



Two Linear Transformations each Tridiagonal with Respect to an Eigenbasis of the other; Comments on the Parameter Array

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Abstract. Let \mathbb{K} denote a field. Let d denote a nonnegative integer and consider a sequence $p = (\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ consisting of scalars taken from \mathbb{K} . We call p a *parameter array* whenever: (PA1) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ if $i \neq j, (0 \leq i, j \leq d)$; (PA2) $\varphi_i \neq 0, \phi_i \neq 0 (1 \leq i \leq d)$; (PA3) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} (\theta_h - \theta_{d-h}) / (\theta_0 - \theta_d) + (\theta_i^* - \theta_0^*) (\theta_{i-1} - \theta_d)$ ($1 \leq i \leq d$); (PA4) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} (\theta_h - \theta_{d-h}) / (\theta_0 - \theta_d) + (\theta_i^* - \theta_0^*) (\theta_{d-i+1} - \theta_0)$ ($1 \leq i \leq d$); (PA5) $(\theta_{i-2} - \theta_{i+1})(\theta_{i-1} - \theta_i)^{-1}, (\theta_{i-2}^* - \theta_{i+1}^*)(\theta_{i-1}^* - \theta_i^*)^{-1}$ are equal and independent of i for $2 \leq i \leq d-1$. In Terwilliger, *Linear Algebra Appl.*, Vol. 330(2001) p. 155 we showed the parameter arrays are in bijection with the isomorphism classes of Leonard systems. Using this bijection we obtain the following two characterizations of parameter arrays. Assume p satisfies PA1 and PA2. Let A, B, A^*, B^* denote the matrices in $\text{Mat}_{d+1}(\mathbb{K})$ which have entries $A_{ii} = \theta_i, B_{ii} = \theta_{d-i}, A_{ii}^* = \theta_i^*, B_{ii}^* = \theta_{d-i}^* (0 \leq i \leq d), A_{i,i-1} = 1, B_{i,i-1} = 1, A_{i-1,i}^* = \varphi_i, B_{i-1,i}^* = \phi_i (1 \leq i \leq d)$, and all other entries 0. We show the following are equivalent: (i) p satisfies PA3–PA5; (ii) there exists an invertible $G \in \text{Mat}_{d+1}(\mathbb{K})$ such that $G^{-1}AG = B$ and $G^{-1}A^*G = B^*$; (iii) for $0 \leq i \leq d$ the polynomial

$$\sum_{n=0}^i \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\varphi_1 \varphi_2 \cdots \varphi_n}$$

is a scalar multiple of the polynomial

$$\sum_{n=0}^i \frac{(\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-n+1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\phi_1 \phi_2 \cdots \phi_n}.$$

We display all the parameter arrays in parametric form. For each array we compute the above polynomials. The resulting polynomials form a class consisting of the q -Racah, q -Hahn, dual q -Hahn, q -Krawtchouk, dual q -Krawtchouk, quantum q -Krawtchouk, affine q -Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito, and Orphan polynomials. The Bannai/Ito polynomials can be obtained from the q -Racah polynomials by letting q tend to -1 . The Orphan polynomials have maximal degree 3 and exist for $\text{char}(\mathbb{K})=2$ only. For each of the polynomials listed above we give the orthogonality, 3-term recurrence, and difference equation in terms of the parameter array.

Keywords: Leonard pair, tridiagonal pair, Askey–Wilson polynomial, q -Racah polynomial

