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$\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality

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Abstract A code $\mathcal{C}$ is $\mathbb{Z}_2\mathbb{Z}_4$-additive if the set of coordinates can be partitioned into two subsets X and Y such that the punctured code of $\mathcal{C}$ by deleting the coordinates outside X (respectively, Y) is a binary linear code (respectively, a quaternary linear code). In this paper $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are studied. Their corresponding binary images, via the Gray map, are $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, which seem to be a very distinguished class of binary group codes. As for binary and quaternary linear codes, for these codes the fundamental parameters are found and standard forms for generator and parity-check matrices are given. In order to do this, the appropriate concept of duality for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes is defined and the parameters of their dual codes are computed.

Keywords Binary linear codes · Duality · Quaternary linear codes · $\mathbb{Z}_2\mathbb{Z}_4$-additive codes · $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

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1 Introduction

Let $\mathbb{Z}_2$ and $\mathbb{Z}_4$ be the ring of integers modulo 2 and 4, respectively. Let $\mathbb{Z}_2^n$ denote the set of all binary vectors of length n and let $\mathbb{Z}_4^n$ be the set of all n-tuples over the ring $\mathbb{Z}_4$. In this paper, the elements of $\mathbb{Z}_4^n$ will also be called quaternary vectors of length n. Any non-empty subset C of $\mathbb{Z}_2^n$ is a binary code and a subgroup of $\mathbb{Z}_2^n$ is called a *binary linear code* or a $\mathbb{Z}_2$*-linear code*. Equivalently, any non-empty subset $\mathcal{C}$ of $\mathbb{Z}_4^n$ is a quaternary code and a subgroup of $\mathbb{Z}_4^n$ is called a *quaternary linear code*.

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Quaternary codes can be viewed as binary codes under the Gray map defined as $\phi(0) = (0,0)$, $\phi(1) = (0,1)$, $\phi(2) = (1,1)$, $\phi(3) = (1,0)$. If $\mathcal{C}$ is a quaternary linear code, then the binary code $C = \phi(\mathcal{C})$ (coordinatewise extended) is said to be a $\mathbb{Z}_4$-*linear* code. The notion of quaternary dual code of a quaternary linear code $\mathcal{C}$, denoted by $\mathcal{C}^{\perp}$, as well as the notion of binary $\mathbb{Z}_4$-dual code, denoted by $C_{\perp} = \phi(\mathcal{C}^{\perp})$, are defined in the standard way in Sect. 4 following [12] and [18].

Since 1994, quaternary linear codes have became significant due to its relationship to some classical well-known binary codes as the Nordstrom–Robinson, Kerdock, Preparata, Goethals or Reed–Muller codes [12]. It was proved that the Kerdock code and some Preparata-like code are $\mathbb{Z}_4$-linear codes and, moreover, the $\mathbb{Z}_4$-dual code of the Kerdock code is a Preparata-like code. Lately, more families of quaternary linear codes, related to the Reed–Muller codes, have been studied in [2,3,21].

Additive codes were first defined by Delsarte in 1973 in terms of association schemes [10,11]. In general, an additive code, in a translation association scheme, is defined as a subgroup of the underlying abelian group. On the other hand, translation invariant propelinear codes were first defined in 1997 [20], where it is proved that all these binary codes are group-isomorphic to subgroups of $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta} \times \mathbb{Q}_8^{\sigma}$, being $\mathbb{Q}_8$ the non-abelian quaternion group on eight elements. In the special case when the association scheme is the binary Hamming scheme, that is, when the underlying abelian group is of order 2^n, the additive codes coincides with the abelian translation invariant propelinear codes. Hence, as it is pointed out in [11], the only structures for the abelian group are those of the form $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$, with $\alpha + 2\beta = n$. Therefore, the subgroups $\mathcal{C}$ of $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ are the only additive codes in the binary Hamming scheme. In order to distinguish them from additive codes over finite fields [1], from now on, we will call them $\mathbb{Z}_2\mathbb{Z}_4$-*additive codes*.

The binary image of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code under the extended Gray map defined in Sect. 2 is called $\mathbb{Z}_2\mathbb{Z}_4$-*linear code*. There are $\mathbb{Z}_2\mathbb{Z}_4$-linear codes in several important classes of binary codes. For example, $\mathbb{Z}_2\mathbb{Z}_4$-linear perfect single error-correcting codes (or 1-perfect codes) are found in [20] and fully characterized in [8]. Also, in subsequent papers [7,15,19,23], $\mathbb{Z}_2\mathbb{Z}_4$-linear extended 1-perfect and Hadamard codes are studied and classified.

As we have seen, the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes belong to the more general family of additive codes. However, note that one could think of other families of codes with an algebraic structure that also include the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes; such as mixed group codes [6,13,17] and translation invariant propelinear codes [20,22].

Most of the concepts on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes described in this paper have been implemented by the authors as a new package [5] in MAGMA [9]. A beta version of this new package for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and the manual with the description of all functions can be downloaded from the web page http://www.ccg.uab.cat (for any comment or further information about this package, you can send an e-mail to support-ccg@deic.uab.cat).

