I)$ and $\Phi(v_J)$ be any two of these binary vectors such that $I \neq J$. If $\Phi(v_I) \in K(C) + \Phi(v_J)$, then $\Phi(v_I) + \Phi(v_J) \in K(C)$ and, by Lemma 6, $\Phi(v_I + v_J) \in K(C)$. We also have that $v_I + v_J = v_{I \triangle J} + 2v_{I \cap J}$. Hence, $\Phi(v_{I \triangle J} + 2v_{I \cap J}) = \Phi(v_{I \triangle J}) + \Phi(2v_{I \cap J}) \in K(C)$ and $\Phi(v_{I \triangle J}) \in K(C)$, which is a contradiction, since $I \triangle J \subseteq \{1, \ldots, \bar{k}\}$ and $I \triangle J \neq \emptyset$. □

It is important to note that if C is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code, then $K(C)$ is a $\mathbb{Z}_2\mathbb{Z}_4$-linear subcode of C, by Lemma 6. The *kernel* of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ of type $(\alpha, \beta; \gamma, \delta; \kappa)$, denoted by $\mathcal{K}(\mathcal{C})$, can be defined as $\mathcal{K}(\mathcal{C}) = \Phi^{-1}(K(C))$, where $C = \Phi(\mathcal{C})$ is the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. By Lemma 5, $\mathcal{K}(\mathcal{C}) = \{u \in \mathcal{C} \mid 2u * v \in \mathcal{C}, \forall v \in \mathcal{C}\}$ and it is easy to see that $\mathcal{K}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive subcode of $\mathcal{C}$ of type $(\alpha, \beta; \gamma + \bar{k}, \delta - \bar{k}; \kappa)$. The following theorem follows.

Theorem 4 *Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $ker(C) = \gamma + 2\delta - \bar{k}$, where $\bar{k} \in \{0\} \cup \{2, \ldots, \delta\}$. Then, $K(C)$ is a $\mathbb{Z}_2\mathbb{Z}_4$-linear subcode of C of type $(\alpha, \beta; \gamma + \bar{k}, \delta - \bar{k}; \kappa)$.*

Note that replacing ones with twos in the first α coordinates, we can see $\mathbb{Z}_2\mathbb{Z}_4$-additive codes as quaternary linear codes. Let χ be the map from $\mathbb{Z}_2$ to $\mathbb{Z}_4$, which is the usual inclusion from the additive structure in $\mathbb{Z}_2$ to $\mathbb{Z}_4$: $\chi(0) = 0$, $\chi(1) = 2$. This map can be extended to the map $(\chi, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \to \mathbb{Z}_4^{\alpha+\beta}$, which will also be denoted by χ. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix $\mathcal{G}$, then $\chi(\mathcal{C})$ is a quaternary linear code of length $\alpha + \beta$ and type $2^\gamma 4^\delta$ with generator matrix $\mathcal{G}_{\chi(\mathcal{C})} = \chi(\mathcal{G})$. Note that $\mathcal{K}(\mathcal{C}) = \chi^{-1}\mathcal{K}(\chi(\mathcal{C}))$ and $\mathcal{K}(\chi(\mathcal{C}))^\perp$ is the quaternary linear code generated by the matrix

$$\begin{pmatrix} \mathcal{H}_{\chi(\mathcal{C})} \\ 2\mathcal{G}_{\chi(\mathcal{C})} * \mathcal{H}_{\chi(\mathcal{C})} \end{pmatrix},$$

where $\mathcal{H}_{\chi(\mathcal{C})}$ is the generator matrix of the quaternary dual code of $\chi(\mathcal{C})$ and $2\mathcal{G}_{\chi(\mathcal{C})} * \mathcal{H}_{\chi(\mathcal{C})}$ is the matrix obtained computing the component-wise product $2u * v$ for all $u \in \mathcal{G}_{\chi(\mathcal{C})}$, $v \in \mathcal{H}_{\chi(\mathcal{C})}$.