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1. Introduction

In this paper we continue to develop the theory of Leonard pairs and Leonard systems [6, 13–19]. We briefly summarize our results so far. In [13] we introduced the

notion of a Leonard pair and the closely related notion of a Leonard system (see Section 2). We classified the Leonard systems. In the process we introduced the split decomposition for Leonard systems. Moreover, we showed that every Leonard pair satisfies two cubic polynomial relations which we call the tridiagonal relations. The tridiagonal relations generalize both the cubic q -Serre relations and the Dolan–Grady relations. In [6] we introduced a generalization of a Leonard pair (resp. system), which we call a tridiagonal pair (resp. system). We extended some of our results on Leonard pairs and systems to tridiagonal pairs and systems. For instance we showed that every tridiagonal system has a split decomposition. Moreover, we showed that every tridiagonal pair satisfies an appropriate pair of tridiagonal relations. We did not get a classification of tridiagonal systems and to our knowledge this remains an open problem. In [14] we introduced the tridiagonal algebra. This is an associative algebra on two generators subject to a pair of tridiagonal relations. We showed that every tridiagonal pair induces on the underlying vector space the structure of an irreducible module for a tridiagonal algebra. Given an irreducible finite dimensional module for a tridiagonal algebra, we displayed sufficient conditions for it to be induced from a tridiagonal pair in this fashion. We also showed each sequence of Askey–Wilson polynomials gives a basis for an appropriate infinite dimensional irreducible tridiagonal algebra module. In [15] we began with an arbitrary Leonard pair, and exhibited 24 bases for the underlying vector space which we found attractive. For each of these bases we computed the matrices which represent the Leonard pair. We found each of these matrices is tridiagonal, diagonal, upper bidiagonal or lower bidiagonal. We computed the transition matrix for sufficiently many ordered pairs of bases in our set of 24 to enable one to readily find the transition matrix for any ordered pair of bases in our set of 24. In the survey [16] we gave a number of examples of Leonard pairs. We used these examples to illustrate how Leonard pairs arise in representation theory, combinatorics, and the theory of orthogonal polynomials. The paper [17] is another survey. In [18] we introduced the notion of a parameter array. We showed that the classification of Leonard systems mentioned above gives a bijection from the set of isomorphism classes of Leonard systems to the set of parameter arrays. We introduced the TD - D canonical form and the LB - UB canonical form for Leonard systems. For a Leonard system in TD - D canonical form the associated Leonard pair is represented by a tridiagonal and diagonal matrix, subject to a certain normalization. For a Leonard system in LB - UB canonical form the associated Leonard pair is represented by a lower bidiagonal and upper bidiagonal matrix, subject to a certain normalization. We showed every Leonard system is isomorphic to a unique Leonard system which is in TD - D canonical form and a unique Leonard system which is in LB - UB canonical form. We described these canonical forms using the associated parameter array. In [19] we obtained two characterizations of Leonard pairs based on the split decomposition.

We now give an overview of the present paper. We first review our bijection between the set of isomorphism classes of Leonard systems and the set of parameter arrays. We then use this bijection to obtain two characterizations of Leonard systems. The first characterization involves bidiagonal matrices and is given

in Theorem 3.2. The second characterization involves polynomials and is given in Theorem 4.1. We view Theorem 4.1 as a variation on a theorem of Leonard [2, p.260], [9]. In Section 5 we display all the parameter arrays. For each parameter array we display the corresponding polynomials from our second characterization. These corresponding polynomials form a class consisting of the q -Racah, q -Hahn, dual q -Hahn, q -Krawtchouk, dual q -Krawtchouk, quantum q -Krawtchouk, affine q -Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito, and Orphan polynomials. The Bannai/Ito polynomials can be obtained from the q -Racah polynomials by letting q tend to -1 . The Orphan polynomials have maximal degree 3 and exist for $\text{char}(\mathbb{K})=2$ only. For each of the polynomials listed above we give the orthogonality, 3-term recurrence, and difference equation in terms of the parameter array. We conclude the paper with an open problem.

We now recall the definition of a parameter array. For the rest of this paper \mathbb{K} will denote a field.

