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On binary self-dual codes of lengths 60, 62, 64 and 66 having an automorphism of order 9

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Abstract A method for constructing binary self-dual codes having an automorphism of order p^2 for an odd prime p is presented in (S. Bouyuklieva et al. IEEE. Trans. Inform. Theory, 51, 3678–3686, 2005). Using this method, we investigate the optimal self-dual codes of lengths $60 \leq n \leq 66$ having an automorphism of order 9 with six 9-cycles, t cycles of length 3 and f fixed points. We classify all self-dual [60,30,12] and [62,31,12] codes possessing such an automorphism, and we construct many doubly-even [64,32,12] and singly-even [66,33,12] codes. Some of the constructed codes of lengths 62 and 66 are with weight enumerators for which the existence of codes was not known until now.

Keywords Self-dual Codes · Automorphims · Optimal codes

AMS Classification 94B05

1 Introduction

A linear $[n, k]$ code C is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the finite field of q elements. The elements of C are called codewords, and the (Hamming) weight of a codeword is the number of its non-zero coordinates. The minimum weight d of C is the smallest weight among all non-zero codewords of C, and C is called an $[n, k, d]$ code. A matrix whose rows form a basis of C is called a generator matrix of this code. The weight enumerator $W(y)$ of a code C is given by $W(y) = \sum_{i=0}^{n} A_i y^i$ where A_i is the number of codewords of weight i in C. Unless otherwise stated, the inner product we use will be the

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ordinary inner product given by $(u, v) = \sum_{i=1}^{n} u_i v_i$ computed in $\mathbb{F}_q$, where $u, v \in \mathbb{F}_q^n$. The dual code of C is $C^\perp = \{u \in \mathbb{F}_q^n : (u, v) = 0$ for all $v \in C\}$. $C^\perp$ is a linear $[n, n-k]$ code. If $C \subseteq C^\perp$, C is termed self-orthogonal, and if $C = C^\perp$, C is self-dual. If C is self-dual, then $k = \frac{1}{2}n$.

A doubly-even code is a binary code for which all weights are divisible by four. A self-dual code with some codeword of weight not divisible by four is called singly-even. Self-dual doubly-even codes exist if and only if n is a multiple of eight. The codes with the largest minimum weight among all self-dual codes of given length are named optimal self-dual codes.

Two binary codes are equivalent if one can be obtained from the other by a permutation of coordinates. The permutation $\sigma \in S_n$ is an automorphism of the binary code C if $C = \sigma(C)$. The set of all automorphisms of C forms the automorphism group $Aut(C)$ of C.

Two codes over $\mathbb{F}_q$ are (monomially) equivalent if one can be obtained from the other by a coordinate permutation followed by multiplying some (or no) coordinates by a nonzero element of $\mathbb{F}_q$.

Huffman and Yorgov (cf. [13,19,20]) developed a method for constructing binary self-dual codes with an automorphism of odd prime order. Dontcheva, van Zanten and Dodunekov extended the method for automorphisms of odd composite order [6]. A method for constructing binary self-dual codes having an automorphism of order p^2 for an odd prime p is presented in [3], and all self-dual optimal codes possessing an automorphism of order 9 with six 9-cycles without cycles of length 3 are obtained there. In this work we continue the investigations for binary optimal self-dual codes with an automorphism of order 9 with six 9-cycles and cycles of length 3. We classify all self-dual [60,30,12] and [62,32,12] codes possessing such an automorphism. We construct many doubly-even [64,32,12] and singly-even [66,33,12] codes. Some of the constructed codes of lengths 62 and 66 have weight enumerators for which the existence of codes was not known before. We give the description of the method used in Sect. 2. The authors suggest the reader consult [3] for more details.

2 Construction method

We will use the notations from [3]. Let C be a binary self-dual code of length n, and σ be an automorphism of C of type $9-(c, t, f)$, i.e. σ has c independent 9-cycles, t independent cycles of length 3 and f fixed points. Obviously, $n = 9c + 3t + f$. Then σ^3 is an automorphism of type $3-(3c, 3t+f)$, and the parameter c must be even. Without loss of generality we can assume that

$$\sigma = \Omega_1 \dots \Omega_c \Omega_{c+1} \dots \Omega_{c+t} \Omega_{c+t+1} \dots \Omega_{c+t+f} \quad (1)$$

where $\Omega_i = (9i-8, \dots, 9i)$, $i = 1, \dots, c$ are the cycles of length 9, $\Omega_{c+i} = (9c + 3(i-1)+1, \dots, 9c+3i)$, $i = 1, \dots, t$ are the cycles of length 3, and $\Omega_{c+t+i} = (9c+3t+i)$, $i = 1, \dots, f$ are the fixed points.

Let $F_\sigma(C) = \{v \in C : v\sigma = v\}$ and $E_\sigma(C) = \{v \in C : wt(v|\Omega_i) \equiv 0(mod\ 2), i = 1, \dots, c+t+f\}$, where $v|\Omega_i$ is the restriction of v on Ω_i. Then $C = F_\sigma(C) \oplus E_\sigma(C)$.

Each vector $v \in F_\sigma(C)$ is constant on any cycle of σ. Let $\pi : F_\sigma(C) \to \mathbb{F}_2^{c+t+f}$ be the projection map where if $v \in F_\sigma(C)$, $(v\pi)_i = v_j$ for some $j \in \Omega_i$, $i = 1, 2, \dots, c+t+f$. It is known that the "contracted" code $C_\pi = \pi(F_\sigma(C))$ is a binary self-dual code of length $c+t+f$. The code $F_\sigma(C)$ is uniquely determined by the code C_π.

