Des. Codes Cryptogr. (2008) 46:343–352
DOI 10.1007/s10623-007-9163-5

Williamson matrices up to order 59

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Received: 28 August 2007 / Revised: 1 December 2007 / Accepted: 11 December 2007 /
Published online: 5 January 2008
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Abstract A recent result of Schmidt has brought Williamson matrices back into the spotlight. In this article, a new algorithm is introduced to search for hard to find Williamson matrices. We find all nonequivalent Williamson matrices of odd order n up to $n = 59$. It turns out that there are none for $n = 35, 47, 53, 59$ and it seems that the Turyn class may be the only infinite class of these matrices.

Keywords Symmetric circulant matrices · Williamson matrices · Hadamard matrices

AMS Classification 05B20

1 Introduction

Williamson (1944) introduced a class of matrices now known as Williamson matrices [11]. Four symmetric circulant $(-1, 1)$-matrices A, B, C, D of order n satisfying

$$AA^t + BB^t + CC^t + DD^t = 4nI_n,$$

are called *Williamson matrices* of order n. Using such matrices in the array

Communicated by C.J. Colbourn.

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$$W = \begin{pmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{pmatrix}$$

results in a Hadamard matrix of order $4n$.

The method of Williamson has been extended in a number of directions. For instance, one may replace the symmetric circulant matrices A, B, C, D with abelian group matrices and, in order to gain more structure, impose a condition such as $AB^t - BA^t + CD^t - DC^t = 0$. In doing so, one needs to change the array W above appropriately. Such matrices are said to be of *Williamson type*. See [8] for details. In this article, however, we concentrate only on Williamson matrices.

Let $X = \mathrm{circ}(x_0, x_1, \ldots, x_{n-1})$ be a circulant matrix with the first row $x_0, x_1, \ldots, x_{n-1}$. Let σ be an automorphism of the additive group $\mathbb{Z}_n$ and define $\sigma X = \mathrm{circ}(x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(n-1)})$. If A, B, C, D are Williamson matrices of order n, then it is easy to see that $\sigma A, \sigma B, \sigma C, \sigma D$ are also Williamson matrices. Two sets of four Williamson matrices of order n are called *equivalent* if either set can be obtained from the other by applying an automorphism of $\mathbb{Z}_n$ and/or by negating matrices.

Williamson matrices were introduced quite early in the development of the theory of orthogonal matrices in [11]. Baumert and Hall [1] did the first exhaustive search for these matrices up to $n = 23$. The only known infinite class of Williamson matrices are those of order $(q + 1)/2$, where $q \equiv 1 \pmod 4$ is a prime power [2,5,9,10,12]. Also all equivalence classes of Williamson matrices have been determined only for odd orders up to 39, see [3,6]. It seems that the only known order $n > 39$, which is not included in the above infinite class is 43 [1]. Most researchers working in the area had hoped that Williamson matrices of every order must exist. However, it was quite disappointing when it was shown that there was none of order 35 [3]. A recent result of Schmidt [8] that Williamson type Hadamard matrices are Ito group Hadamard matrices has brought Williamson matrices back into the spotlight. Generally speaking the search for Williamson matrices of order n is easier whenever n is a composite number. For such orders there is a specific algorithm which reduces the search to matrices of smaller orders, see [3,7]. This method has been used successfully by Doković and van Vliet for orders 45, 51 [3,7]. There seems to be no published work in which the search for Williamson matrices of orders 45, 51 is claimed to have been exhaustive. Doković [3] writes that "In the case $n = 35$ our computer search did not produce any solutions of Eq. 1. Thus, we claim that there are no Williamson matrices of order $4 \cdot 35$. Although we are confident about the correctness of this claim, an independent verification of it is highly desirable since this is the first odd integer, found so far, with this property".

Baumert and Hall [1] used the classical Williamson method in their search. Both methods of Williamson and Doković are computationally prohibitive for matrices of order larger then 39. In this article we develop a new algorithm which makes it computationally possible to search for all Williamson matrices of odd order up to order 59. Using this algorithm, we find all equivalence classes of Williamson matrices of odd order n, $n \leq 59$. Our computation confirms the previous exhaustive searches for Williamson matrices of orders $n \leq 39$, $n = 45, 51$. For the order $n = 43$, we find a new class. Furthermore, it turns out that there are no Williamson matrices of order 47, 53, or 59. The fact that we can advance our computation to order 59 is an indication of the power of the algorithm. In light of the statement above by Doković and a new result of Schmidt that Williamson type Hadamard matrices are Ito Hadamard groups, we felt that it was essential to conduct this search. Our result is convincing enough that Williamson's method is almost fruitless. This strongly suggests that researchers should look for Williamson type matrices instead.