Definition 1.1. Let d denote a nonnegative integer. By a *parameter array over \mathbb{K} of diameter d* we mean a sequence of scalars $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ taken from \mathbb{K} , which satisfy the following conditions (PA1)–(PA5):

$$(PA1) \quad \theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad \text{if } i \neq j, \quad 0 \leq i, j \leq d,$$

$$(PA2) \quad \varphi_i \neq 0, \quad \phi_i \neq 0, \quad 1 \leq i \leq d,$$

$$(PA3) \quad \varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d), \quad 1 \leq i \leq d,$$

$$(PA4) \quad \phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0), \quad 1 \leq i \leq d,$$

(PA5) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \tag{1}$$

are equal and independent of i for $2 \leq i \leq d-1$.

We now turn our attention to Leonard systems.

2. Parameter Arrays and Leonard Systems

We recall the notion of a Leonard system and discuss how these objects are related to parameter arrays. Our account will be brief; for more detail see [13, 15, 16, 18]. Let d denote a nonnegative integer. Let $\text{Mat}_{d+1}(\mathbb{K})$ denote the \mathbb{K} -algebra consisting of all $d+1$ by $d+1$ matrices which have entries in \mathbb{K} . We index the rows and columns by $0, 1, \dots, d$. Let \mathcal{A} denote a \mathbb{K} -algebra isomorphic to $\text{Mat}_{d+1}(\mathbb{K})$. An element $A \in \mathcal{A}$ is called *multiplicity-free* whenever it has $d+1$ mutually distinct eigenvalues in \mathbb{K} . Let A denote a multiplicity-free element of \mathcal{A} . Let $\theta_0, \theta_1, \dots, \theta_d$

denote an ordering of the eigenvalues of A , and for $0 \leq i \leq d$ put

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j},$$

where I denotes the identity of \mathcal{A} . We observe $AE_i = \theta_i E_i$ ($0 \leq i \leq d$); (ii) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq d$); (iii) $\sum_{i=0}^d E_i = I$. Let \mathcal{D} denote the subalgebra of \mathcal{A} generated by A . Using (i)–(iii) we find E_0, E_1, \dots, E_d form a basis for the \mathbb{K} -vector space \mathcal{D} . We call E_i the *primitive idempotent* of A associated with θ_i . By a *Leonard system* in \mathcal{A} we mean a sequence $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$, which satisfies the following (i)–(v):

- (i) each of A, A^* is a multiplicity-free element of \mathcal{A} ,
- (ii) E_0, E_1, \dots, E_d is an ordering of the primitive idempotents of A ,
- (iii) $E_0^*, E_1^*, \dots, E_d^*$ is an ordering of the primitive idempotents of A^* ,
- (iv) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1, \end{cases} \quad (0 \leq i, j \leq d),$
- (v) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1, \end{cases} \quad (0 \leq i, j \leq d).$

We call \mathcal{A} the *ambient algebra* of Φ and say Φ is *over* \mathbb{K} [13, Definition 1.4].

Let $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in \mathcal{A} . Then each of the following is a Leonard system in \mathcal{A} :

$$\begin{aligned} \Phi^* &:= (A^*; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d), \\ \Phi^\downarrow &:= (A; A^*; \{E_i\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d), \\ \Phi^\Downarrow &:= (A; A^*; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d). \end{aligned}$$

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{2}$$

$$\Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \downarrow \Downarrow = \Downarrow \downarrow. \tag{3}$$

The group generated by the symbols $*, \downarrow, \Downarrow$ subject to the relations (2), (3) is the dihedral group D_4 . We recall D_4 is the group of symmetries of a square, and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all Leonard systems.

Let $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in \mathcal{A} . In order to describe Φ we define some parameters. For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We call $\theta_0, \theta_1, \dots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \dots, \theta_d^*$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ . Let V denote an irreducible left \mathcal{A} -module.