Denote by $E_\sigma(C)^*$ the code $E_\sigma(C)$ with the last f coordinates deleted. So $E_\sigma(C)^*$ is a binary self-orthogonal $[9c+3t, 4c+t]$ code. For $v \in E_\sigma(C)^*$ we identify $v|\Omega_i = (v_0, v_1, \cdots, v_8)$ with the polynomial $v_0 + v_1x + \cdots + v_8x^8$ from $\mathcal{T}$ for $i = 1, \ldots, c$, and $v|\Omega_i = (v_0, v_1, v_2)$ with the polynomial $v_0 + v_1x + v_2x^2$ from $\mathcal{P}$ for $i = c+1, \ldots, c+t$, where $\mathcal{T}$ and $\mathcal{P}$ are the sets of even-weight polynomials in $\mathbb{F}_2[x]/(x^9-1)$ and $\mathbb{F}_2[x]/(x^3-1)$, respectively. Thus we obtain the map $\phi : E_\sigma(C)^* \to \mathcal{T}^c \times \mathcal{P}^t$.

Definition 1 [3] A linear code $\mathcal{C} \subset \mathcal{T}^c \times \mathcal{P}^t$ is a subset of $\mathcal{T}^c \times \mathcal{P}^t$ such that $v + w \in \mathcal{C}, \forall\, v, w \in \mathcal{C}$ and $xv \in \mathcal{C}, \forall\, v \in \mathcal{C}$.

Then $C_\phi = \phi(E_\sigma(C)^*)$ is a linear code in $\mathcal{T}^c \times \mathcal{P}^t$. Following [15] we define Hermitian inner products over $\mathcal{T}$ and $\mathcal{P}$ as $\langle v, w\rangle = \sum_{i=1}^{c} v_i(x)w_i(x^{-1}) = \sum_{i=1}^{c} v_i(x)w_i(x^8)$, $v, w \in \mathcal{T}^c$ and $\langle v', w'\rangle = \sum_{i=1}^{t} v_i'(x)w_i'(x^{-1}) = \sum_{i=1}^{t} v_i'(x)w_i'(x^8)$, $v', w' \in \mathcal{P}^t$. Using these two inner products we can define the inner product in $\mathcal{T}^c \times \mathcal{P}^t$ in the following way:

$$\langle (v_1, v_2), (w_1, w_2)\rangle = \langle v_1, w_1\rangle + (x^6 + x^3 + 1)\langle v_2, w_2\rangle \tag{2}$$

for all vectors $v_1, w_1 \in \mathcal{T}^c$ and $v_2, w_2 \in \mathcal{P}^t$. If $\mathcal{C}$ is a linear code in $\mathcal{T}^c \times \mathcal{P}^t$ we define its dual code as the set $\mathcal{C}^\perp$ of all vectors $w \in \mathcal{T}^c \times \mathcal{P}^t$ such that $\langle v, w\rangle = 0$ for all vectors $v \in \mathcal{C}$. It is easy to prove that the dual of a linear code in $\mathcal{T}^c \times \mathcal{P}^t$ is linear, too. If $\mathcal{C}$ coincides with its dual code, it is called self-dual. The next theorem is the main tool in our investigation, and it is a direct consequence of Theorem 1 in [3].

Theorem 1 *The binary code C having an automorphism σ defined in (1) is self-dual iff C_π is a binary self-dual code and $C_\phi = \phi(E_\sigma(C)^*)$ is a self-dual code in $\mathcal{T}^c \times \mathcal{P}^t$ with respect to the inner product (2).*

As 2 is a primitive root modulo 9, the factorization of the polynomial $x^9 - 1$ into irreducible factors over $\mathbb{F}_2$ is given by $x^9 - 1 = (x-1)(x^2+x+1)(x^6+x^3+1)$. Then $\mathcal{P}$ is a field with four elements, and $\mathcal{T} = I_1 \oplus I_2$, where I_1 and I_2 are the cyclic codes with parity check polynomials x^2+x+1 and x^6+x^3+1, respectively. Hence $I_1 \cong GF(4)$ and $I_2 \cong GF(2^6)$, and for every element $a(x) \in \mathcal{T}$, we have $a = a_1(x) + a_2(x)$ where $a_1 \in I_1, a_2 \in I_2$.

Let A be the largest subcode of $E_\sigma(C)^*$ which is zero on the last $3t$ coordinates corresponding to the cycles of length 3, and let B be the largest subcode of $E_\sigma(C)^*$ which is zero on the first $9c$ coordinates corresponding to the 9-cycles. Denote by A^* the code A with the last $3t$ coordinates deleted, and by B^* the code B with the first $9c$ coordinates deleted. Then A^* is a binary linear code of length $9c$ having an automorphism of order 9 with c independent 9-cycles. Then $M = \phi(A^*) = M_1 \oplus M_2$, where $M_j = \{u \in \phi(A^*) | u_i \in I_j, i = 1, \ldots, c\}$, $j = 1, 2$ is a linear code over I_j. The code $B_\phi = \phi(B^*)$ is a linear code over $\mathcal{P}$ of length t and dimension k_t. So we can consider a generator matrix for C_ϕ in the form

$$G_\phi = \begin{pmatrix} gen_{I_1} & M_1 & 0 \\ gen_{I_2} & M_2 & 0 \\ 0 & & gen_{\mathcal{P}} B_\phi \\ D_c & & D_t \end{pmatrix}$$ and by [3] the following theorem holds:

Theorem 2 *If C is a binary self-dual $[n, \frac{n}{2}, d]$ code with automorphism of type $9-(c, t, f)$. Then:*

(1) *M_2 is a self-dual $[c, \frac{c}{2}]$ code over I_2.*
(2) *$dim_{I_1} M_1 = \frac{c-t}{2} + k_t$ and $rank(D_c) = rank(D_t) = t - 2k_t$.*
(3) *The rows of $gen_{I_1} M_1$ and D_c generate the code $M_1^\perp$ and the rows of $gen_P B_\phi$ and D_t generate the code $B_\phi^\perp$.*

As c is even, so are the parameters t and $f = n - 9c - 3t$. All optimal self-dual codes with an automorphism of type $9 - (c, t, f)$ for $c = 2, 4$ and for $c = 6$, $t = 0$ are constructed in [2] and [3]. In this work we continue the investigations for $c = 6$ and $t \neq 0$.

The following theorem is a particular case of the results in [[3], Theorem 5]; here $n(k, d)$ is the minimum length n for which a binary linear $[n, k, d]$ code exists.

