Des Codes Crypt (2007) 43:1–8
DOI 10.1007/s10623-007-9046-9

Constructions of intriguing sets of polar spaces from field reduction and derivation

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Received: 27 July 2006 / Revised: 28 October 2006 /
Accepted: 2 February 2007 / Published online: 3 March 2007

Abstract The concepts of a tight set of points and an m-ovoid of a generalised quadrangle were unified recently by Bamberg, Law and Penttila under the title of intriguing sets. This unification was subsequently extended to polar spaces of arbitrary rank. The first part of this paper deals with a method of constructing intriguing sets of one polar space from those of another via field reduction. In the second part of this paper, we generalise an ovoid derivation of Payne and Thas to a derivation of intriguing sets.

Keywords m-ovoid · Tight set · Derivation · Field reduction

AMS Classification 05B25 · 51A50 · 51E12 · 51E20

1 Introduction

Tight sets and m-ovoids of generalised quadrangles have been given much attention (see [2] and the references within) and there are close connections between these two objects and projective two-weight codes and strongly regular graphs (described in [3] and more recently in J. Bamberg et al., Submitted data). The concept of an m-ovoid of a generalised quadrangle was extended to polar spaces of arbitrary rank in [7] where it was defined to be a set of points such that each maximal meets it in m-points. The concept of a *tight set* of points of a generalised quadrangle was extended to polar spaces of arbitrary rank in Drudge's Ph.D. thesis [4]. A set of points $\mathcal{I}$ of a finite polar space $\mathcal{S}$ is tight if the average number of points of $\mathcal{I}$ collinear with a given point of $\mathcal{S}$ equals the maximum possible value. Both tight sets and m-ovoids have

Communicated by S. Ball.

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two intersection numbers with respect to degenerate hyperplanes $P^{\perp}$ depending on whether P is in the set or not and this leads to a unification of the two concepts under that of an *intriguing set*.

We say that a set of points $\mathcal{I}$ of a polar space is *intriguing* if

$$|P^{\perp} \cap \mathcal{I}| = \begin{cases} h_1 & \text{if } P \in \mathcal{I}, \\ h_2 & \text{if } P \notin \mathcal{I} \end{cases} \tag{1}$$

for some integers h_1 and h_2. It is shown in (J. Bamberg et al. (submitted data, Theorem 6) that an intriguing set is an m-ovoid if $h_1 < h_2$, and is a tight set if $h_1 > h_2$ (equality never occurs). If the set is tight, its size is i times the number of points on a maximal. In this case, the set is said to be *i-tight*. This paper describes two methods of constructing new intriguing sets from known ones.

The first method reworks the idea of field reduction in a similar manner to the way it is used to define the fifth Aschbacher class $\mathcal{C}_5$ of maximal subgroups of the finite classical groups [1] (see also [5] for more detail). This involves taking the formed space (V, κ) defining a polar geometry $\mathcal{S}$ and constructing a new form κ' from κ on V considered as a vector space over a subfield of the field of V, thus defining a new polar space $\mathcal{S}'$ with higher rank over a smaller field. The subspaces $\mathcal{I}$ of $\mathcal{S}$ then canonically define a subset $\mathfrak{R}(\mathcal{I})$ of the subspaces of $\mathcal{S}'$ (more formally described in Sect. 3) and using this correspondence, structures in $\mathcal{S}$ sometimes give rise to structures in $\mathcal{S}'$. The main result of Sect. 3 is the following theorem.