The aim of this paper is a general study of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. It is organized as follows. In Sect. 2, we give the definition of $\mathbb{Z}_2\mathbb{Z}_4$-additive and $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, we find the fundamental parameters and we discuss about the automorphism groups of these codes. In Sect. 3, we deduce a standard form for generator matrices of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. Section 4 is devoted to the duality concept for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes defining the appropriate inner product. In Sect. 5, we show how the generator and parity-check matrices are related and we also compute the parameters of the dual code. Finally, in Sect. 6, we give some conclusions and discuss about further research.

2 Definitions

From now on, we will focus on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}$, which are subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. We will take an extension of the usual Gray map: $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$, where $n = \alpha + 2\beta$, given by

$$\Phi(x, y) = (x, \phi(y_1), \ldots, \phi(y_\beta)) \quad \forall x \in \mathbb{Z}_2^\alpha,\ \forall y = (y_1, \ldots, y_\beta) \in \mathbb{Z}_4^\beta;$$

where $\phi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2^2$ is the usual Gray map, that is,

$$\phi(0) = (0, 0),\ \phi(1) = (0, 1),\ \phi(2) = (1, 1),\ \phi(3) = (1, 0).$$

This Gray map is an isometry which transforms Lee distances defined in a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ to Hamming distances defined in the binary code $C = \Phi(\mathcal{C})$. Note that the length of the binary code C is $n = \alpha + 2\beta$.

Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, it is also isomorphic to an abelian structure like $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Therefore, $\mathcal{C}$ is of type $2^\gamma 4^\delta$ as a group, it has $|\mathcal{C}| = 2^{\gamma+2\delta}$ codewords and the number of order two codewords in $\mathcal{C}$ is $2^{\gamma+\delta}$.

Let X (respectively Y) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$) coordinate positions, so $|X| = \alpha$ and $|Y| = \beta$. Unless otherwise stated, the set X corresponds to the first α coordinates and Y corresponds to the last β coordinates. Call $\mathcal{C}_X$ (respectively $\mathcal{C}_Y$) the punctured code of $\mathcal{C}$ by deleting the coordinates outside X (respectively Y). Let $\mathcal{C}_b$ be the subcode of $\mathcal{C}$ which contains all order two codewords and let κ be the dimension of $(\mathcal{C}_b)_X$, which is a binary linear code. For the case $\alpha = 0$, we will write $\kappa = 0$.

Considering all these parameters, we will say that $\mathcal{C}$ (or equivalently $C = \Phi(\mathcal{C})$) is of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Notice that $\mathcal{C}_Y$ is a quaternary linear code of type $(0, \beta; \gamma_Y, \delta; 0)$, where $0 \leq \gamma_Y \leq \gamma$, and $\mathcal{C}_X$ is a binary linear code of type $(\alpha, 0; \gamma_X, 0; \gamma_X)$, where $\kappa \leq \gamma_X \leq \kappa + \delta$.

Definition 1 Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, that is, a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. We say that the binary image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-*linear code* of length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$, where γ, δ and κ are defined as above.

Note that $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are a generalization of binary linear codes and $\mathbb{Z}_4$-linear codes. When $\beta = 0$, the binary code $C = \mathcal{C}$ corresponds to a binary linear code. On the other hand, when $\alpha = 0$, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is a quaternary linear code and its corresponding binary code $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_4$-linear code.

Two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ both of type $(\alpha, \beta; \gamma, \delta; \kappa)$ are said to be *monomially equivalent*, if one can be obtained from the other by permutating the coordinates and (if necessary) changing the signs of certain $\mathbb{Z}_4$ coordinates. Two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are said to be *permutation equivalent* if they differ only by a permutation of coordinates. The *monomial automorphism group* of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$, denoted by $MAut(\mathcal{C})$, is the group generated by all permutations and sign-changes of the $\mathbb{Z}_4$ coordinates that preserves the set of codewords of $\mathcal{C}$, while the *permutation automorphism group* of $\mathcal{C}$, denoted by $PAut(\mathcal{C})$, is the group generated by all permutations that preserves the set of codewords of $\mathcal{C}$ [14].

If two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are monomially equivalent, then, after the Gray map, the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes $C_1 = \Phi(\mathcal{C}_1)$ and $C_2 = \Phi(\mathcal{C}_2)$ are isomorphic as binary codes. Note that the inverse statement is not always true.

3 Generator matrices of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Although $\mathcal{C}$ is not a free module, every codeword is uniquely expressible in the form

$$\sum_{i=1}^{\gamma} \lambda_i u^{(i)} + \sum_{j=\gamma+1}^{\gamma+\delta} \mu_j v^{(j)},$$

where $\lambda_i \in \mathbb{Z}_2$ for $1 \leq i \leq \gamma$, $\mu_j \in \mathbb{Z}_4$ for $\gamma + 1 \leq j \leq \gamma + \delta$ and $u^{(i)}$, $v^{(j)}$ are vectors in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ of order two and order four, respectively. Vectors $u^{(i)}$, $v^{(j)}$ give us a generator matrix $\mathcal{G}$ of size $(\gamma + \delta) \times (\alpha + \beta)$ for the code $\mathcal{C}$. Moreover, we can write $\mathcal{G}$ as

$$\mathcal{G} = \left(\begin{array}{c|c} B_1 & 2B_3 \\ \hline B_2 & Q \end{array}\right), \tag{1}$$

where B_1, B_2 are matrices over $\mathbb{Z}_2$ of size $\gamma \times \alpha$ and $\delta \times \alpha$, respectively; B_3 is a matrix over $\mathbb{Z}_4$ of size $\gamma \times \beta$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and Q is a matrix over $\mathbb{Z}_4$ of size $\delta \times \beta$ with quaternary row vectors of order four.