Theorem 3 *If C is a binary self-dual $[n, \frac{n}{2}, d]$ code with automorphism of type $9-(6, t, f)$ then we have the following inequalities:*

(1) $54 \geq n(18, d)$.
(2) *If $3t + f > 18$, the following inequality holds: $3t + f \geq n(\frac{3t+f}{2} - 9, d)$.*
(3) *If $f > 6 + t$, the following inequality holds: $f \geq n(\frac{f-t}{2} - 3, d)$.*

3 Self-dual codes with an automorphism of order 9 with $c = 6$

As described in Sect. 2, $\mathcal{T} = I_1 \oplus I_2$. $I_1 = \{0, x^s e_1 | s = 0, 1, 2\}$ is a field of four elements with identity $e_1 = x^8 + x^7 + x^5 + x^4 + x^2 + x$, and I_2 is a field of 2^6 elements with identity $e_2 = x^6 + x^3$. The element $\alpha = (x + 1)e_2$ is a primitive element of I_2. The element $\delta = \alpha^9 = x^2 + x^4 + x^5 + x^7$ has multiplicative order 7 in I_2 and $I_2 = \{0, x^s \delta^k |$ for $0 \leq s \leq 8$ and $0 \leq k \leq 6\}$.

Let C be a binary optimal self-dual $[n, k, d]$ code having an automorphism σ of order 9 defined in (1) with six independent 9-cycles, $t \neq 0$ independent 3-cycles and f fixed points. Hence $n \geq 60$ and so $d \geq 12$. Then $C_\phi = M_1 \oplus M_2$ where M_2 is a Hermitian self-dual [6, 3] code over the field I_2.

To actually construct a generator matrix of the code C we use four matrices: $gen\ C_\pi$, $gen_{I_2} M_2$, $S = \left(gen_{I_1} M_1 / D_6\right)$ and D_t. To narrow our calculations we use the following transformations which preserve the decomposition and send the code C to an equivalent one:

(i) a permutation of the last f fixed coordinates.
(ii) a permutation of the t 3-cycles coordinates.
(iii) a permutation of the six 9-cycles coordinates.
(iv) a substitution $x \to x^2$ in C_ϕ.
(v) a cyclic shift to each 9-cycle independently. This action preserves $genC_\pi$, and it is equivalent to multiplication of the coordinates of $gen_{I_2} M_2$ and S by x^k for $k = 0, 1, \ldots, 8$ and by x^k for $k = 0, 1, 2$, respectively.
(vi) a cyclic shift to each 3-cycle independently. This action also preserves $gen\ C_\pi$.

There exist four monomially nonequivalent possibilities for M_2 with generator matrices

$$L_1 = \begin{pmatrix} e_2 & 0 & 0 & \delta & \delta^3 & 0 \\ 0 & e_2 & 0 & \delta^2 & e_2 & \delta^2 \\ 0 & 0 & e_2 & \delta^5 & \delta^3 & \delta^6 \end{pmatrix}, L_2 = \begin{pmatrix} e_2 & 0 & 0 & e_2 & \delta & \delta \\ 0 & e_2 & 0 & \delta^2 & \alpha^2 & \alpha^{10} \\ 0 & 0 & e_2 & \delta^2 & \alpha^{14} & \alpha^{39} \end{pmatrix},$$

$$L_3 = \begin{pmatrix} e_2 & 0 & 0 & e_2 & \delta & \delta \\ 0 & e_2 & 0 & \delta^3 & \alpha^{12} & \alpha^{38} \\ 0 & 0 & e_2 & \delta^3 & \alpha^{50} & \alpha^{46} \end{pmatrix} \text{ and } L_4 = \begin{pmatrix} e_2 & 0 & 0 & e_2 & \delta & \delta \\ 0 & e_2 & 0 & \delta^5 & e_2 & \delta^5 \\ 0 & 0 & e_2 & \delta^5 & \delta & \delta^2 \end{pmatrix}.$$

The code $\phi^{-1}(M_2)$ is a linear [54, 18, 12] code at each one of these possibilities. Hence the minimum weight of the code is 12 and $60 \leq n \leq 68$. So the following lemma holds:

Lemma 1 *The minimum weight of a binary self-dual optimal $[n, \frac{n}{2}]$ code having an automorphism of type $9 - (6, t, f)$ is 12 and $60 \leq n \leq 68$. The possibilities for the parameters* *t and f are either:* (1) $t = 2, f = 0$ *or* (2) $t = 2, f = 2$ *or* (3) $t = 2, f = 4$ *or* (4) $t = 4, f = 0$ *or* (5) $t = 4, f = 2$.

In this work we investigate optimal self-dual codes of lengths $60 \le n \le 66$ i.e. all cases in Lemma 1 except case 5). As the parameter t is at most 4, $k_t = dim\ B_\phi = 0$, and we can take D_t to be the identity matrix over the field $\mathcal{P}$.

We can fix $gen_{I_2} M_2 = L_i$, for $i = 1, \dots, 4$ and D_t. First we determine all possibilites for the matrix $H = \begin{pmatrix} gen\ C_\pi \\ gen_{I_2} M_2\ 0 \end{pmatrix}$. After that we add the matrices S and D_t and check the constructed codes for equivalence, using the program **Q-extension** [1].

3.1 $t = 2$

In our construction, $D_2 = \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix}$ where $e_3 = x + x^2$ is the identity element of $\mathcal{P}$. Since the minimum weight of C is 12, M_1 is a [4, 2, 4] self-orthogonal code over the field I_1. Applying the orthogonal condition (2) and row reducing, we obtain a unique possibility for the matrix S up to a permutation of the coordinates followed by multiplying the coordinates by x^k for $k = 0, 1, 2$, and it is

$$S = \left(\begin{array}{cccccc} e_1 & 0 & e_1 & e_1 & e_1 & 0 \\ 0 & e_1 & 0 & xe_1 & xe_1 & e_1 \\ \hline 0 & 0 & e_1 & e_1 & 0 & x^2 e_1 \\ 0 & 0 & e_1 & 0 & e_1 & x^2 e_1 \end{array}\right). \tag{3}$$

3.1.1 $f = 0$, [60, 30, 12] *codes*

The possible weight enumerators were derived in [4] and [8]:

$$W_{60,1} = 1 + (2555 + 64\beta)y^{12} + (33600 - 384\beta)y^{14} + \cdots$$

and

$$W_{60,2} = 1 + 3451y^{12} + 24128y^{14} + 336081y^{16} + \cdots,$$

where β is an integer with $0 \le \beta \le 10$. An optimal self-dual code with weight enumerator $W_{60,2}$ was constructed in [4]. For weight enumerators of type $W_{60,1}$, self-dual codes were constructed with $\beta = 0$, 1, 7 and 10 (see [3,5,11,16]).