Theorem 1.1 *Let $\mathcal{I}$ be an intriguing set of a polar space $\mathcal{S}$ and let $n = |P^{\perp} \cap \mathcal{I}|$ be the number of points of $\mathcal{I}$ collinear with an arbitrary point P of $\mathcal{S}$ outside $\mathcal{I}$ (where $\perp$ is the polarity associated with $\mathcal{S}$). If $|Q^{\perp} \cap \mathcal{I}| = n$ for every point Q of the ambient projective space outside $\mathcal{S}$ then the set $\mathcal{I}' = \mathfrak{R}(\mathcal{I})$ (defined in Sect. 3) is an intriguing set of $\mathcal{S}'$. Furthermore, if $\mathcal{I}$ is i-tight then $\mathcal{I}'$ is i-tight and if $\mathcal{I}$ is an m-ovoid of $\mathcal{S}$ then $\mathcal{I}'$ is an $m|\mathfrak{R}(P)|$-ovoid of $\mathcal{S}'$, where $|\mathfrak{R}(P)|$ is the number of $\mathcal{S}'$-points in the image $\mathfrak{R}(P)$ of P under $\mathfrak{R}$.*

A summary of the polar space mappings and the induced intriguing sets (given an initial intriguing set of the initial polar space) is given in Table 1.

The second method is a generalisation of the derivation of a non-classical ovoid of $\mathsf{H}(3, q^2)$ (the unitary geometry over $\mathrm{GF}(q^2)^4$) from a classical one described by Thas and Payne in [8]. We remind the reader here of Payne and Thas derivation. Let $\mathcal{O}$ be a classical ovoid of $\mathsf{H}(3, q^2)$, that is, the set of totally singular points in a non-degenerate hyperplane $P^{\perp}$, where P is a point that is not totally singular. The unique line ℓ through any two points of $\mathcal{O}$ will be secant to $\mathsf{H}(3, q^2)$, that is, intersect it in $q + 1$ points. Removing these points from $\mathcal{O}$ and adding the totally singular points of $\ell^{\perp}$ (the polar line to ℓ) results in a new ovoid

$$\mathcal{O}' = (\mathcal{O} \backslash \ell) \cup (\ell^{\perp} \cap \mathsf{H}(3, q^2))$$

that is not contained in a hyperplane and hence, not classical.

We generalise this method in Sect. 4 to a derivation of one m-ovoid from another in certain polar spaces of arbitrary rank. As well as providing a method of constructing m-ovoids this construction allows us to show that there exist $\frac{q^{2(r-1)}-1}{q^2-1}$-ovoids of $\mathsf{H}(2r-1, q^2)$ and $\mathsf{Q}^{+}(2r-1, q^2)$ (the unitary and hyperbolic quadric polar spaces over $\mathrm{GF}(q^2)^{2r}$) that are not classical. That is, that are not the intersection of a

Table 1 Polar space mappings and induced intriguing sets

Mapping $\mathcal{S} \to \mathcal{S}'$	Intriguing set of $\mathcal{S}$	Intriguing set of $\mathcal{S}'$
$\mathsf{H}(2r, q^{2e}) \to \mathsf{H}(e(2r+1)-1, q^2)$, e odd	m-ovoid	$m\frac{q^{2e-1}-1}{q^2-1}$-ovoid
$\mathsf{H}(2r, q^{2e}) \to \mathsf{W}(2e(r+1)-1, q)$	m-ovoid	$m\frac{q^{2e}-1}{q-1}$-ovoid
$\mathsf{H}(2r, q^{2e}) \to \mathsf{Q}^-(2e(r+1)-1, q)$	m-ovoid	$m\frac{q^{2e}-1}{q-1}$-ovoid
$\mathsf{H}(2r-1, q^{2e}) \to \mathsf{H}(2er-1, q^2)$, e odd	i-tight	i-tight
$\mathsf{H}(2r, q^{2e}) \to \mathsf{W}(4er-1, q)$	i-tight	i-tight
$\mathsf{H}(2r, q^{2e}) \to \mathsf{Q}^+(4er-1, q)$	i-tight	i-tight
$\mathsf{W}(2r-1, q^e) \to \mathsf{W}(2er-1, q)$	m-ovoid	$m\frac{q^e-1}{q-1}$-ovoid
$\mathsf{W}(2r-1, q^e) \to \mathsf{W}(2er-1, q)$	i-tight	i-tight
$\mathsf{Q}^+(2r-1, q^e) \to \mathsf{Q}^+(2er-1, q)$	i-tight	i-tight
$\mathsf{Q}^-(2r+1, q^e) \to \mathsf{Q}^-(e(2r+2)-1, q)$	m-ovoid	$m\frac{q^e-1}{q-1}$-ovoid

non-degenerate hyperplane with the polar space. The main result of Sect. 4 is the following theorem.