2 Preliminaries

Let $A_i, 0 \leq i \leq 3$ be Williamson matrices of order n. Then by definition,

$$\sum_{i=0}^{3} A_i A_i^t = 4nI. \tag{1}$$

Consider a common basis consisting of eigenvectors of the commuting symmetric matrices $A_i A_i^t$. Let $\lambda_{i,j} (0 \leq j \leq n-1)$ denote the nonnegative eigenvalues of $A_i A_i^t$ corresponding to this set of eigenvectors. It follows from (1) that

$$\sum_{i=0}^{3} \lambda_{i,j} = 4n, \quad 0 \leq j \leq n-1. \tag{2}$$

Note that the eigenvalues of a circulant matrix $X = \mathrm{circ}(x_0, x_1, \ldots, x_{n-1})$ are calculated by the formula

$$\lambda_j = \sum_{k=0}^{n-1} x_k \omega_j^k, \quad 0 \leq j \leq n-1,$$

where $\{\omega_j\}$ are the nth roots of unity. From this identity, one can easily find the eigenvalues of XX^t in terms of the inner products of rows of X. In fact for odd n, the eigenvalues of XX^t are given by

$$\mu_j = p_0 + \sum_{k=0}^{(n-1)/2} 2p_k \cos\left(\frac{2jk\pi}{n}\right), \quad 0 \leq j \leq n-1,$$

where p_k is the inner product of rows 0 and k of X.

The identities (2) are useful in pruning the search space in our algorithm. Since the eigenvalues are all nonnegative, we must have

$$\sum_{i=0}^{l} \lambda_{i,j} \leq 4n$$

for all $0 \leq j \leq n-1$ and $0 \leq l \leq 3$. But we only need to use these inequalities for $l \leq 1$. Let s_i be the constant row sum of A_i. Then s_i^2 is the eigenvalue of $A_i A_i^t$ corresponding to the eigenvector with every component 1. So it follows from (2) that

$$\sum_{i=0}^{3} s_i^2 = 4n. \tag{3}$$

This identity is specially useful in the pruning process. One of the main features of our algorithm is based on the following theorem of Williamson.

Theorem 1 [11] *Let* $A_i = circ(a_{i,0}, a_{i,1}, \ldots, a_{i,n-1})$ $(0 \leq i \leq 3)$ *be Williamson matrices of odd order n. Then* $a_{0,j}a_{1,j}a_{2,j}a_{3,j} = -a_{0,0}a_{1,0}a_{2,0}a_{3,0}$ *for all* $1 \leq j \leq n-1$.

3 The algorithm

In this section, we present our new search algorithm for Williamson matrices of odd orders. Let n be odd and suppose that we are looking for Williamson matrices $A_i = \mathrm{circ}(a_{i,0}, a_{i,1}, \ldots, a_{i,n-1})$ for $0 \leq i \leq 3$, with the constant row sum s_i, respectively. Without loss of generality, we may assume that $a_{i,0} = 1$ for all i. By (3), we have $\sum_{i=0}^{3} s_i^2 = 4n$, where $s_i \equiv n \pmod 4$. We may also assume that $|s_0| \geq |s_1| \geq |s_2| \geq |s_3|$. This assumption is made to have fewer number of choices for A_0 and A_1. First, we find the set M of all symmetric circulant $(-1, 1)$-matrices with the row sum s_0. We then apply the automorphism group of $\mathbb{Z}_n$ to the set M and select a representative X from each of the orbits and check if all the eigenvalues of XX^t are at most $4n$. This process eliminates some of the unnecessary cases resulting in a set M' of all possible candidates for A_0. For each element A_0 from M' we proceed to find all possible corresponding A_1. We do this by finding the set N of all symmetric circulant $(-1, 1)$-matrices Y with the row sum s_1 in such a way that no eigenvalue of $A_0A_0^t + YY^t$ exceeds $4n$. Now we choose an element A_1 of N and having A_0 and A_1 in hand we then proceed to find A_2 and A_3 simultaneously using Theorem 1 applying the following procedure.