In this case C_π is a binary [8,4] self-dual code, equivalent either to C_2^4 or to the extended Hamming code H_8. When $C_\pi \approx C_2^4$, we obtain exactly five nonequivalent self-dual [60, 30, 12] codes with weight enumerators $W_{60,1}$ for $\beta = 0$. All of them have automorphism group of order 18. When $C_\pi \approx H_8$, we construct exactly three nonequivalent self-dual [60, 30, 12] codes with weight enumerators $W_{60,1}$ for $\beta = 1$ and 8 codes for $\beta = 10$. These codes have automorphism groups of orders either 9, 18 or 54. The following theorem summarizes these results:

Theorem 4 *There exist exactly* 16 *nonequivalent binary self-dual* [60, 30, 12] *codes having an automorphism of order* 9 *with* 6 9*-cycles and* 2 3*-cycles. All of them have weight enumerator* $W_{60,1}$ *for* $\beta = 0, 1,$ *and* 10.

The first known self-dual [60, 30, 12] code with weight enumerator $W_{60,1}$ for $\beta = 1$ has been constructed in [3] via an automorphism of type 9-(6,0,6), and it is equivalent to one of

Table 1 New self-dual [60, 30, 12] codes with weight enumerators $W_{60,1}$ for $\beta = 1$

$genM_2$	S	C_π	$\lvert Aut \rvert$
L_2	$\begin{pmatrix} e_1 & 0 & 0 & x^2e_1 & xe_1 & xe_1 \\ 0 & e_1 & xe_1 & e_1 & x^2e_1 & 0 \\ 0 & 0 & e_1 & x^2e_1 & 0 & xe_1 \\ 0 & 0 & e_1 & 0 & xe_1 & xe_1 \end{pmatrix}$	$\begin{pmatrix} 1\,1\,0\,1\,1\,0 \mid 0\,0 \\ 1\,0\,1\,1\,0\,1 \mid 0\,0 \\ 1\,1\,1\,0\,0\,0 \mid 1\,0 \\ 0\,0\,0\,1\,1\,1 \mid 0\,1 \end{pmatrix}$	18
L_4	$\begin{pmatrix} e_1 & 0 & e_1 & xe_1 & x^2e_1 & 0 \\ 0 & xe_1 & xe_1 & x^2e_1 & 0 & e_1 \\ 0 & 0 & e_1 & 0 & x^2e_1 & x^2e_1 \\ 0 & 0 & 0 & xe_1 & x^2e_1 & x^2e_1 \end{pmatrix}$	$\begin{pmatrix} 1\,1\,1\,0\,1\,0 \mid 0\,0 \\ 1\,0\,1\,1\,0\,1 \mid 0\,0 \\ 1\,1\,0\,1\,0\,0 \mid 1\,0 \\ 0\,0\,1\,0\,1\,1 \mid 0\,1 \end{pmatrix}$	9

our codes. Thus we conclude that there exist at least three self-dual [60, 30, 12] codes with weight enumerators $W_{60,1}$ for $\beta = 1$.

The new self-dual [60, 30, 12] codes with $\beta = 1$ are presented in Table 1 with the order of their automorphism groups $|Aut|$.

3.1.2 $f = 2$, [62, 31, 12] *codes*

There are two possible forms for the weight enumerator of an optimal self-dual code of length 62 [5]:

$$W_{62,1} = 1 + 2308y^{12} + 23767y^{14} + 279405y^{16} + \cdots$$

and

$$W_{62,2} = 1 + (1860 + 32\beta)y^{12} + (28055 - 160\beta)y^{14} + (255533 + 96\beta)y^{16} + \cdots,$$

where β is an integer with $0 \le \beta \le 93$. Only codes with weight enumerator $W_{62,2}$ where $\beta =$ 0, 10, 15 are known until now (see [5,10]). In this work we construct self-dual [62, 32, 12] codes with weight enumerator $W_{62,1}$ and codes with weight enumerator $W_{62,2}$ for $\beta = 0, 9$. All obtained optimal codes of length 62 have automorphism groups of order 9.

The code C_π in this case is a binary [10,5,2] self-dual code. Up to equivalence there are two such codes – C_2^5 and $C_2 \oplus H_8$. We obtain self-dual [62, 32, 12] codes only when $C_\pi \approx C_2 \oplus H_8$. There are two possibilities for its generator matrix

$$G_{\pi,1} = \begin{pmatrix} 110000|00|00 \\ 001000|01|11 \\ 000100|10|11 \\ 000010|11|01 \\ 000001|11|10 \end{pmatrix} \text{ and } G_{\pi,2} = \begin{pmatrix} 111100|00|00 \\ 000001|10|00 \\ 110010|01|00 \\ 011010|00|10 \\ 101010|00|01 \end{pmatrix}$$

up to a permutation of the 9-cycles coordinates and a permutation of the last four coordinates corresponding to the 3-cycles and the fixed points.

(1) Let $C_\pi \approx <G_{\pi,1}>$. We did not obtain optimal codes when M_2 is generated by L_1. In the other cases for M_2 we found exactly 3 nonequivalent [62, 32, 12] codes with weight enumerators $W_{62,2}$. We have one code for $\beta = 0$ and two codes for $\beta = 9$. The constructed codes with $W_{62,2}$ for $\beta = 9$ are the first known codes with this weight enumerator. They are generated by the matrices $G_{62,1}$ and $G_{62,2}$.