Theorem 1.2 *Let $\mathcal{O}$ be an m-ovoid of a polar space $\mathcal{S}$ of rank r in $\mathsf{PG}(2r-1, q)$.*

(1) *Given a dimension $(r-1)$ subspace ℓ of $\mathsf{PG}(2r-1, q^2)$ that satisfies $\ell^\perp \cap \mathcal{O} \subseteq \ell$,*

$$\mathcal{O}' = (\mathcal{O} \backslash \ell) \cup (\ell^\perp \cap \mathcal{S})$$

is also an m-ovoid of $\mathcal{S}$.

(2) *If $\mathcal{S}$ is $\mathsf{H}(2r-1, q^2)$ or $\mathsf{Q}^+(2r-1, q^2)$ and $\mathcal{O}$ is a classical $\frac{q^{2(r-1)}-1}{q^2-1}$-ovoid then there exists a dimension $(r-1)$ subspace ℓ of $\mathsf{PG}(2r-1, q^2)$ that satisfies $\ell^\perp \cap \mathcal{O} \subseteq \ell$. Furthermore, $\mathcal{O}'$ is not a classical $\frac{q^{2(r-1)}-1}{q^2-1}$-ovoid.*

2 Notation

In this section, we outline the notation used in this paper.

By $\mathsf{PG}(d, q)$ we denote the projective geometry consisting of the subspaces of $\mathrm{GF}(q)^{d+1}$ the $(d+1)$-dimensional vector space over the finite field with q elements. The polar geometries consisting of the totally isotropic subspaces of an alternating non-degenerate bilinear form on $\mathrm{GF}(q)^{d+1}$ will be denoted $\mathsf{W}(d, q)$. The geometry consisting of the totally singular subspaces of a non-degenerate quadratic form on $\mathrm{GF}(q)^{d+1}$ will be denoted $\mathsf{Q}^-(d, q), \mathsf{Q}(d, q), \mathsf{Q}^+(d, q)$ depending on whether the Witt index is $\frac{d-1}{2}, \frac{d}{2}$ or $\frac{d+1}{2}$, respectively. Sometimes $\mathsf{Q}^\varepsilon(d, q)$ may be used to refer to $\mathsf{Q}^+(d, q), \mathsf{Q}(d, q)$ and $\mathsf{Q}^-(d, q)$ concurrently. The polar geometries consisting of the totally isotropic subspaces of an Hermitian non-degenerate bilinear form on $\mathrm{GF}(q^2)^{d+1}$ will be denoted $\mathsf{H}(d, q^2)$. The term polar space will always refer to a polar geometry of one of the kinds mentioned here. The term *rank* will refer to the Witt index of the formed space. We will use the unqualified term *dimension* to refer to the projective dimension of a subspace and *algebraic dimension* otherwise.

3 Constructing intriguing sets via field reduction

The construction is based on a method of embedding various classical groups into others via field reduction. We describe the method briefly here to set notation for this section.