Let I_k be the identity matrix of size $k \times k$. In [12], it was shown that any quaternary linear code of type $(0, \beta; \gamma, \delta; 0)$ is permutation equivalent to a quaternary linear code with a generator matrix of the form

$$\mathcal{G}_S = \left(\begin{array}{|ccc} 2T & 2I_\gamma & \mathbf{0} \\ \hline S & R & I_\delta \end{array}\right), \tag{2}$$

where R, T are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$, of size $\delta \times \gamma$ and $\gamma \times (\beta - \gamma - \delta)$, respectively; and S is a matrix over $\mathbb{Z}_4$ of size $\delta \times (\beta - \gamma - \delta)$. In this section, we will generalize this result for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, that is, we will give a canonical generator matrix for these codes [4].

First, note that replacing ones with twos in the coordinates over $\mathbb{Z}_2$, we can see the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes as quaternary linear codes. Let χ be the map from $\mathbb{Z}_2$ to $\mathbb{Z}_4$, which is the usual inclusion from the additive structure in $\mathbb{Z}_2$ to $\mathbb{Z}_4$: $\chi(0) = 0$, $\chi(1) = 2$. This map can be extended to the map $(\chi, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta$, which will also be denoted by χ.

Theorem 1 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathcal{C}$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with canonical generator matrix of the form*

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_q & R & I_\delta \end{array}\right), \tag{3}$$

where T_b, S_b are matrices over $\mathbb{Z}_2$; T_1, T_2, R are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and S_q is a matrix over $\mathbb{Z}_4$.

Proof Since κ is the dimension of the matrix B_1 over $\mathbb{Z}_2$ given in (1), the code $\mathcal{C}$ is permutation equivalent to a code with a generator matrix of the form

$$\left(\begin{array}{cc|c} I_\kappa & \bar{B}_1 & 2\bar{B}_3 \\ \mathbf{0} & \mathbf{0} & 2\bar{B}_4 \\ \hline \mathbf{0} & \bar{B}_2 & \bar{Q} \end{array}\right),$$

where $\bar{B}_1$, $\bar{B}_2$ are matrices over $\mathbb{Z}_2$ of size $\kappa \times (\alpha - \kappa)$ and $\delta \times (\alpha - \kappa)$, respectively; $\bar{B}_3$, $\bar{B}_4$ are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$ of size $\kappa \times \beta$ and $(\gamma - \kappa) \times \beta$, respectively; and $\bar{Q}$ is a matrix over $\mathbb{Z}_4$ of size $\delta \times \beta$.

The quaternary linear code $\mathcal{C}^-$ of type $(0, \alpha - \kappa + \beta; \gamma - \kappa, \delta; 0)$ generated by the matrix

$$\left(\begin{array}{|cc} \mathbf{0} & 2\bar{B}_4 \\ \hline 2\bar{B}_2 & \bar{Q} \end{array}\right)$$

is permutation equivalent to a quaternary linear code with generator matrix of the form

$$\mathcal{G}^- = \left(\begin{array}{|cccc} \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline 2S_b & S_q & R & I_\delta \end{array}\right),$$

where the permutation of coordinates fixes the first $\alpha - \kappa$ coordinates, [12], (2). So, the quaternary linear code $\chi(\mathcal{C})$ generated by the matrix

$$\left(\begin{array}{|ccc} 2I_\kappa & 2\bar{B}_1 & 2\bar{B}_3 \\ \mathbf{0} & \mathbf{0} & 2\bar{B}_4 \\ \hline \mathbf{0} & 2\bar{B}_2 & \bar{Q} \end{array}\right)$$

is permutation equivalent to a quaternary linear code with generator matrix of the form

$$\mathcal{G}_\chi = \left(\begin{array}{|ccccc} 2I_\kappa & 2T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & 2S_b & S_q & R & I_\delta \end{array}\right).$$

Finally, $\mathcal{C}$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix $\chi^{-1}(\mathcal{G}_\chi) = \mathcal{G}_S$. □

Example 1 Let $\mathcal{C}_1$ denote the $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(1, 3; 1, 2; 1)$ with generator matrix

$$\mathcal{G} = \left(\begin{array}{c|ccc} 1 & 2 & 2 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 3 \end{array}\right).$$

The code $\mathcal{C}_1$ can also be generated by the matrix

$$\left(\begin{array}{c|ccc} 1 & 2 & 2 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 3 \end{array}\right).$$

The quaternary linear code $\mathcal{C}^-$ generated by $\left(\begin{array}{|ccc} 1 & 1 & 0 \\ 1 & 0 & 3 \end{array}\right)$ is permutation equivalent (indeed, equal) to a quaternary linear code with generator matrix $\mathcal{G}^- = \left(\begin{array}{|ccc} 1 & 1 & 0 \\ 3 & 0 & 1 \end{array}\right)$. So, the quaternary linear code $\chi(\mathcal{C})$ generated by

$$\left(\begin{array}{|cccc} 2 & 2 & 2 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 3 \end{array}\right)$$

is permutation equivalent to a quaternary linear code with generator matrix

$$\mathcal{G}_\chi = \left(\begin{array}{|cccc} 2 & 2 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right).$$

Therefore, the code $\mathcal{C}_1$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with canonical generator matrix

$$\mathcal{G}_S = \chi^{-1}(\mathcal{G}_\chi) = \left(\begin{array}{c|ccc} 1 & 2 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right).$$

Example 2 Let $\mathcal{C}_2$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type (3, 4; 3, 1; 3) with generator matrix

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

By Theorem 1, $\mathcal{C}_2$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with canonical generator matrix

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

4 Duality of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

For linear codes over finite fields or finite rings, there exists the well-known concept of duality. In this section, we will study the duality for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, taking advantage of their abelian group structure.