(2) Let $C_\pi \approx <G_{\pi,2}>$. In this case we obtain exactly 67 nonequivalent self-dual [62, 32, 12] codes with weight enumerator $W_{62,1}$. These codes are the first known codes with this weight enumerator.

We summarize the results in the following theorems:

Theorem 5 *There exist exactly* 70 *nonequivalent binary self-dual* [62,31,12] *codes having an automorphism of order* 9 *with* 6 9*-cycles and* 2 3*-cycles. One of them has weight enumerator $W_{62,2}$ with $\beta = 0$, two codes have weight enumerator $W_{62,2}$ with $\beta = 9$ and* 67 *codes have weight enumerator $W_{62,1}$.*

Theorem 6 *There exist at least* 67 *self-dual* [62, 31, 12] *codes with weight enumerators $W_{62,1}$ and at least two codes with weight enumerators $W_{62,2}$ for $\beta = 9$.*

In Table 2 we give examples for generator matrices of the self-dual [62,31,12] codes with weight enumerators $W_{62,1}$. These codes are determined by $C_\pi = <\tau(G_{\pi,2})>$, $\mu(x^{i_1}, x^{i_2}, x^{i_3}, x^{i_4}, x^{i_5}, x^{i_6})(S)$ and the matrix $genM_2$ where τ and μ are permutations from the symmetric groups S_8 and S_6, respectively. The notation $\mu(x^{i_1}, x^{i_2}, x^{i_3}, x^{i_4}, x^{i_5}, x^{i_6})(S)$ means that first we permute the columns of the matrix S, defined in (3), by μ, and then we multiply the columns by x^{i_j}, $i_j = 0, 1, 2$, for $j = 1, \ldots, 6$.

$G_{62,1}=$

111111111111111111000000000000000000000000000000000000	000000	00
000000000000000000111111111000000000000000000000000000	000111	11
000000000000000000000000000111111111000000000000000000	111000	11
000000000000000000000000000000000000111111111000000000	111111	01
000000000000000000000000000000000000000000000111111111	111111	10
000100100000000000000000000000100100001011010001011010	000000	00
000010010000000000000000000000010010000101101000101101	000000	00
000001001000000000000000000000001001100010110100010110	000000	00
100000100000000000000000000100000100010001011010001011	000000	00
010000010000000000000000000010000010101000101101000101	000000	00
001000001000000000000000000001000001110100010110100010	000000	00
000000000000100100000000000000111001110001001000111001 11	000000	00
000000000000010010000000000101110011000010010101110011	000000	00
000000000000001001000000000110111001000001001110111001	000000	00
000000000100000100000000000111011100100000100111011100	000000	00
000000000010000010000000000011101110010000010011101110	000000	00
000000000001000001000000000001110111001000001001110111	000000	00
000000000000000000000100100011100111001011010010011001	000000	00
000000000000000000000010010101110011000101101101001100	000000	00
000000000000000000000001001110111001100010110010100110	000000	00
000000000000000001000001001110111000100010110010100110 11	000000	00
000000000000000001000001001110111010100010110010100 1	000000	00
000000000000000001000001001110111110100010110010100	000000	00
011011011000000000110110110000000001101101101011011 01	000000	00
101101101000000000101101101000000000110110111101101 10	000000	00
000000000101101101000000000110110110110110111101101 10	000000	00
000000000110110110000000000101101101101101101011011011	000000	00
000000000000000000110110111101101101101101100000000000	011000	00
000000000000000000101101101011011011011011010000000000	101000	00
000000000000000000110110111101101100000000000101101101	000011	00
000000000000000000101101101011011010000000000110110110	000101	00

$$
G_{62,2}=\left(\begin{array}{c|c|c}
111111111000000000000000000000000000111111111000000000 & 000000 & 00\\
000000000111111111000000000000000000000000000000000000 & 111111 & 01\\
000000000000000000111111111000000000000000000000000000 & 000111 & 11\\
000000000000000000000000000111111111000000000000000000 & 111000 & 11\\
000000000000000000000000000000000000000000000111111111 & 111111 & 10\\
\hline
000100100000000000000000000000100100001011010001011010 & 000000 & 00\\
000010010000000000000000000000010010000101101000101101 & 000000 & 00\\
000001001000000000000000000000001001100010110100010110 & 000000 & 00\\
100000100000000000000000000100000100010001011010001011 & 000000 & 00\\
010000010000000000000000000010000010101000101101000101 & 000000 & 00\\
001000001000000000000000000001000001110100010110100010 & 000000 & 00\\
000000000000100100000000000010011001000101101001110111 & 000000 & 00\\
000000000000010010000000000101001100100010110100111011 & 000000 & 00\\
000000000000001001000000000010100110010001011110011101 & 000000 & 00\\
000000000100000100000000000001010011101000101111001110 & 000000 & 00\\
000000000010000010000000000100101001110100010011100111 & 000000 & 00\\
000000000001000001000000000110010100011010001101110011 & 000000 & 00\\
000000000000000000000100100010011001010010000011001010 & 000000 & 00\\
000000000000000000000010010101001100001001000001100101 & 000000 & 00\\
000000000000000000000001001010100110000100100100110010 & 000000 & 00\\
000000000000000000100000100001010011000010010010011001 & 000000 & 00\\
000000000000000000010000010100101001000001001101001100 & 000000 & 00\\
000000000000000000001000001110010100100000100010100110 & 000000 & 00\\
011011011000000000011011011110110110110110110000000000 & 000000 & 00\\
101101101000000000101101101011011011011011011000000000 & 000000 & 00\\
000000000101101101000000000011011011011011011110110110 & 000000 & 00\\
000000000110110110000000000101101101101101101011011011 & 000000 & 00\\
\hline
000000000000000000011011011110110110000000000101101101 & 011000 & 00\\
000000000000000000101101101011011011000000000110110110 & 101000 & 00\\
000000000000000000011011011000000000110110110101101101 & 000011 & 00\\
000000000000000000101101101000000000011011011110110110 & 000101 & 00
\end{array}\right)
$$

3.1.3 $f = 4$, [64, 32, 12] *codes*

There exist singly-even [64, 32, 12] codes and more than 3,250 doubly-even self-dual codes with these parameters constructed from 2-designs and double circulant matrices (see [14] and the references given therein). C_π is a binary self-dual $[12, 6, d \geq 2]$ code equivalent either to $C_2^2 \oplus H_8$ or B_{12}. In the first case we proved that the code $F_\sigma(C)$ is a doubly-even subcode. As the subcode $E_\sigma(C)^*$ is also doubly-even, so are the obtained codes C. We constructed 10,637 nonequivalent doubly-even [64, 32, 12] codes, and we stopped our calculations. All obtained codes have automorphism groups of order either 9 or 18. When $C_\pi \approx B_{12}$, there does not exist optimal codes of length 64. Hence all the [64,32,12] self-dual codes with an automorphism of order 9 with 6 9-cycles and 2 3-cycles are doubly-even.