Let (V, κ) be a non-degenerate formed space with associated polar space $\mathcal{S}$ where $V = \mathrm{GF}(q^e)^{d+1}$ and κ is a sesquilinear (quadratic) form. The set V can be considered as an $e(d+1)$-dimensional vector space $V' \cong \mathrm{GF}(q)^{e(d+1)}$ via the inclusion $\mathrm{GF}(q) \subset \mathrm{GF}(q^e)$. Composition of κ with the trace map $T\colon \mathrm{GF}(q^e) \to \mathrm{GF}(q)$ which sends z to $\sum_{i=1}^{e} z^{q^i}$ provides a new form $T\kappa$ on V' mapping into $\mathrm{GF}(q)$ and so we obtain a new formed space $(V', T\kappa)$. This is not always non-degenerate but can be made non-degenerate in some cases by multiplying κ by a suitable choice of scalar $\lambda \in \mathrm{GF}(q^e)$ resulting in the form $\kappa' = T\lambda\kappa$. If our new formed space (V', κ') is non-degenerate, then it has an associated polar space $\mathcal{S}'$. The isomorphism types and various conditions are presented in Table 3 which is a modified form of the table appearing in Ref. [6].

Now each point in $\mathsf{PG}(d, q^e)$ corresponds to a 1-space in V which in turn corresponds to an e-space in V', an $(e-1)$-subspace of $\mathsf{PG}(e(d+1)-1, q)$ and hence, a set of points of $\mathsf{PG}(e(d+1)-1, q)$. Extending this map from points of $\mathsf{PG}(d, q^e)$ to subsets of points of $\mathcal{S}$ we obtain a map

$$\mathfrak{R}\colon 2^{\mathsf{PG}(d,q^e)} \to 2^{\mathsf{PG}(e(d+1)-1,q)}$$

from subsets of points of $\mathsf{PG}(d, q^e)$ to the subsets of points of $\mathsf{PG}(e(d+1)-1, q)$ where a set of points $\mathcal{I}$ in $\mathsf{PG}(d, q^e)$ is sent to the set of points $\mathcal{I}' = \mathfrak{R}(\mathcal{I})$ in $\mathsf{PG}(e(d+1)-1, q)$ consisting of all the points in $(e-1)$-subspaces of $\mathsf{PG}(e(d+1)-1, q)$ corresponding to points of $\mathcal{I}$. One immediate consequence of the definitions is that the image of the pointset of the original polar space $\mathcal{S}$ is contained in the new polar space $\mathcal{S}'$ (but is not necassarily equal to it).

The focus of this section is to consider the question: Given an intriguing set $\mathcal{I}$ of $\mathcal{S}$, when is $\mathcal{I}' = \mathfrak{R}(\mathcal{I})$ an intriguing set of $\mathcal{S}'$? This question is answered in Theorem 1.1, whose proof will be delayed to the end of this section as it relies on the following two lemmas.

We denote by $\perp$ the polarities associated with both $\mathcal{S}$ and $\mathcal{S}'$, with context making clear which is intended. The $\mathrm{GF}(q^e)$-span of $v \in V$ is denoted by $\langle v \rangle_{q^e}$ and the $\mathrm{GF}(q)$-span of $v \in V'$ by $\langle v \rangle_q$. Furthermore, if κ is a quadratic form then β will denote the associated bilinear form. If κ is bilinear then $\beta = \kappa$.

Table 2 Isomorphism types and conditions for field reduction

$\mathcal{S}$	$\mathcal{S}'$	κ'	Conditions/notes
$\mathsf{H}(n-1, q^{2e})$	$\mathsf{H}(en-1, q^2)$	$T\kappa$	e odd
$\mathsf{H}(n-1, q^2)$	$\mathsf{W}(2n-1, q)$	$T\lambda\kappa$	$\lambda \neq 0$, $T\lambda = 0$
$\mathsf{H}(\frac{n-2}{2}, q^2)$	$\mathsf{Q}^-(n-1, q)$	$T\kappa$	$\frac{n}{2}$ is odd
$\mathsf{H}(\frac{n-2}{2}, q^2)$	$\mathsf{Q}^+(n-1, q)$	$T\kappa$	$\frac{n}{2}$ is even
$\mathsf{W}(n-1, q^e)$	$\mathsf{W}(en-1, q)$	$T\kappa$	
$\mathsf{Q}^\varepsilon(n-1, q^e)$	$\mathsf{Q}^\varepsilon(en-1, q)$	$T\kappa$	
$\mathsf{Q}^\varepsilon(\frac{n-2}{2}, q^2)$	$\mathsf{Q}^\varepsilon(n-1, q)$	$T\kappa$	$\frac{qn}{2}$ odd