For an exhaustive search, it is necessary to solve the system of equations of (1) which in expanded form becomes

$$\sum_{j=0}^{n-1}(a_{2,j}a_{2,j+k} + a_{3,j}a_{3,j+k}) = -\sum_{j=0}^{n-1}(a_{0,j}a_{0,j+k} + a_{1,j}a_{1,j+k}), \quad 1 \leq k \leq \frac{n-1}{2}, \quad (4)$$

where the subscripts are taken modulo n. Noting that $a_{3,j} = -a_{2,j}a_{1,j}a_{0,j}$ (by Theorem 1) and $a_{2,j} = a_{2,n-j}$ for $1 \leq j \leq n-1$, the actual variables in the system (4) to be found are $a_{2,j}$, $1 \leq j \leq (n-1)/2$. In dealing with the equations we first make the observation that for $1 \leq j, k \leq (n-1)/2$, we have

$$a_{2,j}a_{2,j+k} + a_{2,n-j-k}a_{2,n-j} + a_{3,j}a_{3,j+k} + a_{3,n-j-k}a_{3,n-j} = 2a_{2,j}a_{2,j+k} + 2a_{3,j}a_{3,j+k},$$

and so by Theorem 1 we get

$$2a_{2,j}a_{2,j+k} + 2a_{3,j}a_{3,j+k} = \begin{cases} 0 & \text{if } a_{0,j}a_{1,j}a_{0,j+k}a_{1,j+k} = -1, \\ 4a_{2,j}a_{2,j+k} & \text{otherwise.} \end{cases} \quad (5)$$

Therefore, given A_0 and A_1, by using (5) and $a_{2,j} = a_{2,n-j} (1 \leq j \leq n-1)$, the system (4) becomes a system in variables $a_{2,j} (1 \leq j \leq (n-1)/2)$ and all its equations are divisible by 4. We divide all equations by 4. Using the known fact that $xy = x + y - 1 \pmod 4$ for $x, y \in \{-1, 1\}$, we convert the system to a system of linear equations and proceed to find all solutions modulo 4. To do this we transform the system into the standard reduced echelon form. In practice, for small values of n the standard reduced echelon form allows all $(-1, 1)$-solutions to be found very quickly. In fact, the number of solutions found were between 1 and 2^8 with highest frequency at 1. For example, for $n = 47, 53$ there was always a unique solution. This is an indication that our algorithm works best for prime orders. Next, we check all obtained solutions in the original system (4) to find all possible Williamson matrices. In the final step, the obtained sets of matrices are checked for equivalence and a representative for each equivalence class is recorded. A summary of the algorithm is as follows.

An algorithm for finding all Williamson matrices of odd order n

1. Find all the integer solutions of the equation

$$\sum_{i=0}^{3} s_i^2 = 4n, \quad |s_0| \geq |s_1| \geq |s_2| \geq |s_3| \quad \text{and} \quad s_i \equiv n \pmod 4.$$

Table 1 Number of Williamson matrices of order 1–59

Order:	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29
Number:	1	1	1	2	3	1	4	4	4	6	7	1	10	6	1
Order:	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59
Number:	2	5	0	4	1	1	2	1	0	1	2	0	1	1	0

2. For any solution s_1, s_2, s_3, s_4, of part 1 do the following:
   (a) Set $S = \emptyset$.
   (b) Find the set M of all symmetric circulant $(-1, 1)$-matrices with the row sum s_0 and the first entry in the first row being 1.
   (c) Find a set M' of representatives X of orbits of the action of the automorphism group of $\mathbb{Z}_n$ on M such that none of the eigenvalues of XX^T exceed $4n$.
   (d) For each element A_0 of M' do the following:
      (i) Find the set N of all symmetric circulant $(-1, 1)$-matrices Y with the row sum s_1, the first entry in the first row being 1 and no eigenvalue of $A_0A_0^t + YY^T$ exceeding $4n$.
      (ii) For each element A_1 of N do the following:
         A. Write down the equations of the system $\sum_{i=0}^{3} A_iA_i^t = 4nI$ with the unknowns being only the entries of A_2 (using Theorem 1).
         B. Divide all equations of the system by 4.
         C. Convert the system into a linear system of equations modulo 4.
         D. Transform the system to the standard reduced echelon form and then find all $(-1, 1)$-solutions of the system.
         E. Check the obtained solutions in D for validity in the original system in part A and add them to S, as a set of Williamson matrices A_0, A_1, A_2, A_3.
   (e) Check the elements of S for equivalence and retain a representative from each equivalence class.