It is well-known that any finite abelian group G is isomorphic to a direct sum of cyclic groups, each one of order a prime power [16]. Say $G = \langle a_1 \rangle \oplus \cdots \oplus \langle a_k \rangle$, where a_i is of order a prime power $p_i^{\alpha_i}$, for any $i \in \{1, \ldots, k\}$. The set of elements $a_i \in G$ are called a basis of G and fully determine the algebraic structure of G.

Given a basis of G, every element $u \in G$ can be represented by the k-tuple of integers, $u = (u_1, u_2, \ldots, u_k)$ with $u = \sum_{i=1}^{k} u_i a_i$. This expression is unique in the sense that any other expression like $u = \sum_{i=1}^{k} u'_i a_i$ means that, for all indices $i \in \{1, \ldots, k\}$, $u_i \equiv u'_i \pmod{p_i^{\alpha_i}}$. Note that the exponent m of the group G can be computed as $m = lcm\{p_i^{\alpha_i} \mid i = 1, \ldots, k\}$ and is divisible by any $p_i^{\alpha_i}$. Therefore, taking $m = s_i p_i^{\alpha_i}$ we obtain s_i, which has order $p_i^{\alpha_i}$ in $\mathbb{Z}_m$. Given a basis of G and fixed elements $\bar{s}_i$ of order $p_i^{\alpha_i}$ in $\mathbb{Z}_m$ (e.g., the above s_i), the inner product of elements $u = (u_1, u_2, \ldots, u_k)$, $v = (v_1, v_2, \ldots, v_k) \in G$ is defined as the equivalence class of $\sum_{i=1}^{k} u_i v_i \bar{s}_i$ in $\mathbb{Z}_m$ and denoted by $\langle u, v \rangle$. Hence,

$$\langle u, v \rangle = \sum_{i=1}^{k} u_i v_i \bar{s}_i \in \mathbb{Z}_m. \tag{4}$$

Now, consider the specific case of the finite abelian group $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ of exponent $m = 4$, whose elements are vectors of $\alpha + \beta$ coordinates (the first α over $\mathbb{Z}_2$ and the last β over $\mathbb{Z}_4$). Take as generators $a_i = 1 \in \mathbb{Z}_2$, for $1 \le i \le \alpha$, and $a_i \in \{1, 3\} \in \mathbb{Z}_4$, for $\alpha + 1 \le i \le \alpha + \beta$. Also take the values $\bar{s}_i = 2$, for $1 \le i \le \alpha$, which is the only possible value of order two in $\mathbb{Z}_4$, and $\bar{s}_i = 1 \in \{1, 3\} \subset \mathbb{Z}_4$, for $\alpha + 1 \le i \le \alpha + \beta$. The inner product given by (4) will be called *standard inner product* and can be written as

$$\langle u, v\rangle = 2\left(\sum_{i=1}^{\alpha} u_i v_i\right) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \ \in \ \mathbb{Z}_4,$$

where $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and the computations are made taking the zeros and ones in the α binary coordinates as quaternary zeros and ones, respectively.

Note that although $\bar{s_i}$ is uniquely defined for $1 \le i \le \alpha$, the value of $\bar{s_i}$, for $\alpha + 1 \le i \le \alpha + \beta$, can be chosen from $\{1, 3\}$ and so, we can produce several different presentations for the inner product. Also note that all these different presentations of the inner product can be reduced to the standard one, as long as in the computation of $\langle u, v\rangle$ we take the representation of vector u using the given generators a_i and the representation of vector v using the generators $a_i' = a_i \in \mathbb{Z}_2$, for $1 \le i \le \alpha$, and $a_i' = \bar{s_i} a_i \in \mathbb{Z}_4$, for $\alpha + 1 \le i \le \alpha + \beta$.

We can also write the standard inner product as

$$\langle u, v\rangle = u \cdot J_n \cdot v^t,$$

where $J_n = \left(\begin{array}{c|c} 2I_\alpha & \mathbf{0} \\ \hline \mathbf{0} & I_\beta \end{array}\right)$ is a diagonal matrix over $\mathbb{Z}_4$. Note that when $\alpha = 0$ the inner product is the usual one for $\mathbb{Z}_4$-vectors (i.e., vectors over $\mathbb{Z}_4$) and when $\beta = 0$ it is twice the usual one for $\mathbb{Z}_2$-vectors.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. The *additive orthogonal code* of $\mathcal{C}$, denoted by $\mathcal{C}^\perp$, is defined in the standard way

$$\mathcal{C}^\perp = \{v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \mid \langle u, v\rangle = 0 \text{ for all } u \in \mathcal{C}\}.$$

We will also call $\mathcal{C}^\perp$ the *additive dual code* of $\mathcal{C}$. The corresponding binary code $\Phi(\mathcal{C}^\perp)$ is denoted by $C_\perp$ and called $\mathbb{Z}_2\mathbb{Z}_4$-*dual code* of C. In the case that $\alpha = 0$, that is, when $\mathcal{C}$ is a quaternary linear code, $\mathcal{C}^\perp$ is also called the *quaternary dual code* of $\mathcal{C}$ and $C_\perp$ the $\mathbb{Z}_4$-*dual code* of C.