Lemma 3.1 *Let $P \in \mathsf{PG}(d, q^e)$ and $Q \in \mathcal{S}$ such that $\beta(u, v) \neq 0$ for all non-zero vectors $u \in Q, v \in P$. Then each $P' \in \mathfrak{R}(P)$ is collinear with exactly $\frac{q^{e-1}-1}{q-1}$ points of $\mathfrak{R}(Q)$.*

Proof The restriction of the map $w \mapsto \beta(w, v)$ to Q is a bijection onto $\mathrm{GF}(q^e)$. The $\mathrm{GF}(q)$-linear map L:$\mathrm{GF}(q^e) \to \mathrm{GF}(q)$ defined by $z \mapsto T\lambda z$ has full rank (for suitable λ as in Table 3) so there are q^{e-1} vectors in the kernel and so there are exactly q^{e-1} vectors $w \in Q$ that satisfy $T\lambda\beta(w, v) = 0$. By $\mathrm{GF}(q)$-linearity of $T\lambda\beta$, these vectors form a $\mathrm{GF}(q)$-subspace W and since there are q^{e-1} of them, the rank of W over $\mathrm{GF}(q)$ is $e - 1$. Hence, W contains $\frac{q^{e-1}-1}{q-1}$ points of $\mathcal{S}'$. □

Lemma 3.2 *Let $P \in \mathsf{PG}(d, q^e)$, $P' \in \mathfrak{R}(P) \cap \mathcal{S}'$ and let $A \subseteq \mathcal{S}$ be a set of points. Set $n = |P^\perp \cap A|$ and let $m = |A|$. Then*

$$|P'^\perp \cap \mathfrak{R}(A)| = nq^{e-1} + m\frac{q^{e-1} - 1}{q - 1}.$$

Proof If a point Q is in $P^\perp$ then all points of $\mathfrak{R}(Q)$ are in $P'^\perp$ by definition of κ'. Since the set $\mathfrak{R}(Q)$ forms a totally isotropic subspace of $\mathcal{S}'$ there are $\frac{q^e-1}{q-1}$ points of $\mathcal{S}'$ in $\mathfrak{R}(Q) \cap \mathcal{S}' = \mathfrak{R}(Q)$. If a point Q is not in $P^\perp$, then by Lemma 3.1, there are $\frac{q^{e-1}-1}{q-1}$ points of $\mathfrak{R}(Q)$ in $P'^\perp$. Since there are n points of A in $P^\perp$ and $m - n$ points of A not in $P^\perp$ this gives $n\frac{q^e-1}{q-1} + (m - n)\frac{q^{e-1}-1}{q-1} = nq^{e-1} + m\frac{q^{e-1}-1}{q-1}$, the number of $\mathcal{S}'$ points in $\mathfrak{R}(A)$ collinear with P'. □

We now proceed with the proof of Theorem 1.1.