Moreover, by Proposition 2, given a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ with generator matrix $\mathcal{G}$, there exists a set $\{v_1, v_2, \ldots, v_{\bar{k}}\}$ of row vectors of order four in $\mathcal{G}$, such that

$$\mathcal{C} = \bigcup_{I \subseteq \{1, \ldots, \bar{k}\}} (\mathcal{K}(\mathcal{C}) + v_I).$$

Lemma 8 *Let A be a symmetric matrix over $\mathbb{Z}_2$ of odd order and with zeroes in the main diagonal. Then, $\det(A) = 0$.*

Proof Let n be the order of the matrix A. The map $f : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ defined by $f(u, v) = uAv^t$ is an alternating bilinear form and A is a symplectic matrix [19, 435]. It is known that the rank r of a symplectic matrix is always even [19, 436]. Therefore, since the order n of A is an odd number, $r < n$ and $\det(A) = 0$.   □

Proposition 3 *Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of binary length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $ker(C) = \gamma + 2\delta - \bar{k}$, where*

$$\begin{cases} \bar{k} = 0, & if \quad s = 0, \\ \bar{k} \in \{0\} \cup \{2, \ldots, \delta\} \quad and \quad \bar{k} \quad even, & if \quad s = 1, \\ \bar{k} \in \{0\} \cup \{2, \ldots, \delta\}, & if \quad s \geq 2, \end{cases}$$

and $s = \beta - (\gamma - \kappa) - \delta$.

Proof For $s = 0$, by Proposition 1 we have that $rank(C) = \gamma + 2\delta$, so C is a binary linear code and $ker(C) = \gamma + 2\delta$. For $s \geq 2$, by Lemma 7 we have that $ker(C) \in \{\gamma + \delta, \ldots, \gamma + 2\delta - 2, \gamma + 2\delta\}$.

Now, we will prove the result for $s = 1$. By Theorem 1, $\mathcal{C}$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by

$$\mathcal{G}_S = \begin{pmatrix} I_\kappa & T' & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S' & S & R & I_\delta \end{pmatrix},$$

where S is a matrix over $\mathbb{Z}_4$ of size $\delta \times 1$. Let $\{u_i\}_{i=1}^\gamma$ and $\{v_j\}_{j=1}^\delta$ be the row vectors in $\mathcal{G}_S$ of order two and four, respectively.

If $\delta < 3$, then it is easy to see that $ker(C) = \gamma + 2\delta - 2$ or $ker(C) = \gamma + 2\delta$, by Lemma 7. If $\delta \geq 3$ we will show that, given four vectors $v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}$ such that $2v_{j_1} * v_{j_2} \notin \mathcal{C}$ and $2v_{j_3} * v_{j_4} \notin \mathcal{C}$, then $2v_{j_1} * v_{j_2} + 2v_{j_3} * v_{j_4} \in \mathcal{C}$. Let e_k, $1 \leq k \leq \gamma - \kappa$, denote the row vector of length $\gamma - \kappa$, with a one in the kth coordinate and zeroes elsewhere. Then, we can write $2v_{j_1} * v_{j_2} = (\mathbf{0}, \mathbf{0}, 2c, 2e_I, \mathbf{0})$, where $c \in \{0, 1\}$ and $I \subseteq \{\alpha + 2, \ldots, \alpha + \gamma - \kappa + 1\}$, and $2v_{j_3} * v_{j_4} = (\mathbf{0}, \mathbf{0}, 2c', 2e_J, \mathbf{0})$, where $c' \in \{0, 1\}$ and $J \subseteq \{\alpha + 2, \ldots, \alpha + \gamma - \kappa + 1\}$. We denote by u_I (resp. u_J) the row vector obtained by adding the row vectors of order two in $\mathcal{G}_S$ with 2 in the coordinate positions given by I (resp. J). Then, $u_I = (\mathbf{0}, \mathbf{0}, 2d, 2e_I, \mathbf{0}) \in \mathcal{C}$ with $d \in \{0, 1\}$ (resp. $u_J = (\mathbf{0}, \mathbf{0}, 2d', 2e_J, \mathbf{0}) \in \mathcal{C}$ with $d' \in \{0, 1\}$). Since $2v_{j_1} * v_{j_2} \notin \mathcal{C}$ (resp. $2v_{j_3} * v_{j_4} \notin \mathcal{C}$) we have $2v_{j_1} * v_{j_2} = u_I + (\mathbf{0}, \mathbf{0}, 2, \mathbf{0}, \mathbf{0})$ (resp. $2v_{j_3} * v_{j_4} = u_J + (\mathbf{0}, \mathbf{0}, 2, \mathbf{0}, \mathbf{0})$). Therefore, $2v_{j_1} * v_{j_2} + 2v_{j_3} * v_{j_4} = u_I + u_J \in \mathcal{C}$