4 The results

Using the algorithm described in the previous section, we found all Williamson matrices of odd order $n \leq 59$. Our results confirm those of all previous authors up to the order 39 (see [3,4,6] and the references therein). Our search turned in exactly one solution for orders $n = 41, 45, 49, 55, 57$ all being included in the only known infinite class of Williamson matrices. For $n = 47, 53, 59$, no Williamson matrices were found. For $n = 43$, there are exactly two solutions, one of them previously known [1]. For $n = 51$, there are exactly two solutions, also a previously known fact. These results are summarized in Table 1.

The computations were done twice on two different hardware systems. The first time a cluster of sixteen 2.6 GHz PC was used, while the second time we used a cluster of five dual processor 3.06 GHz PC. The order 57 and 59 computations were done just on the latter cluster. The computational times scaled to a single 1 GHz machine were approximately 12, 24, 100, 600, and 900 days for the orders 51, 53, 55, 57, and 59, respectively.

Christos Koukouvinos maintains an elaborate web page for selected Williamson matrices of small order at http://www.math.ntua.gr/people/ckoukouv/en_index.html. For

Table 2 All Williamson matrices of order 3–25

Order	*A**	*B**	*C**	*D**	Row sums			
3	+	−	−	−	3	−1	−1	−1
5	−−	−−	+−	−+	−3	−3	1	1
7	−−−	+−−	−+−	−−+	−5	−1	−1	−1
7	++−	−++	−++	−−+	3	3	3	−1
9	+−−−	−−+−	−+−−	−−−+	−3	−3	−3	−3
9	−+++	+−−−	−++−	−++−	5	−3	1	1
9	−+++	−−−+	+−+−	−−++	5	−3	1	1
11	−−−−+	−+−++	−−+++	+−−+−	−5	3	3	−1
13	−−−−−+	++−+−−	−++−−+	−+−−++	−7	1	1	1
13	−+++−+	−++−−−	−++−−−	+−−−+−	5	−3	−3	−3
13	−+++−+	+−−+−−	−−−+−+	−−−−++	5	−3	−3	−3
13	++−+−+	−−++++	−+++−−	−++−−+	5	5	1	1
15	−+−−−−+	−+−−−−+	−−++−++	++−−+−−	−5	−5	3	−1
15	−++++−+	++−++−−	−−−++−+	−+−−−++	7	3	−1	−1
15	−+++−++	++−−+−+	−+−−++−	−−−−+++	7	3	−1	−1
15	−++−+++	−++−−++	+−++−−−	−+−−−++	7	3	−1	−1
17	−−−−−++−	+−+−+−−−	−−+++−−−	−++−+−−+	−7	−3	−3	1
17	−−−−−++−	+−+−−+−−	−−−++−+−	−+−−−+++	−7	−3	−3	1
17	−−−−−++−	++−−−−+−	−−−+−+−+	−−+−+++−	−7	−3	−3	1
17	−++−+++−	−−+−++++	−−−+++−−	+−+−−−+−	5	5	−3	−3
19	−−++++−++	+−+−−+−−−	++−−−+−+−	+−+−−−++−	7	−5	−1	−1
19	−−++−++++	−−−+++−−−	−+++−−−+−	+−+−−+−+−	7	−5	−1	−1
19	−−++++−++	++−−−−+−−	−+−−−++−+	−+−−−++−+	7	−5	−1	−1