Proof of Theorem 3.2 The points in $\mathcal{S}'$ are of three kinds, those in $\mathfrak{R}(\mathcal{I})$, those in $\mathfrak{R}(\mathcal{S})\backslash\mathfrak{R}(\mathcal{I})$ and those in $\mathcal{S}'\backslash\mathfrak{R}(\mathcal{S})$. By definition every point in $\mathcal{I}'$ is of the first kind and so by Lemma 3.2 each point in $\mathcal{I}'$ is collinear with the same number of other points in $\mathcal{I}'$. The points not in $\mathcal{I}'$ are of the latter two kinds and so, again by Lemma 3.2, every point not in $\mathcal{I}'$ is collinear with the same number of points of $\mathcal{I}'$ if and only if $|P^\perp \cap \mathcal{I}| = |Q^\perp \cap \mathcal{I}|$ for any two points $P \in \mathsf{PG}\backslash\mathcal{S}$ and $Q \in \mathcal{S}\backslash\mathcal{I}$. Note that $|P^\perp \cap \mathcal{I}|$ and $|Q^\perp \cap \mathcal{I}|$ correspond to n in the statement of Lemma 3.2. Thus, each point in $\mathcal{I}'$ is collinear with the same number h'_1 of points of $\mathcal{I}'$ and each point not in $\mathcal{I}'$ is collinear with the same number h'_2 of points of $\mathcal{I}'$. Now let $h_1 = |P^\perp \cap \mathcal{I}|$ for a point $P \in \mathcal{I}$ and $h_2 = |P^\perp \cap \mathcal{I}|$ for any point $P \in \mathcal{S}\backslash\mathcal{I}$. By Lemma 3.2, $h_1 < h_2$ if and only if $h'_1 < h'_2$ so by J. Bamberg et al. (Sumbitted data, Theorem 6) m-ovoids correspond to m'-ovoids and i-tight sets correspond to i'-tight sets.

To see that $i = i'$ in the case where $\mathcal{I}$ is i-tight we can either obtain i' from the size of $\mathcal{I}'$ or calculate the size of $P^\perp \cap \mathcal{I}'$ using Lemma 3.2. We see similarly that $m' = m|\mathfrak{R}(P)|$ where P is any point of $\mathcal{S}$ and $|\mathfrak{R}(P)|$ the number of $\mathcal{S}'$-points in its image $\mathfrak{R}(P)$ under $\mathfrak{R}$. □

We now apply Theorem 1.1 to each case individually. As all the proofs follow the same lines, only the first case will be proved.

Corollary 3.3 *If $\mathsf{H}(2r, q^{2e})$ admits an m-ovoid $\mathcal{O}$ then*

$\mathsf{H}(e(2r + 1) - 1, q^2)$ *admits an* $m\frac{q^{2e}-1}{q^2-1}$*-ovoid (if e is odd),*

$\mathsf{W}(2e(2r + 1) - 1, q)$ *admits an* $m\frac{q^{2e}-1}{q-1}$*-ovoid, and*

$\mathsf{Q}^-(2e(2r + 1) - 1, q)$ *admits an* $m\frac{q^{2e}-1}{q-1}$*-ovoid.*

Proof In all cases we use field reduction to obtain a set $\mathcal{O}' = \mathfrak{R}(\mathcal{O})$ in the listed polar space (see Table 3). For a point $P \in \mathsf{PG}(2r, q^{2e})\backslash\mathsf{H}(2r, q^{2e})$ we have $|P^{\perp} \cap \mathcal{O}| = m(1 + q^{e(2r-1)})$ by the *mi*-intersection result (J. Bamberg et al., Submitted data, Corollary 5) since $P^{\perp} \cap \mathsf{H}(2r, q^{2e})$ is $(1 + q^{e(2r-1)})$-tight. For a point $Q \in \mathsf{H}(2r, q^{2e})\backslash\mathcal{O}$ we have $|Q^{\perp} \cap \mathcal{O}| = m(1 + q^{e(2r-1)})$. Hence, we can apply Theorem 1.1. □

Corollary 3.4

(1) *If* $\mathsf{H}(2r-1, q^{2e})$ *admits an i-tight set then*

$\mathsf{H}(2er-1, q^2)$ *admits an i-tight set (if e odd),*
$\mathsf{W}(4er-1, q)$ *admits an i-tight set, and*
$\mathsf{Q}^+(4er-1, q)$ *admits an i-tight set.*