Table 2 New self-dual [62, 32, 12] codes with weight enumerators $W_{62,1}$

τ	$\mu(i_1, i_2, i_3, i_4, i_5, i_6)$	$genM_2$	τ	$\mu(i_1, i_2, i_3, i_4, i_5, i_6)$	$genM_2$
(1,6)(8,9)	(2,3)(5,6)(0,0,1,1,2,1)	L_1	(1,6)(8,9)	(2,3)(5,6)(0,1,1,2,2,1)	L_1
(1,6)(7,8,9)	(3,6)(0,0,2,0,1,2)	L_1	(1,6)(8,10)	(3,4)(5,6)(0,0,1,1,0,2)	L_1
(1,6)(7,8,10)	(3,6,4)(0,1,1,0,1,2)	L_1	(1,6,4)	(4,5,6)(0,0,1,2,2,2)	L_1
(1,6,4)(7,8,9)	(3,6,4)(0,0,2,2,1,1)	L_1	(1,6,4)(7,8,10)	(3,4)(5,6)(0,0,0,0,1,1)	L_1
(1,6,3,4)	(3,4,5,6)(0,0,2,0,0,1)	L_1	(1,6,3,4)(7,8)	(3,4)(5,6)(0,0,2,0,0,0)	L_1
(1,6,3,4)(8,9)	(3,5,6,4)(0,1,0,0,0,0)	L_1	(1,6,2)(3,4)(8,9)	(3,5,6,4)(0,1,0,0,0,0)	L_1
(1,6,2)(3,4)(7,8,9)	(2,3)(0,0,1,0,0,0)	L_1	(1,6,2)(3,4)(7,8,9)	(2,3)(0,1,1,2,2,1)	L_1
(7,8,10)	id(0,2,2,0,0,1)	L_2	(1,6)(8,10)	(4,5,6)(0,0,0,2,1,1)	L_2
(1,6,4)(8,10)	(3,5,6,4)(0,1,2,1,2,0)	L_2	(1,6,4)(8,10)	(23)(56)(0,1,1,1,0,2)	L_2
(1,6,4)(8,10)	(2,3)(4,5,6)(0,2,1,0,0,2)	L_2	(1,6,4)(8,10)	(2,3)(4,5,6)(0,2,2,0,0,2)	L_2
(1,6,3,4)(8,9)	(3,6)(0,1,2,2,1,0)	L_2	(1,6,3,4)(7,8,9)	(4,5,6)(0,1,2,0,2,1)	L_2
id	(3,5,4)(0,0,1,2,0,1)	L_3	id	(3,6,5,4)(0,2,2,1,0,1)	L_3
id	(23)(56)(0,1,0,0,0,0)	L_3	(7,8)	(2,3)(5,6)(0,2,1,0,2,0)	L_3
(8,9)	(3,4)(5,6)(0,0,0,2,2,1)	L_3	(8,10)	(3,6,4)(0,0,1,2,2,2)	L_3
(8,10)	(3,6,4)(0,1,1,0,1,0)	L_3	(7,8,10)	(3,4)(5,6)(0,1,1,2,2,1)	L_3
(7,8,10)	(3,5,6,4)(0,0,2,0,0,0)	L_3	(1,6)	id(0,1,2,2,1,1)	L_3
(1,6)(8,9)	(23)(56)(0,1,0,1,1,1)	L_3	(1,6)(7,8,9)	(3,4,5,6)(0,2,2,0,1,2)	L_3
(1,6)(8,10)	(4,5,6)(0,0,2,1,1,2)	L_3	(1,6)(8,10)	(4,5,6)(0,0,2,2,1,0)	L_3
(1,6)(8,10)	(2,3)(0,1,2,2,2,1)	L_3	(1,6)(7,8,10)	(4,5,6)(0,0,2,2,2,2)	L_3
(1,6)(7,8,10)	(3,4)(5,6)(0,0,0,1,2,0)	L_3	(1,6,4)	(5,6)(0,0,2,1,2,0)	L_3
(1,6,4)	(4,5,6)(0,0,2,1,1,2)	L_3	(1,6,4)	(3,4)(0,1,0,2,2,1)	L_3
(1,6,4)	(3,4)(0,1,1,1,1,1)	L_3	(1,6,4)(7,8)	(4,5,6)(0,1,0,0,1,0)	L_3
(1,6,4)(8,9)	(3,5,6)(0,1,1,2,1,2)	L_3	(1,6,4)(7,8,9)	(3,5,6)(0,2,2,2,2,2)	L_3
(1,6,4)(7,8,9)	(3,6,4)(0,1,0,1,1,0)	L_3	(1,6,4)(8,10)	(3,5,6,4)(0,1,0,2,1,2)	L_3
(1,6,4)(8,10)	(3,5,6,4)(0,2,1,0,2,0)	L_3	(1,6,4)(7,8,10)	(2,3)(5,6)(0,2,1,1,2,2)	L_3
(1,6,4)(7,8,10)	(2,3)(4,5,6)(0,0,1,2,0,1)	L_3	(1,6,4)(7,8,10)	(2,3)(4,5,6)(0,2,2,0,1,2)	L_3
(1,6,3,4)(7,8,9)	(3,5,6,4)(0,0,2,0,1,1)	L_3	(1,6,3,4)(7,8,9)	(3,5,6,4)(0,0,2,2,0,0)	L_3
(1,6,3,4)(7,8,9)	(3,6)(0,1,0,2,1,1)	L_3	(1,6,3,4)(8,10)	(5,6)(0,0,0,2,2,0)	L_3
(1,6,3,4)(8,10)	(2,3)(0,1,2,0,1,0)	L_3	(1,6,3,4)(8,10)	(2,3)(4,5,6)(0,2,1,2,2,1)	L_3
(1,6,3,4)(7,8,10)	(2,3)(0,0,1,1,2,2)	L_3	(1,6,2)(3,4)	(3,6)(0,0,1,2,1,2)	L_3
(1,6,2)(3,4)(7,8)	(3,6,4)(0,0,1,2,2,1)	L_3	(1,6,2)(3,4)(7,8)	(3,6,4)(0,1,1,1,1,0)	L_3
(1,6,2)(3,4)(7,8)	(3,6,4)(0,1,2,0,2,0)	L_3	(1,6,2)(3,4)(7,8)	(3,6)(0,2,0,2,1,1)	L_3
(1,6,2)(3,4)(7,8,9)	(3,5,6,4)(0,2,2,2,2,2)	L_3	(1,6,2)(3,4)(8,10)	(23)(56)(0,0,0,2,1,2)	L_3
(1,6,2)(3,4)(7,8,10)	(4,5,6)(0,0,2,1,0,2)	L_3			