The additive dual code $\mathcal{C}^\perp$ is also a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, that is, a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Moreover, as it is pointed out in [11], $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are the additive codes in the binary Hamming association scheme, in the sense of Delsarte [10]. The weight distribution of $\mathcal{C}$ and $\mathcal{C}^\perp$ are related to each other by the MacWilliams identities in the usual sense [11, p. 2501], [20]. Note that the weight distribution of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ refers to the Lee weight, which coincides with the Hamming weight of the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$, after applying the Gray map. The codes C and $C_\perp$ are not necessarily linear, so they are not dual in the binary linear sense, but the weight enumerator polynomial of $C_\perp$ is the MacWilliams transform of the weight enumerator polynomial of C. This remarkable relationship was first established for the specific case of $\mathbb{Z}_4$-linear codes in [12], where it is pointed out that the Kerdock code is the additive dual of some Preparata like code.

Lemma 1 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and $\mathcal{C}^\perp$ its additive dual code. Then, $|\mathcal{C}||\mathcal{C}^\perp| = 2^n$, where $n = \alpha + 2\beta$.*

Proof Let $W_\mathcal{C}(x, y)$ be the weight enumerator polynomial of $\mathcal{C}$. That is,

$$W_\mathcal{C}(x, y) = \sum_{c \in \mathcal{C}} x^{n - wt(c)} y^{wt(c)},$$

where $wt(c)$ stands for the Lee weight of codeword $c \in \mathcal{C}$ or, equivalently, the Hamming weight of $\Phi(c)$.

From MacWilliams Identity,

$$W_{\mathcal{C}^\perp}(x,y) = \frac{1}{|\mathcal{C}|} W_{\mathcal{C}}(x+y, x-y).$$

Taking $x = y$ we obtain,

$$|\mathcal{C}^\perp| x^n = \frac{1}{|\mathcal{C}|}(2x)^{n-wt(\mathbf{0})}$$

and hence $|\mathcal{C}^\perp||\mathcal{C}| = 2^n$. □

Finally, note again that one could think on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes (or $\mathbb{Z}_2\mathbb{Z}_4$-linear codes) only as quaternary linear codes (or $\mathbb{Z}_4$-linear codes), under the χ map; that is, replacing ones with twos in the coordinates over $\mathbb{Z}_2$. However, if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and $\mathcal{C}^\perp$ is its additive dual code, considering the standard inner product defined in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, then $\chi(\mathcal{C}^\perp)$ and $\chi(\mathcal{C})$ are not necessarily quaternary dual codes. Take, for example, $\alpha = \beta = 1$ and the vectors $v = (1,3)$ and $w = (1,2)$. It is easy to check that $\langle v, w\rangle = 0$, so v and w are orthogonal. If we replace the ones with twos in the coordinates over $\mathbb{Z}_2$ of these vectors we get $v' = (2,3)$ and $w' = (2,2)$, which are not orthogonal in the quaternary sense.

5 Parity-check matrices of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

In this section, first we will prove two different methods to construct the additive dual code of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and we will compute the type of this additive dual code. Then, we will apply one of these two methods to show how to construct a parity-check matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, or equivalently a generator matrix of its additive dual code, when the $\mathbb{Z}_2\mathbb{Z}_4$-additive code is generated by a canonical generator matrix as in (3).

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. The code $\mathcal{C}$ is generated by γ vectors of order two and δ vectors of order four, which can be written as the row vectors of a generator matrix $\mathcal{G}$. The codewords of $\mathcal{C}^\perp$, the additive dual code of $\mathcal{C}$, are the vectors in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ which are orthogonal to $\mathcal{C}$. By linearity, we can take $\mathcal{G}$ as a parity check matrix for $\mathcal{C}^\perp$. Analogously, a generator matrix of $\mathcal{C}^\perp$ can be seen as a parity check matrix for $\mathcal{C}$.

Example 3 The code $\mathcal{C}_1$ (or the corresponding $C_1 = \Phi(\mathcal{C}_1)$) in Example 1 is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code (or a $\mathbb{Z}_2\mathbb{Z}_4$-linear code) of type $(1,3;1,2;1)$ with generator matrix

$$\mathcal{G}_1 = \left(\begin{array}{c|ccc} 1 & 2 & 2 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 3 \end{array}\right).$$

The generator matrix $\mathcal{G}_1$ for $\mathcal{C}_1$ can be also viewed as a parity-check matrix for its additive dual code $\mathcal{C}_1^\perp$. Notice also that $|\mathcal{C}_1| = |C_1| = 2 \times 4^2 = 32$, so by Lemma 1, $|\mathcal{C}_1^\perp| = 2^7/32 = 4$.