(2) *If* $\mathsf{W}(2r-1, q^e)$ *admits an m-ovoid (resp. i-tight set) then* $\mathsf{W}(2re-1, q)$ *admits an* $m\frac{q^e-1}{q-1}$*-ovoid (resp. i-tight set).*
(3) *If* $\mathsf{Q}^+(2r-1, q^e)$ *admits an i-tight set then* $\mathsf{Q}^+(2er-1, q)$ *admits an i-tight set.*
(4) *If* $\mathsf{Q}^-(2r+1, q^e)$ *admits an m-ovoid then* $\mathsf{Q}^-(e(2r+2)-1, q)$ *admits a* $\frac{q^e-1}{q-1}$*-ovoid.*

It is perhaps interesting to note that the image $\mathfrak{R}(\mathcal{S})$ of $\mathcal{S}$ under $\mathfrak{R}$ only covers $\mathcal{S}'$ if $\mathcal{S}$ is symplectic. If $\mathcal{S}$ is not symplectic then the improper intriguing set consisting of the whole polar space $\mathcal{S}$ is mapped by $\mathfrak{R}$ to a proper intriguing set.

4 A generalised derivation

We now present the construction that generalises the Payne and Thas construction [8] of an ovoid of $\mathsf{H}(3, q^2)$. Recall that given a classical ovoid $\mathcal{O}$ of $\mathsf{H}(3, q^2)$ and a secant line ℓ through two points of $\mathcal{O}$ the set

$$\mathcal{O}' = (\mathcal{O}\backslash\ell) \cup (\ell^{\perp} \cap \mathsf{H}(3, q^2))$$

is a non-classical ovoid.

Now let V be a totally isotropic $(r-3)$-space in $\mathsf{H}(2r-1, q^2)$. Then $V^{\perp}$ is a $(r+1)$-space of $\mathsf{PG}(2r-1, q^2)$ which meets $\mathsf{H}(2r-1, q^2)$ in a cone H with vertex V. Furthermore, H subtends an $\mathsf{H}(3, q^2)$. The basic idea is to perform the Payne and Thas derivation on the $\mathsf{H}(3, q^2)$ subtended by H. As it turns out, we can weaken the conditions in this more general setting as well. We begin by proving the first part of Theorem 1.2 which is restated as Proposition 4.1.

Proposition 4.1 *Let $\mathcal{O}$ be an m-ovoid of a polar space $\mathcal{S}$ of rank r in* $\mathsf{PG}(2r-1, q)$ *and let ℓ be a dimension $(r-1)$ subspace of* $\mathsf{PG}(2r-1, q)$ *that satisfies* $\ell^{\perp} \cap \mathcal{O} \subseteq \ell$. *Then*

$$\mathcal{O}' = (\mathcal{O}\backslash\ell) \cup (\ell^{\perp} \cap \mathcal{S})$$

is also an m-ovoid of $\mathcal{S}$.

Proof We show that for an arbitrary maximal M the number of points in $M \cap \mathcal{O}'$ is the same as the number in $M \cap \mathcal{O}$. Since $\mathcal{O}$ is an m-ovoid, the result follows. For an arbitrary maximal M the number of points in $M \cap \mathcal{O}'$ is

$$|M \cap \mathcal{O}'| = |M \cap \mathcal{O}| - |M \cap \ell| + |M \cap \ell^{\perp}| - |M \cap \ell^{\perp} \cap (\mathcal{O}\backslash\ell)|.$$

It follows from properties of $\perp$ and M being a maximal that $\dim M \cap \ell^{\perp} = \dim M \cap \ell$ and so $|M \cap \ell^{\perp}| - |M \cap \ell| = 0$. Furthermore, by assumption $\ell^{\perp} \cap \mathcal{O} \subseteq \ell$ and so $\ell^{\perp} \cap (\mathcal{O} \backslash \ell)$ is empty. Hence $|M \cap \mathcal{O}'| = |M \cap \mathcal{O}|$ for every maximal M and so $\mathcal{O}'$ is an m-ovoid. □

We now show the second part of Theorem 1.2 which is restated in the following proposition.