Table 2 continued

Order	*A**	*B**	*C**	*D**	Row sums			
19	−+++++−−+	++−+++−−−	−−+++−+−+	−++−−+−++	7	3	3	3
19	−+++++−−+	+−++++−−−	−+++−−+−+	−+−−++−++	7	3	3	3
19	−−+−+++++	−+++−−++−	−+++−−+−+	++−+−−−++	7	3	3	3
21	−−−−−+−+−+	−−++−++++−	+−+−−−++−−	−++−+++−−−	−7	5	−3	1
21	−−−+−−−+−+	−−−+++++−+	+−−−+−−+−+	−++++−−−+−	−7	5	−3	1
21	−−−−−+−+−+	+−−+−−++++	+−+−−−++−−	++−−+−+−−+	−7	5	−3	1
21	++++−−++−−	−++++−+−−+	−+−+++++−−	−−+−+−−++−	5	5	5	−3
21	−++++−++−−	−+−++−−+++	−++++−++−−	+−+−−++−−−	5	5	5	−3
21	++++−−−+−+	−+−++−−+++	++−−++−+−+	+−−++−+−−−	5	5	5	−3
21	−++−−+++++	−++−+−−++−	−+++−+−−−+	+−−−−+−+++	9	1	1	1
23	−−−+++++++−	−−++−−−+−+−	−+++++−−−+−	+−+−++−+−−+	7	−5	3	3
25	+−−−−+−−+−−+	−−−−−−+++−−+	−−++++−+−+−−	−+−−−+−++−++	−7	−7	1	1
25	−+−−−+−−+−−+	−−−−−−−++−++	+−−+−+−+++−−	−−+−++++−−−+	−7	−7	1	1
25	−−−−−+−++−−+	−−−−−+−++−−+	−++−+−+++−−−	+−−+−+−−−+++	−7	−7	1	1
25	−−−−−+−++−−+	+++−+−−+++−−	−−+−+++−+++−	−−++++−+−+−−	−7	5	5	1
25	−−−−−+−++−−+	−+−++−++++−−	+−+−++−−−+++	−−−−++−+++−+	−7	5	5	1
25	−−−−−+−+−++−	++−−++−+−++−	+++−−−−+++−+	++−+−++−−−+−	−7	5	5	1
25	++−++++−−+−−	−−++++++−−−+	+−++−+−+++−−	+−+−+−++−++−	5	5	5	5
25	+++−++−+−+−−	−−−+++++−++−	−+++−+−−+++−	−++++−−+−−++	5	5	5	5
25	++−−+++−+−+−	−−+−−++++−++	++−+−+−++−−+	++−−−−++−+++	5	5	5	5
25	−++−−−++++++	+−−+−−++−+−−	−+−−−+−−−+++	−+−−+−++−−++	9	−3	−3	1

Table 3 All Williamson matrices of order 27–57

Order	*A** *C**	*B** *D**	Row sums Row sums
27	−−−−−−++−+−−+	+−−+−−+++−−−−	−9 −5
	−−+−+−+++−−−+	−+−−−+−−+−+++	−1 −1
27	−−−−−−−++−−++	++−+−−−−−+−+−	−9 −5
	−++−−−+−+++−−	−+−−++−−++−+−	−1 −1
27	−+−−−++++++−+	−−+−+−+−−++−−	7 −5
	−−−++−++−−−−+	+−−−+−−+−++++	−5 3
27	++−+−++−+++−−	−−−++−++−++++	7 7
	+−+−−++−−−+++	+−−+−+−−−+−++	3 −1
27	−+−+++++−−−++	−−−++−+−+++++	7 7
	−−−++++−++−+−	+−+−−+−−++−−+	3 −1
27	−−−+++++−+−++	−−−+++++−+−++	7 7
	++−++−+−−−++−	−−+−−+−+++−−+	3 −1
29	−−−+++−++++++−	++++−−++−−−+−+	9 5
	−+−−−−++++−−+−	−+−+−−+−++−++−	−3 1
31	−−−+++−+−++++−+	+++−−−+−+−−−−+−	7 −5
	−++−−−−−+−++−−+	−++−−−−−+−++−−+	−5 −5
31	++−+−−+−−+++−++	−−−++−−+++++−++	7 7
	−+−−−+−−+−−+−++	−+++−−−−+++−+−−	−5 −1
33	−−−−−−−−++−++−−+	−−−+−−−++++−+−+−	−11 −3
	+−−−+−++++−+−−+−	−++−−+−+−−−++++−	1 1
33	−++−−+−+−−−−+−+−	−−+++−+−−−+−−−−+	−7 −7
	+−+−−−++−−++++−+	−−−−−−+++++−+−−+	5 −3
33	−−+++−−−+−++++++	+−+−−−+−−−+−−+−+	9 −7
	+−++−+−++−−−++−−	++−+−−−−+++−+−−+	1 1
33	−−++−++++++−−−++	−+++++−++−++−−−−	9 5
	+−+−+−−+−++++−−+	−−−+++−−++−+−+−+	5 1
33	−−+++−−+−−++++++	+++++−−−−++−+−−+	9 5
	+−+−+−+−−+++−+−+	+−−+−+−−++−+++−−	5 1