In order to construct the additive dual code of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, we will need the following maps: ξ from $\mathbb{Z}_4$ to $\mathbb{Z}_2$ which is the usual modulo two map, that is $\xi(0) = 0$, $\xi(1) = 1$, $\xi(2) = 0$, $\xi(3) = 1$; and the identity map ι from $\mathbb{Z}_2$ to $\mathbb{Z}_4$, that is $\iota(0) = 0$, $\iota(1) = 1$. These maps can be extended to the maps $(\xi, Id) : \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and $(\iota, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta$, which will also be denoted by ξ and ι, respectively. Recall also the map χ from $\mathbb{Z}_2$ to $\mathbb{Z}_4$ which is the normal inclusion from the additive structure in $\mathbb{Z}_2$ to $\mathbb{Z}_4$, that is $\chi(0) = 0$, $\chi(1) = 2$; and its extension $(\chi, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta$, denoted also by χ. We denote by $\langle \cdot, \cdot \rangle_4$ the standard inner product for quaternary vectors.

Lemma 2 *If* $u \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, $v \in \mathbb{Z}_4^{\alpha+\beta}$, *then* $\langle \chi(u), v\rangle_4 = \langle u, \xi(v)\rangle$.

Proof $\langle \chi(u), v\rangle_4 = \sum_{i=1}^{\alpha}(2u_i)v_i + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j = \sum_{i=1}^{\alpha}(2u_i)(v_i \bmod 2) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j = \langle u, \xi(v)\rangle$. □

Corollary 1 *If* $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, *then* $\langle \chi(u), \iota(v)\rangle_4 = \langle u, v\rangle$.

Proof By Lemma 2, $\langle \chi(u), \iota(v)\rangle_4 = \langle u, \xi(\iota(v))\rangle = \langle u, v\rangle$. □

Proposition 1 *Let* $\mathcal{C}$ *be a* $\mathbb{Z}_2\mathbb{Z}_4$*-additive code of type* $(\alpha, \beta; \gamma, \delta; \kappa)$. *Then,*

$$\mathcal{C}^\perp = \xi(\chi(\mathcal{C})^\perp).$$

Proof We know that if $v \in \mathcal{C}^\perp$, then $\langle u, v\rangle = 0$, for all $u \in \mathcal{C}$. By Corollary 1, $\langle u, v\rangle = \langle \chi(u), \iota(v)\rangle_4 = 0$. Therefore, $\xi(\iota(v)) = v \in \xi(\chi(\mathcal{C})^\perp)$ and $\mathcal{C}^\perp \subseteq \xi(\chi(\mathcal{C})^\perp)$. On the other hand, if $v \in \chi(\mathcal{C})^\perp$, then $\langle \chi(u), v\rangle_4 = 0$, for all $u \in \mathcal{C}$. By Lemma 2, $\langle \chi(u), v\rangle_4 = \langle u, \xi(v)\rangle = 0$. Thus, $\xi(\chi(\mathcal{C})^\perp) \subseteq \mathcal{C}^\perp$ and we obtain the equality. □

Proposition 2 *Let* $\mathcal{C}$ *be a* $\mathbb{Z}_2\mathbb{Z}_4$*-additive code of type* $(\alpha, \beta; \gamma, \delta; \kappa)$. *Then,*

$$\mathcal{C}^\perp = \chi^{-1}(\xi^{-1}(\mathcal{C})^\perp).$$

Proof Let $\mathcal{G}$ be a generator matrix of the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ written as in (1). Then, the quaternary linear code $\xi^{-1}(\mathcal{C})$ has a generator matrix of the form

$$\begin{pmatrix} 2I_\alpha & \mathbf{0} \\ B_1 & 2B_3 \\ B_2 & Q \end{pmatrix}. \tag{5}$$

We will show that $v \in \mathcal{C}^\perp$ if and only if $\chi(v) \in \xi^{-1}(\mathcal{C})^\perp$. In fact, for each row vector f in the matrix $(2I_\alpha\ \mathbf{0})$, we have $\langle \chi(v), f\rangle_4 = \sum_{i=1}^{\alpha} f_i 2v_i = 0$ because there is only one index i such that $f_i = 2$. Moreover, by Corollary 1, $0 = \langle v, u\rangle = \langle \chi(v), \iota(u)\rangle_4$, for all $u \in \mathcal{C}$. □

The following question we will settle is the computation of the type of the additive dual code of a given $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$. First, we will remember this well-known result for quaternary linear codes, that is for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes with $\alpha = 0$. Then, we will generalize it for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, not necessarily quaternary linear codes.

Lemma 3 [12] *If* $\mathcal{C}$ *is a quaternary linear code of type* $(0, \beta; \gamma, \delta; 0)$, *then the quaternary dual code* $\mathcal{C}^\perp$ *is of type* $(0, \beta; \gamma, \beta - \gamma - \delta; 0)$.