Proposition 4.2 *Let $\mathcal{S}$ be either $\mathsf{H}(2r-1, q^2)$ or $\mathsf{Q}^+(2r-1, q)$, let P be a point in the ambient projective space not contained in $\mathcal{S}$ and let $\mathcal{O} = P^{\perp} \cap \mathcal{S}$ be the corresponding classical m-ovoid. Then there exists a subspace $\ell \subseteq P^{\perp}$ of dimension $(r-1)$ such that $\ell^{\perp} \cap \mathcal{O} \subseteq \ell$. Furthermore, the m-ovoid $\mathcal{O}'$ of Proposition 4.1 is not a classical m-ovoid.*

Proof The set of points $P^{\perp} \cap \mathcal{S}$ is a rank $r-1$ polar space and so has maximals of (projective) dimension $r-2$. Let V be a subspace of dimension $r-3$ of a maximal of $P^{\perp} \cap \mathcal{S}$. Now $V^{\perp}$ has dimension $r+1$. Since V is totally isotropic, $V \subset V^{\perp}$ and so we can decompose $V^{\perp}$ into $V^{\perp} = V \oplus U$. The form β on U is non-degenerate and so defines a polar space $\mathcal{S}'$. Clearly, there is a hyperbolic line ℓ' in $\mathcal{S}'$ so set $\ell = V + \ell'$.

Suppose that there is a totally singular point Q in $\ell^{\perp} \cap P^{\perp}$ that is not in ℓ. Let R' be a totally singular point of ℓ' and let $R = V + R'$. Note that R is a totally isotropic subspace of dimension $r-2$. Now $Q \in \ell^{\perp} \subset R^{\perp}$, as $R \subset \ell$, so Q is collinear with every point in R. Thus, $R + Q$ is a totally isotropic subspace of dimension $r-1$, that is, a maximal of $\mathcal{S}$. But $P^{\perp} \cap \mathcal{S}$ is a proper m-ovoid and so cannot contain a maximal of $\mathcal{S}$. Hence, $\ell^{\perp} \cap (P^{\perp} \cap \mathcal{S}) \subseteq \ell$.

The only thing left to show is that the new m-ovoid $\mathcal{O}' = (\mathcal{O} \backslash \ell) \cup (\ell^{\perp} \cap \mathcal{S})$ (where $\mathcal{O} = P^{\perp} \cap \mathcal{S}$) is not contained in a hyperplane. To see this, note that we are removing a proper subspace ℓ from another subspace $P^{\perp}$ so the remaining vectors $P^{\perp} \backslash \ell$ still span $P^{\perp}$. So long as there is at least one vector in $\ell^{\perp}$ that is not in $P^{\perp}$ the new m-ovoid will span the whole space and therefore not be classical. That there is such a vector follows from the property $\ell^{\perp} \cap (P^{\perp} \cap \mathcal{S}) \subseteq \ell$ as follows: The sizes of $\mathcal{O}$ and $\mathcal{O}'$ must be the same so there are the same number of totally singular points in ℓ as there are in $\ell^{\perp}$. All the totally singular points in $\ell^{\perp}$ that are in $P^{\perp}$ are contained in ℓ. This cannot be all of them since that would imply that $\ell = \ell^{\perp}$. So there is at least one totally singular point of $\ell^{\perp}$ not in $P^{\perp}$ and therefore $\mathcal{O}'$ spans the whole space. Since $\mathcal{O}'$ is not contained in a hyperplane it is not a classical m-ovoid. □

Acknowledgments The author thanks John Bamberg and Maska Law as well as his honours supervisors Michael Giudici and Tim Penttila for their encouragement, patience, guidance and proofreading.

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