By Proposition 2, there exist $\bar{k}$ row vectors $v_1, v_2, \ldots, v_{\bar{k}}$ in $\mathcal{G}_S$, such that $\Phi(v_I) \notin K(C)$ for any nonempty subset $I \subseteq \{1, \ldots, \bar{k}\}$ and $ker(C) = \gamma + 2\delta - \bar{k}$. Assume $\bar{k}$ is odd. We will show that there exists a subset $I \subseteq \{1, \ldots, \bar{k}\}$ such that $\Phi(v_I) \in K(C)$. Since this is a contradiction, $\bar{k}$ can not be an odd number and the assertion will be proved.

By Lemma 5, in order to prove that $\Phi(v_I) \in K(C)$, it is enough to prove that $2v_I * v_j \in \mathcal{C}$ for all $j \in \{1, \ldots, \bar{k}\}$. That is, following the above remark, for each $j \in \{1, \ldots, \bar{k}\}$ the number of $i \in I$ such that $2v_i * v_j \notin \mathcal{C}$ is even. We define a symmetric matrix $A = (a_{ij})$, $1 \leq i, j \leq \bar{k}$, in the following way: $a_{ij} = 1$ if $2v_i * v_j \notin \mathcal{C}$ and 0 otherwise. Therefore, A is a symmetric matrix of odd order and with zeroes in the main diagonal. Lemma 8 shows that $\det(A) = 0$ and hence there exists a linear combination of some rows, $i_1, \ldots, i_l$, of A equal to $\mathbf{0}$. The vector $\Phi(v_I)$, where $I = \{i_1, \ldots, i_l\}$, belongs to $K(C)$. This completes the proof. $\qquad\square$

Example 5 Continuing with Example 1, the dimension of the kernel for a Hadamard $\mathbb{Z}_4$-linear code H was computed in [23,17] and the dimension of the kernel for an extended 1-perfect $\mathbb{Z}_4$-linear code C in [7]. Specifically,

$$ker(H) = \begin{cases} \gamma + \delta + 1 & \text{if } \delta \geq 3 \\ \gamma + 2\delta & \text{if } \delta = 1, 2 \end{cases}$$

and

$$ker(C) = \begin{cases} \bar{\gamma} + \bar{\delta} + 1 & \text{if } \delta \geq 3 \\ \bar{\gamma} + \bar{\delta} + 2 & \text{if } \delta = 2 \\ \bar{\gamma} + \bar{\delta} + t & \text{if } \delta = 1. \end{cases}$$

Example 6 Continuing with Example 2, the dimension of the kernel for a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear code H was computed in [23] and the dimension of the kernel for an extended 1-perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C in [7]. Specifically,

$$ker(H) = \begin{cases} \gamma + \delta & \text{if } \delta \geq 2 \\ \gamma + 2\delta & \text{if } \delta = 0, 1 \end{cases}$$

and

$$ker(C) = \begin{cases} \bar{\gamma} + \bar{\delta} + 1 & \text{if } \delta \geq 1 \\ \bar{\gamma} + 2\bar{\delta} & \text{if } \delta = 0. \end{cases}$$

Note that the kernel dimension of the Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes satisfies the lower bound.

Example 7 Let $\overline{QRM}(r, m)$ be the class of $\mathbb{Z}_4$-linear Reed-Muller codes defined in [3], as in Example 3. The dimension of the kernel of any code $C \in \overline{QRM}(r, m)$ is

$$ker(C) = \sum_{i=0}^{r} \binom{m}{i} + 1 = \delta + 1,$$

except for $r = m$ (in this case, $C = \mathbb{Z}_2^{2^{m+1}}$), [3].