Table 3 continued

Order	A^* C^*	B^* D^*	Row sums Row sums
37	−−+++−+−−−−−+−−−−+	−−−++−++−−+−+−++−−	−11 −3
	++++−+−−−−−++−−−−+	−−+−+−+−++−−−+−−++	−3 −3
37	−+++−−−+−−+−−−+−+−	−−−−−−−++−+++−−+−+	−7 −7
	−+−−−+++++−+−−+++−	++−−+−−−+−++−++++−	5 5
37	−+−+−−−+−++−−−+−+−	−−−−−−−++−++++−+−−	−7 −7
	++−−+++−+−−+−−+++−	−++−−−−++−++−−++++	5 5
37	−−−++++−−++−+++−++	+++−−−−++−−+−−−+−−	9 −7
	+−−−+−++−++−+−+−−−	+−−−+−++−++−+−+−−−	−3 −3
39	−−−+−−−+−++−++−−−−+	+−−+++−+−−++−+−−−−−	−9 −5
	−−−+−−++++−−−−+−+−+	−++−−−−−−++−−+−+−++	−5 −5
41	−+−−−+++−++−+−−+++++	−+−−−+++−++−+−−+++++	9 9
	−++++−+−−−+−−+−−+++−	+−−−−+−+++−++−++−−−+	1 1
43	++−+−++−−+−+−++++−+−−	−−−++−−++++−+−+++−++−	7 7
	+−++++++−−−−+−+−−++−+	+−−−++++−+−−+−−++−−−−	7 −5
43	−−+−−+++−+−+++−++++−+	−+−−−−−−++++−+++−−+−−	11 −5
	+−−+−−+−−−+++−+−−−+−+	−−−−+−+−−++−++++−−−++	−5 −1
45	−−++−−+−+++−+−+−−+++++	++−−++−+−−−+−+−++−−−−−	9 −7
	+−−−−+−++++−+++−−++−+−	+−−−−+−++++−+++−−++−+−	5 5
49	−−−−+−+−+++++−++−++−−+++	−−−−+−+−+++++−++−++−−+++	9 9
	+−++−−+−+−−+++−+++−−−−++	−+−−++−+−++−−−+−−−++++−−	5 −3
51	−−−+++−++−+−+++−−+++++−−+	+++−−−+−−+−+−−−++−−−−−++−	11 −9
	−++−++++−+−−−+−+−−−−−++−+	−++−++++−+−−−+−+−−−−−++−+	−1 −1
51	−−−++++−−+−++−++−+−+−++++	−−+−−+++−−−+−+++−+++−+++−	11 7
	−−++−++−−−++++−+−+−−+−−−−	++++−−−−+−−−+++−+−−+−+++−	−5 3
55	−−−+−−++++−+−−+++++−+++−+−+	+++−++−−−−+−++−−−−−+−−−+−+−	11 −9
	−+−−+++−−+++−+−−−−−+++−++−+	−+−−+++−−+++−+−−−−−+++−++−+	3 3
57	−−−++−+−−++++−+++−++−−−+−+++	+−+−+−−−++−++−+−−+++−−−−−+−−	9 −7
	+++−−+−++−−−−+−−−+−−+++−+−−−	+−+−+−−−++−++−+−−+++−−−−−+−−	−7 −7

completeness, we present representatives of all equivalence classes of Williamson matrices of odd orders up to 57 in Tables 2 and 3. In the tables, as is conventional, $+$ and $-$ denote 1 and -1, respectively. Matrices are given in *truncated* form. For $\mathrm{circ}(x_0, x_1, \ldots, x_{2m})$, the truncated form is just $x_1 x_2 \cdots x_m$; throughout $x_0 = 1$.

Acknowledgments We are grateful to NSERC for equipment grants which made our extensive searches possible. H. Kharaghani is supported by an NSERC operating grant and B. Tayfeh-Rezaie is supported by a grant from IPM.

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