Theorem 2 *Let* $\mathcal{C}$ *be a* $\mathbb{Z}_2\mathbb{Z}_4$*-additive code of type* $(\alpha, \beta; \gamma, \delta; \kappa)$. *The additive dual code* $\mathcal{C}^\perp$ *is then of type* $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$, *where*

$$\begin{aligned} \bar{\gamma} &= \alpha + \gamma - 2\kappa, \\ \bar{\delta} &= \beta - \gamma - \delta + \kappa, \\ \bar{\kappa} &= \alpha - \kappa. \end{aligned}$$

Proof Let $\mathcal{G}$ be a generator matrix of the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ written as in (1). Then, the matrix (5) is a generator matrix for the quaternary linear code $\xi^{-1}(\mathcal{C})$, which is of type $(0, \alpha + \beta; \gamma', \delta'; 0)$, where $\gamma' = \alpha + \gamma - 2\kappa$ and $\delta' = \delta + \kappa$. The value of δ' comes from the fact that the κ independent binary vectors of $(\mathcal{C}_b)_X$ are in B_1 and, so, the number of independent quaternary vectors of order four becomes $\delta + \kappa$. The value of γ' comes from the fact that the cardinality of the quaternary linear code $\xi^{-1}(\mathcal{C})$ is $2^{\gamma'+2\delta'} = 2^{\gamma+2\delta+\alpha}$.

By Lemma 3, the quaternary dual code $\xi^{-1}(\mathcal{C})^{\perp}$ is of type $(0, \alpha+\beta; \bar{\gamma}, \bar{\delta}; 0)$, where $\bar{\gamma}=\gamma'$ and $\bar{\delta}=\alpha+\beta-\gamma'-\delta'=\alpha+\beta-(\gamma+\alpha-2\kappa)-(\delta+\kappa)=\beta-\gamma-\delta+\kappa$.

Note that the $\bar{\delta}$ independent vectors in $\xi^{-1}(\mathcal{C})^{\perp}$, restricted to the first α coordinates, are vectors of order two, because in $\xi^{-1}(\mathcal{C})$ there are the row vectors of the matrix $(2I_{\alpha}\ \mathbf{0})$. Finally, applying χ^{-1} we obtain the additive dual code of $\mathcal{C}$. For this additive dual code $\mathcal{C}^{\perp}$, the value of $\bar{\kappa}$ can be easily computed from the fact that, again, the additive dual coincides with $\mathcal{C}$. □

There are two different methods to obtain the additive dual code $\mathcal{C}^{\perp}$, one given by Proposition 1 and another one by Proposition 2. Using any of these two methods, we can construct a generator matrix of $\mathcal{C}^{\perp}$, or equivalently a parity-check matrix of $\mathcal{C}$, starting from a generator matrix of $\mathcal{C}$. In Example 4, we consider the canonical generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and apply these two methods to obtain a generator matrix of its additive dual code. Note that the process to obtain this matrix is different using both methods but, in this case, the generator matrices obtained coincide.

Theorem 3 shows how to construct the parity-check matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by a canonical generator matrix as in (3). This result is proved using the method given by Proposition 1. Notice also that we can apply any of the two methods to any generator matrix, not necessary a canonical generator matrix, to get a parity-check matrix.

Lemma 4 [12] *If $\mathcal{C}$ is a quaternary linear code of type* $(0, \beta; \gamma, \delta; 0)$ *with canonical generator matrix* (2)*, then the generator matrix of $\mathcal{C}^{\perp}$ is*

$$\mathcal{H}_{\mathcal{S}}=\left(\begin{array}{|ccc}\mathbf{0} & 2I_{\gamma} & 2R^t\\ \hline I_{\beta-\gamma-\delta} & T^t & -(S+RT)^t\end{array}\right), \qquad (6)$$

where R, T are matrices over $\mathbb{Z}_4$ with all entries in $\{0,1\}\subset\mathbb{Z}_4$ of size $\delta\times\gamma$ and $\gamma\times(\beta-\gamma-\delta)$, respectively; and S is a matrix over $\mathbb{Z}_4$ of size $\delta\times(\beta-\gamma-\delta)$.

Theorem 3 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type* $(\alpha, \beta; \gamma, \delta; \kappa)$ *with canonical generator matrix (3). Then, the generator matrix of $\mathcal{C}^{\perp}$ is*

$$\mathcal{H}_{\mathcal{S}}=\left(\begin{array}{cc|ccc}T_b^t & I_{\alpha-\kappa} & \mathbf{0} & \mathbf{0} & 2S_b^t\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & 2R^t\\ \hline T_2^t & \mathbf{0} & I_{\beta+\kappa-\gamma-\delta} & T_1^t & -\left(S_q+RT_1\right)^t\end{array}\right), \qquad (7)$$

where T_b, T_2 are matrices over $\mathbb{Z}_2$; T_1, R, S_b are matrices over $\mathbb{Z}_4$ with all entries in $\{0,1\}\subset\mathbb{Z}_4$; and S_q is a matrix over $\mathbb{Z}_4$. Moreover, T_2 and S_b are obtained from the matrices of (3) with the same name after applying ι^{-1} and ξ^{-1}, respectively.

Proof By Lemma 4, if $\bar{\mathcal{C}}$ is a quaternary linear code with generator matrix

$$\bar{\mathcal{G}}=\left(\begin{array}{|ccccc}2T_b & 2T_2 & 2I_{\kappa} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 2T_1 & \mathbf{0} & 2I_{\gamma-\kappa} & \mathbf{0}\\ \hline 2S_b & S_q & \mathbf{0} & R & I_{\delta}\end{array}\right),$$

then the quaternary dual code $\bar{\mathcal{C}}^{\perp}$ has generator matrix

$$\bar{\mathcal{H}}=\left(\begin{array}{|ccccc}\mathbf{0} & \mathbf{0} & 2I_{\kappa} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & 2R^t\\ \hline I_{\alpha-\kappa} & \mathbf{0} & T_b^t & \mathbf{0} & 2S_b^t\\ \mathbf{0} & I_{\beta-\gamma-\delta+\kappa} & T_2^t & T_1^t & -\left(S_q+RT_1\right)^t\end{array}\right).$$