3.2 $t = 4$, $f = 0$

There exist three possible forms for the weight enumerator of an optimal self-dual code of length 66 [7]:

$$W_{66,1} = 1 + 1690y^{12} + 7990y^{14} + \cdots,$$
$$W_{66,2} = 1 + (858 + 8\beta)y^{12} + (18678 - 24\beta)y^{14} + \cdots,$$

where $0 \le \beta \le 778$ and

$$W_{66,3} = 1 + (858 + 8\beta)y^{12} + (18166 - 24\beta)y^{14} + (255533 + 96\beta)y^{16} + \cdots,$$

where $14 \le \beta \le 756$.

Table 3 New self-dual [66, 33, 12] codes with weight enumerators $W_{66,2}$

β	τ	$G_{\pi,i}$	μ $(i_1, i_2, i_3, i_4, i_5, i_6)$	S_j	$genM_2$
2	(2, 3)(8, 9, 10)	$G_{\pi,3}$	(2, 5) (0, 2, 0, 1, 2, 2)	S_3	$L2$
5	(9, 10)	$G_{\pi,5}$	(1, 2, 3) (0, 2, 1, 0, 2, 2)	S_3	$L2$
6	id	$G_{\pi,4}$	(2, 4, 6, 5) (0, 0, 1, 2, 1, 0)	S_3	$L1$
9	id	$G_{\pi,4}$	(2, 5)(4, 6) (0, 0, 1, 0, 0, 2)	S_3	$L1$
11	(9, 10)	$G_{\pi,6}$	(2, 6, 5, 4, 3) (0, 0, 0, 0, 1, 1)	S_1	$L4$
18	id	$G_{\pi,4}$	(2, 5, 3)(4, 6) (0, 2, 2, 0, 0, 0)	S_3	$L1$
20	id	$G_{\pi,3}$	(2, 5)(4, 6) (0, 2, 1, 2, 2, 0)	S_3	$L1$
23	id	$G_{\pi,5}$	id (0, 1, 2, 0, 0, 1)	S_3	$L1$
27	id	$G_{\pi,4}$	(2, 5)(4, 6) (0, 1, 1, 0, 0, 0)	S_3	$L1$
29	id	$G_{\pi,5}$	(2, 5, 4, 3) (0, 0, 2, 0, 1, 1)	S_1	$L3$
32	id	$G_{\pi,5}$	id (0, 1, 2, 0, 0, 2)	S_3	$L1$
33	id	$G_{\pi,4}$	(2, 3)(4, 6, 5) (0, 1, 0, 0, 0, 0)	S_3	$L1$
35	id	$G_{\pi,6}$	(2, 3, 4, 5) (0, 0, 0, 0, 1, 2)	S_2	$L1$
42	id	$G_{\pi,4}$	(2, 3)(4, 5) (0, 2, 2, 1, 1, 2)	S_3	$L1$
44	id	$G_{\pi,6}$	(2, 3, 4, 5) (0, 1, 2, 0, 1, 0)	S_2	$L1$
47	id	$G_{\pi,3}$	(2, 5)(3, 4, 6) (0, 0, 0, 1, 0, 0)	S_2	$L1$
50	id	$G_{\pi,5}$	(3, 5, 4) (0, 1, 0, 1, 1, 2)	S_3	$L1$
51	id	$G_{\pi,4}$	(2, 4, 5, 3) (0, 0, 1, 0, 0, 1)	S_3	$L1$
53	id	$G_{\pi,6}$	(2, 4, 6, 5) (0, 0, 1, 1, 1, 1)	S_2	$L1$
54	id	$G_{\pi,4}$	(1, 2, 6, 4, 5, 3) (0, 1, 2, 0, 1, 1)	S_3	$L1$
56	id	$G_{\pi,3}$	(2, 6, 3, 4, 5) (0, 1, 0, 0, 2, 1)	S_2	$L1$
59	(8, 9)	$G_{\pi,5}$	(3, 5, 4) (0, 2, 1, 0, 2, 0)	S_3	$L1$
60	id	$G_{\pi,4}$	(2, 4, 5, 3) (0, 1, 1, 0, 2, 1)	S_3	$L1$
62	id	$G_{\pi,6}$	(2, 4, 6, 5) (0, 2, 0, 0, 1, 1)	S_2	$L1$
63	(8, 9, 10)	$G_{\pi,4}$	(2, 4, 5t) (0, 2, 1, 2, 0, 2)	S_3	$L1$
65	id	$G_{\pi,3}$	(2, 5)(3, 4, 6) (0, 0, 0, 0, 0, 0)	S_2	$L1$
68	(8, 9)	$G_{\pi,5}$	(3, 4) (0, 1, 0, 0, 1, 2)	S_3	$L1$
69	id	$G_{\pi,4}$	(2, 3, 5, 4) (0, 2, 0, 0, 0, 0)	S_3	$L1$
71	id	$G_{\pi,6}$	(1, 2, 3, 4, 5) (0, 1, 2, 2, 2, 1)	S_2	$L1$
72	(8, 9, 10)	$G_{\pi,4}$	(2, 4, 5, 3) (0, 0, 1, 2, 0, 0)	S_3	$L1$
77	(1, 6)	$G_{\pi,5}$	(1, 2, 6, 5, 4, 3) (0, 2, 2, 2, 2, 2)	S_3	$L1$
83	(8, 9, 10)	$G_{\pi,3}$	(2, 4, 6, 3, 5) (0, 1, 0, 1, 2, 2)	S_2	$L1$
86	(1, 6)	$G_{\pi,5}$	(1, 2, 3)(4, 6, 5) (0, 1, 1, 2, 2, 1)	S_3	$L1$
87	(8, 9)	$G_{\pi,4}$	(1, 2, 6, 5, 4, 3) (0, 2, 2, 1, 1, 2)	S_3	$L1$
92	(7, 8, 10, 9)	$G_{\pi,3}$	(2, 4, 6, 5) (0, 0, 1, 1, 1, 1)	S_2	$L1$