Therefore, $\mathbb{Z}_4$-linear Kerdock-like codes and extended $\mathbb{Z}_4$-linear Preparata-like codes of binary length 4^m have dimension of the kernel $ker(K) = 2m + 1$ and $ker(P) = 2^{2m-1} - 2m + 1$, respectively [3,8].

As in Sect. 3 for the rank, the next point to be solved here is how to construct $\mathbb{Z}_2\mathbb{Z}_4$-linear codes with any dimension of the kernel in the range of possibilities given by Proposition 3.

Theorem 5 *Let $\alpha, \beta, \gamma, \delta, \kappa$ be integers satisfying (5). Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $ker(C) = \gamma + 2\delta - \bar{k}$ if and only if*

$$\begin{cases} \bar{k} = 0, & \text{if } s = 0, \\ \bar{k} \in \{0\} \cup \{2, \dots, \delta\} \text{ and } \bar{k} \text{ even}, & \text{if } s = 1, \\ \bar{k} \in \{0\} \cup \{2, \dots, \delta\}, & \text{if } s \geq 2, \end{cases}$$

and $s = \beta - (\gamma - \kappa) - \delta$.

Proof Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix

$$\mathcal{G} = \begin{pmatrix} I_\kappa & T' & 0 & 0 & 0 \\ 0 & 0 & 0 & 2I_{\gamma-\kappa} & 0 \\ 0 & S' & S_k & 0 & I_\delta \end{pmatrix},$$

where S_k is a matrix over $\mathbb{Z}_4$ of size $\delta \times s$, and $\{u_i\}_{i=1}^{\gamma}$ and $\{v_j\}_{j=1}^{\delta}$ are the row vectors in $\mathcal{G}$ of order two and four, respectively. Let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code with $ker(C) = k = \gamma + 2\delta - \bar{k}$. The necessary conditions for the values of k are given in Proposition 3.

When $s = 0$, the code C is a binary linear code, so $ker(C) = k = \gamma + 2\delta$ and $\bar{k} = 0$. If $s > 0$, then taking S_k the all-zero matrix over $\mathbb{Z}_4$, the code C is also a binary linear code and $\bar{k} = 0$.

When $s = 1$, for each $\bar{k} \in \{2, \dots, \delta\}$ and even, we can construct a matrix S_k over $\mathbb{Z}_4$ of size $\delta \times 1$ with an even number of ones, $\bar{k}$, and zeroes elsewhere. We can assume that $v_1, \dots, v_{\bar{k}}$ are the row vectors of order four with 1 in the column from the matrix S_k, and $v_{\bar{k}+1}, \dots, v_\delta$ the ones with 0 in this column. By Lemma 5, it is easy to see that $\Phi(v_j) \in K(C)$ for all $j \in \{\bar{k} + 1, \dots, \delta\}$. Since the binary images by Φ of all codewords of order two are also in $K(C)$, $ker(C) \geq \gamma + \delta + \delta - \bar{k} = \gamma + 2\delta - \bar{k}$. Let $K'(C)$ be the binary linear subcode of $K(C)$ generated by these $\gamma + 2\delta - \bar{k}$ linearly independent vectors. By the same arguments as in Proposition 2, $C = \bigcup_{I \subseteq \{1, \dots, \bar{k}\}} (K'(C) + \Phi(v_I))$. In order to prove that $ker(C) = \gamma + 2\delta - \bar{k}$, it is enough to show that a codeword from each coset does not belong

to $K(C)$, that is, $\Phi(v_I) \notin K(C)$ for any nonempty subset $I \subseteq \{1, \ldots, \bar{k}\}$. If $|I|$ is even, $2v_I * v_j \notin C$ for any $j \in I$. If $|I|$ is odd, $2v_I * v_j \notin C$ for any $j \in \{1, \ldots, \bar{k}\}\backslash I$. Note that $\{1, \ldots, \bar{k}\}\backslash I$ is a nonempty set, since $\bar{k} \geq 2$ and even. Therefore, $\Phi(v_I) \notin K(C)$ by Lemma 5.