Hence, if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix (3), then the generator matrix of $\chi(\mathcal{C})^\perp$ is

$$\mathcal{H}_\xi = \left(\begin{array}{c|cccc} 2I_\kappa & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & 2R^t \\ \hline T_b^t & I_{\alpha-\kappa} & \mathbf{0} & \mathbf{0} & 2S_b^t \\ T_2^t & \mathbf{0} & I_{\beta-\gamma-\delta+\kappa} & T_1^t & -\left(S_q+RT_1\right)^t \end{array}\right).$$

Finally, by Proposition 1, $\mathcal{H}_\mathcal{S} = \xi(\mathcal{H}_\xi)$ is the generator matrix of $\mathcal{C}^\perp$. □

Note that by Theorems 2 and 3, if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with canonical generator matrix (3), then $\mathcal{C}^\perp$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with canonical generator matrix

$$\left(\begin{array}{cc|ccc} I_{\bar\kappa} & T_b^t & 2S_b^t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2R^t & 2I_{\bar\gamma-\bar\kappa} & \mathbf{0} \\ \hline \mathbf{0} & T_2^t & -\left(S_q+RT_1\right)^t & T_1^t & I_{\bar\delta} \end{array}\right), \tag{8}$$

where T_b, T_2 are matrices over $\mathbb{Z}_2$; T_1, R, S_b are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and S_q is a matrix over $\mathbb{Z}_4$. Moreover, $\bar\gamma = \alpha+\gamma-2\kappa$, $\bar\delta = \beta-\gamma-\delta+\kappa$ and $\bar\kappa = \alpha-\kappa$.

Example 4 Let $\mathcal{C}_{S1}$ denote the $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type (1, 3; 1, 2; 1) with canonical generator matrix

$$\mathcal{G}_S = \left(\begin{array}{c|ccc} 1 & 2 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right).$$

By Theorem 2, the additive dual code $\mathcal{C}_{S1}^\perp$ is of type (1, 3; 0, 1; 0). There are two methods to obtain a parity-check matrix of $\mathcal{C}_{S1}$ from the matrix $\mathcal{G}_S$.

The first one uses Proposition 1. We know that if $\bar{\mathcal{C}}$ is a quaternary linear code with generator matrix $\bar{\mathcal{G}} = \left(\begin{array}{|cccc} 2&2&0&0 \\ \hline 1&0&1&0 \\ 3&0&0&1 \end{array}\right)$, the quaternary dual code $\bar{\mathcal{C}}^\perp$ has generator matrix $\bar{\mathcal{H}} = \left(\begin{array}{|cccc} 0&2&0&0 \\ \hline 1&1&3&1 \end{array}\right)$.

So, the generator matrix of $\chi(\mathcal{C}_{S1})^\perp$ is $\left(\begin{array}{|cccc} 2&0&0&0 \\ \hline 1&1&3&1 \end{array}\right)$ and finally, applying ξ, the generator matrix of $\mathcal{C}_{S1}^\perp = \xi(\chi(\mathcal{C}_{S1})^\perp)$ is

$$\mathcal{H}_\mathcal{S} = \left(1 \middle| 1\ 3\ 1\right).$$

The second method uses Proposition 2. We know that the quaternary linear code $\xi^{-1}(\mathcal{C}_{S1})$ with generator matrix

$$\left(\begin{array}{|cccc} 2&0&0&0 \\ \hline 1&2&0&0 \\ 0&1&1&0 \\ 0&3&0&1 \end{array}\right),$$

or equivalently $\left(\begin{array}{|cccc} \hline 1&2&0&0 \\ 0&1&1&0 \\ 0&3&0&1 \end{array}\right)$, has parity-check matrix $\left(\begin{array}{|cccc} 2&1&3&1 \end{array}\right)$. So, applying χ^{-1}, the generator matrix of $\mathcal{C}_{S1}^\perp = \chi^{-1}(\xi^{-1}(\mathcal{C}_{S1})^\perp)$ is

$$\mathcal{H}_\mathcal{S} = \left(1 \middle| 1\ 3\ 1\right).$$

Example 5 Let $\mathcal{C}_{S2}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type (3, 4; 3, 1; 1) with canonical generator matrix

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

By Theorems 2 and 3, the additive dual code $\mathcal{C}_{S2}^{\perp}$ is of type (3, 4; 0, 3; 0) and has generator matrix

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

6 Conclusions and further research

We have developed a general theory for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes including generator matrices, parity-check matrices and duality. Such class of codes includes classical binary and quaternary linear codes generalizing them. There are some interesting classes of nonlinear binary codes that can be viewed as $\mathbb{Z}_2\mathbb{Z}_4$-linear codes but not as $\mathbb{Z}_4$-linear codes (e.g., some perfect single error-correcting codes). Moreover, $\mathbb{Z}_2\mathbb{Z}_4$-duality shows that $\mathbb{Z}_2\mathbb{Z}_4$-linear codes cannot be considered only as a variant of $\mathbb{Z}_4$-linear codes.

Further research could be done on self-duality. Perhaps enumerator polynomials of additive self-dual codes can be studied and characterized in some way.

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