Codes exist with weight enumerator $W_{66,1}$ [17] and $W_{66,2}$ when $\beta = 0, 3, 8, 10, 14, \ldots, 17$, 22, 24, 26, 31, 36, 38, 41, 43, 45, 46, 52, 59, 66, 73, 74, 76, 78 and 80 ([4,9,12,14]). In this work we construct a number of optimal self-dual [66,33,12] codes with weight enumerators $W_{66,2}$ for 35 new values of the parameter as follows: $\beta =$ 2, 5, 6, 9, 11, 18, 20, 23, 27, 29, 32, 33, 35, 42, 44, 47, 50, 51, 53, 54, 56, 59, 60, 62, 63, 65, 68, 69, 71, 72, 77, 83, 86, 87, 92.

Remark In the second review the authors have been informed that Tsai, Shih, Su, and Chen in [18] find the first examples of codes for $W_{66,3}$ with $\beta =$ 28, 33, and 34. They also find codes for $W_{66,2}$ with $\beta = 40$ and 44.

Let C be a binary self-dual [66, 33, 12] code having an automorphism σ of type 9-(6,4,0) defined in (1). We can fix D_4 to be the identity matrix over the field $\mathcal{P}$. In this case the code M_1 is a $[6, 1, d_1]$ self-orthogonal code over the field I_1. There are many possibilities for the matrix S. We consider three forms up to a permutation of the coordinates followed by multiplying the coordinates by x^k for $k = 0, 1, 2$, denoted by S_1, S_2 and S_3. In this case C_π is a binary [10,5,2] self-dual code equivalent either to C_2^5 or $C_2 \oplus H_8$. We obtain four forms for *gen* C_π, as follows $G_{\pi,3}$, $G_{\pi,4}$ and $G_{\pi,5}$ up to a permutation of the first six coordinates and a permutation of the last four coordinates corresponding to the 3-cycles.

$$S_1 = \begin{pmatrix} e_1 & e_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & e_1 & e_1 & e_1 & 0 \\ 0 & 0 & e_1 & e_1 & 0 & e_1 \\ 0 & 0 & e_1 & 0 & e_1 & e_1 \\ 0 & 0 & 0 & e_1 & e_1 & e_1 \end{pmatrix} \quad G_{\pi,3} = \begin{pmatrix} 110000|0000 \\ 001000|1000 \\ 000100|0100 \\ 000010|0010 \\ 000001|0001 \end{pmatrix} \quad G_{\pi,4} = \begin{pmatrix} 110000|0000 \\ 001000|0111 \\ 000100|1011 \\ 000010|1101 \\ 000001|1110 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} e_1 & e_1 & e_1 & e_1 & 0 & 0 \\ \hline 0 & 0 & e_1 & e_1 & 0 & e_1 \\ e_1 & e_1 & 0 & 0 & e_1 & 0 \\ 0 & e_1 & x^2e_1 & xe_1 & e_1 & e_1 \\ 0 & e_1 & xe_1 & x^2e_1 & e_1 & e_1 \end{pmatrix} \quad G_{\pi,5} = \begin{pmatrix} 111100|0000 \\ 000001|1000 \\ 110010|0100 \\ 011010|0010 \\ 101010|0001 \end{pmatrix} \quad G_{\pi,6} = \begin{pmatrix} 110000|0000 \\ 000000|1111 \\ 001111|0000 \\ 001100|1100 \\ 000110|0110 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} e_1 & e_1 & e_1 & e_1 & e_1 & e_1 \\ \hline 0 & e_1 & xe_1 & x^2e_1 & xe_1 & xe_1 \\ 0 & e_1 & x^2e_1 & xe_1 & x^2e_1 & x^2e_1 \\ 0 & 0 & e_1 & 0 & xe_1 & x^2e_1 \\ 0 & 0 & e_1 & 0 & x^2e_1 & xe_1 \end{pmatrix}$$

Optimal self-dual [66,33,12] codes do not exist for the matrices $G_{\pi,3}$ and S_1, $G_{\pi,4}$ and S_1, $G_{\pi,4}$ and S_2, and $G_{\pi,6}$ and S_3. In the other cases we constructed many self-dual [66,32,12] codes with weight enumerators $W_{66,2}$ for 49 different values of the parameter $\beta \in \{0, 2, 5, 6, 8, 9, 11,14,15,17,18,20,23,24,26,27,29,32,33,35,36,38, 41,42, 44,45,47,50,51,53,54,56, 59,60,62,63, 65,68,69,71,72,74,77,78,80,83,86,87,92\}$. As was mentioned above, for 35 of them the obtained codes are the first known codes with this weight enumerators.

In Table 3 we present one code for each new value of β we have obtained.

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