Finally, when $s \geq 2$, for each $\bar{k} \in \{2, 3, \ldots, \delta\}$, we can construct a matrix S_k over $\mathbb{Z}_4$ of size $\delta \times s$, such that only in the last $\delta - \bar{k}$ row vectors all components are zero and, moreover, in the first $\bar{k}$ coordinates of each column vector there are an even number of ones and zeros elsewhere. In this case, by the same arguments as before, it is easy to prove that $ker(C) = \gamma + 2\delta - \bar{k}$. $\square$

Example 8 By Theorem 5, we know that the possible dimensions of the kernel for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of type $(\alpha, 9; 2, 5; 1)$ are $k \in \{12, 10, 9, 8, 7\}$. For each possible k, we can construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C with $ker(C) = k$, taking the following generator matrix of $\mathcal{C} = \Phi^{-1}(C)$:

$$
\mathcal{G}_S = \begin{pmatrix} 1 & T' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \mathbf{0} & S' & S_k & \mathbf{0} & I_5 \end{pmatrix},
$$

where $S_{12} = (\mathbf{0})$ and S_{10}, S_9, S_8 and S_7 are constructed as follows:

$$
S_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; S_9 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; S_8 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; S_7 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
$$

5 Pairs of rank and kernel dimension of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

In this section, once the dimension of the kernel is fixed, lower and upper bounds on the rank are established. We will show that there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $r = rank(C)$ and $k = ker(C)$ for any possible pair of values (r, k).

Lemma 9 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. If $rank(C) = \gamma + 2\delta + \bar{r}$ and $ker(C) = \gamma + 2\delta - \bar{k}$, with $\bar{k} \geq 2$, then*

$$
1 \leq \bar{r} \leq \binom{\bar{k}}{2}.
$$

Moreover, $\langle C \rangle$ is generated by

$$
\{\Phi(u_i)\}_{i=1}^{\gamma}, \; \{\Phi(v_j), \Phi(2v_j)\}_{j=1}^{\delta} \; and \; \{\Phi(2v_t * v_s)\}_{1 \leq s < t \leq \bar{k}}.
$$

Proof There exist $\{u_i\}_{i=1}^{\gamma}$ and $\{v_j\}_{j=1}^{\delta}$ vectors of order two and four respectively, such that they generate the code $\mathcal{C}$ and $C = \bigcup_{I \subseteq \{1, \ldots, \bar{k}\}}(K(C) + \Phi(v_I))$ by Proposition 2. Note that $\Phi(v_j) \in K(C)$ if and only if $j \in \{\bar{k}+1, \ldots, \delta\}$.

By Lemma 5, for all $j \in \{\bar{k}+1, \ldots, \delta\}$ and $i \in \{1, \ldots, \delta\}$, as $\Phi(v_j) \in K(C)$, $2v_j * v_i \in \mathcal{C}$ and, consequently, $\Phi(2v_j * v_i)$ is a linear combination of $\{\Phi(u_i)\}_{i=1}^{\gamma}$ and $\{\Phi(2v_j)\}_{j=1}^{\delta}$. As a result, $\langle C \rangle$ is generated by $\{\Phi(u_i)\}_{i=1}^{\gamma}$, $\{\Phi(v_j), \Phi(2v_j)\}_{j=1}^{\delta}$ and $\{\Phi(2v_t * v_s)\}_{1 \leq s < t \leq \bar{k}}$ and hence $\bar{r} \leq \binom{\bar{k}}{2}$, by Lemma 3.

Finally, since $\bar{k} \geq 2$, the binary code C is not linear and, therefore, $\bar{r} \geq 1$. $\square$

Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with $ker(C) = \gamma + 2\delta - \bar{k}$ and $rank(C) = \gamma + 2\delta + \bar{r}$. Note that if $\bar{r} = 0$ then, necessarily, $\bar{k} = 0$ (and vice versa) and C is a linear code. The next theorem will determine all possible pairs of rank and dimension of the kernel for nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

Proposition 4 *Let C be a nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear code of binary length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $ker(C) = \gamma + 2\delta - \bar{k}$ and $rank(C) = \gamma + 2\delta + \bar{r}$. Then, for any $\bar{k} \in \{2, \ldots, \delta\}$,*

$$\begin{cases} \bar{r} \in \{2, \ldots, \min(\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2})\}, & \text{if } \bar{k} \text{ is odd,} \\ \bar{r} \in \{1, \ldots, \min(\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2})\}, & \text{if } \bar{k} \text{ is even.} \end{cases}$$

Proof By Proposition 1, $\bar{r} \in \{0, \ldots, \min(\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2})\}$. Moreover, by Lemma 9, for a fixed $\bar{k} \geq 2$, $\bar{r} \leq \binom{\bar{k}}{2}$ and, therefore, if $\bar{k} \in \{2, \ldots, \delta\}$ then $\bar{r} \in \{1, \ldots, \min(\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2})\}$.

In the case $\bar{r} = 1$, C is not linear and, by Lemma 7, $ker(C) = \gamma + 2\delta - \bar{k}$ where $\bar{k} \in \{2, \ldots, \delta\}$. Moreover, by Proposition 2, there exists a generator matrix $\mathcal{G}$ of C with row vectors $\{u_i\}_{i=1}^{\gamma}$ and $\{v_j\}_{j=1}^{\delta}$ of order two and four, respectively, such that the $\bar{k}$ row vectors $v_1, v_2, \ldots, v_{\bar{k}}$ satisfy $C = \bigcup_{I \subseteq \{1, \ldots, \bar{k}\}} (K(C) + \Phi(v_I))$. We will see that if $\bar{r} = 1$, then $\bar{k}$ is necessarily even. Assume $\bar{k}$ is odd. We will prove that there exist $I \subseteq \{1, \ldots, \bar{k}\}$ such that $\Phi(v_I) \in K(C)$, that is, $2v_I * v_j \in C$ for all $j \in \{1, \ldots \bar{k}\}$, which is a contradiction and, therefore, $\bar{k}$ is an even number.

By Lemma 9, $\langle C \rangle$ is generated by $\{\Phi(u_i)\}_{i=1}^{\gamma}$, $\{\Phi(v_j), \Phi(2v_j)\}_{j=1}^{\delta}$ and $\{\Phi(2v_j * v_k)\}_{1 \leq j < k \leq \bar{k}}$. Since $rank(C) = \gamma + 2\delta + 1$, for all $i, j \in \{1, \ldots, \bar{k}\}$ either $2v_i * v_j \in C$ or $2v_i * v_j = w$, for an order two vector $w \notin C$. If there exist $I \subseteq \{1, \ldots, \bar{k}\}$ such that, for each $j \in \{1, \ldots, \bar{k}\}$ the number of $i \in I$ verifying $2v_i * v_j = w \notin C$ is even, then $2v_I * v_j \in C$. In order to prove that there exists such a set I, we define the symmetric matrix $A = (a_{ij})$, $1 \leq i, j \leq \bar{k}$, as in the proof of Proposition 3, and we get the contradiction. $\square$

Theorem 6 *Let $\alpha, \beta, \gamma, \delta, \kappa$ be integers satisfying (5). Then, there exists a nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $ker(C) = \gamma + 2\delta - \bar{k}$ and $rank(C) = \gamma + 2\delta + \bar{r}$ if and only $\bar{k} \in \{2, \ldots, \delta\}$ and*

$$\begin{cases} \bar{r} \in \left\{2, \ldots, \min\left(\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2}\right)\right\}, & \text{if } \bar{k} \text{ is odd,} \\ \bar{r} \in \left\{1, \ldots, \min\left(\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2}\right)\right\}, & \text{if } \bar{k} \text{ is even.} \end{cases}$$

Proof Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix

$$\mathcal{G} = \begin{pmatrix} I_\kappa & T' & 0 & 0 & 0 \\ 0 & 0 & 0 & 2I_{\gamma-\kappa} & 0 \\ 0 & S' & S_{r,k} & 0 & I_\delta \end{pmatrix},$$

where $S_{r,k}$ is a matrix over $\mathbb{Z}_4$ of size $\delta \times (\beta - (\gamma - \kappa) - \delta)$, and let $C = \Phi(\mathcal{C})$ be its corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. The necessary conditions for the values of the pairs $\bar{r}$ and $\bar{k}$ are given in Proposition 4.

Let e_k, $1 \leq k \leq \delta$, denote the column vector of length δ, with a one in the kth coordinate and zeroes elsewhere. For each $\bar{k} \in \{3, \ldots, \delta\}$ and $\bar{r} \in \left\{2, \ldots, \min\left(\beta - (\gamma - \kappa) - \delta, \binom{\bar{k}}{2}\right)\right\}$,

we can construct $S_{r,k}$ as a quaternary matrix where in one column there is the vector $e_1 + \cdots + e_{\bar{k}}$, in $\bar{r} - 1$ columns there are $\bar{r} - 1$ different column vectors $e_k + e_l$ of length δ, $1 \leq k < l \leq \bar{k}$, and in the remaining columns there is the all-zero column vector. It is easy to check that $ker(C) = \gamma + 2\delta - \bar{k}$ and $rank(C) = \gamma + 2\delta + \bar{r}$.

Finally, if $\bar{r} = 1$, we can construct $S_{r,k}$ as a quaternary matrix of size $\delta \times (\beta - (\gamma - \kappa) - \delta)$ with $\bar{k}$ ones in one column and zeroes elsewhere, for each $\bar{k} \in \{2, \ldots, \delta\}$ and even. In this case, it is also easy to check that $rank(C) = \gamma + 2\delta + 1$ and $ker(C) = \gamma + 2\delta - \bar{k}$, for any $\bar{k} \in \{2, \ldots, \delta\}$ and even. $\qquad\square$

Example 9 By Theorem 6, we know that the possible pairs of rank and dimension of the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, C, of type $(\alpha, 9; 2, 5; 1)$ are given in the following table:

$k \setminus r$	12	13	14	15
12	*			
10		*		
9			*	*
8		*	*	*
7			*	*

For each possible pair (r, k), we can construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C with $rank(C) = r$ and $ker(C) = k$, taking the following generator matrix of $\mathcal{C} = \Phi^{-1}(C)$:

$$\mathcal{G}_S = \begin{pmatrix} 1 & T' & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & S' & S_{r,k} & 0 & I_5 \end{pmatrix},$$

where $S_{12,12} = (0)$ and the other possible $S_{r,k}$ are constructed as follows:

$$S_{13,10} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{13,8} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_{14,9} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{14,8} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{14,7} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$S_{15,9} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{15,8} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{15,7} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

6 Conclusion

In this paper we studied two structural properties of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, the rank and dimension of the kernel. Using combinatorial enumeration techniques, we established lower and upper bounds for the possible values of these parameters. We also gave the construction of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with rank r (resp. kernel dimension k) for each feasible value r (resp. k).

Finally, we established the bounds on the rank, once the dimension of the kernel is fixed, and we gave the construction of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with rank r and kernel dimension k for each possible pair (r, k).

The rank, kernel and dimension of the kernel are defined for binary codes and they are specially useful for binary nonlinear codes. We showed that for binary codes which are $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, we can also define the kernel using the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, which are subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. In this case, in order to compute the kernel $K(C)$ of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C is much easier if we consider the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C} = \Phi^{-1}(C)$ and we compute $\mathcal{K}(\mathcal{C}) = \Phi^{-1}(K(C))$ using a generator matrix of $\mathcal{C}$. Moreover, we also proved that if C is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code, then $K(C)$ and $\langle C \rangle$ are also $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. Finally, since $K(C) \subseteq C \subseteq \langle C \rangle$ and C can be written as the union of cosets of $K(C)$, we also have that, equivalently, $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{C} \subseteq \mathcal{S}_\mathcal{C}$, where $\mathcal{S}_\mathcal{C} = \Phi^{-1}(\langle C \rangle)$, and $\mathcal{C}$ can be written as cosets of $\mathcal{K}(\mathcal{C})$.

As a future research in this issue, it would be interesting to establish a characterization of all $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with rank r and dimension of the kernel k, using the canonical generator matrices $\mathcal{G}_S$ of the form (3) and characterizing their submatrices $S_{r,k}$ over $\mathbb{Z}_4$ of size $\delta \times (\beta - \gamma - \delta)$.

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