By a *decomposition* of V we mean a sequence U_0, U_1, \dots, U_d consisting of 1-dimensional subspaces of V such that

$$V = U_0 + U_1 + \dots + U_d, \quad (\text{direct sum}).$$

By [13, Theorem 3.2] there exists a unique decomposition U_0, U_1, \dots, U_d of V such that both

$$(A - \theta_i I)U_i = U_{i+1}, \quad (0 \leq i \leq d-1), \quad (A - \theta_d I)U_d = 0, \quad (4)$$

$$(A^* - \theta_i^* I)U_i = U_{i-1}, \quad (1 \leq i \leq d), \quad (A^* - \theta_0^* I)U_0 = 0. \quad (5)$$

Pick any integer i ($1 \leq i \leq d$). Then $(A^* - \theta_i^* I)U_i = U_{i-1}$ and $(A - \theta_{i-1} I)U_{i-1} = U_i$. Apparently U_i is an eigenspace for $(A - \theta_{i-1} I)(A^* - \theta_i^* I)$ and the corresponding eigenvalue is a nonzero scalar in \mathbb{K} . We denote this eigenvalue by φ_i . We call $\varphi_1, \varphi_2, \dots, \varphi_d$ the *first split sequence* of Φ . We let $\phi_1, \phi_2, \dots, \phi_d$ denote the first split sequence of Φ^\downarrow and call this the *second split sequence* of Φ .

We recall the notion of *isomorphism* for Leonard systems. Let $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in \mathcal{A} and let $\sigma: \mathcal{A} \rightarrow \mathcal{A}'$ denote an isomorphism of \mathbb{K} -algebras. We write $\Phi^\sigma = (A^\sigma; A^{*\sigma}; \{E_i^\sigma\}_{i=0}^d; \{E_i^{*\sigma}\}_{i=0}^d)$ and observe Φ^σ is a Leonard system in \mathcal{A}' . Let Φ and Φ' denote any Leonard systems over \mathbb{K} . By an *isomorphism of Leonard systems* from Φ to Φ' we mean an isomorphism of \mathbb{K} -algebras from the ambient algebra of Φ to the ambient algebra of Φ' such that $\Phi^\sigma = \Phi'$. We say Φ and Φ' are *isomorphic* whenever there exists an isomorphism of Leonard systems from Φ to Φ' .

THEOREM 2.1 [13, Theorem 1.9]. *Let d denote a nonnegative integer and let $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a sequence of scalars taken from \mathbb{K} . Then the following (i), (ii) are equivalent:*

- (i) *the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ is a parameter array over \mathbb{K} ;*
- (ii) *there exists a Leonard system Φ over \mathbb{K} , which has eigenvalue sequence $\theta_0, \theta_1, \dots, \theta_d$, dual eigenvalue sequence $\theta_0^*, \theta_1^*, \dots, \theta_d^*$, first split sequence $\varphi_1, \varphi_2, \dots, \varphi_d$ and second split sequence $\phi_1, \phi_2, \dots, \phi_d$.*

Suppose (i), (ii) hold. Then Φ is unique up to isomorphism of Leonard systems.

Let $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a Leonard system. By the *parameter array* of Φ we mean the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$, where $\theta_0, \theta_1, \dots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \dots, \theta_d^*$) is the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ and $\varphi_1, \varphi_2, \dots, \varphi_d$ (resp. $\phi_1, \phi_2, \dots, \phi_d$) is the first split sequence (resp. second split sequence) of Φ . By Theorem 2.1 the map which sends a given Leonard system to its parameter array induces a bijection from the set of isomorphism classes of Leonard systems over \mathbb{K} to the set of parameter arrays over \mathbb{K} .

Earlier we mentioned an action of D_4 on the set of Leonard systems. The above bijection induces an action of D_4 on the set of parameter arrays. This action is described as follows.

LEMMA 2.2 [13, Theorem 1.11]. Let $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a Leonard system and let $p = (\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote the corresponding parameter array.

- (i) The parameter array of Φ^* is p^* , where $p^* := (\theta_i^*, \theta_i, i = 0 \dots d; \varphi_j, \phi_{d-j+1}, j = 1 \dots d)$;
- (ii) the parameter array of Φ^\downarrow is p^\downarrow , where $p^\downarrow := (\theta_i, \theta_{d-i}^*, i = 0 \dots d; \phi_{d-j+1}, \varphi_{d-j+1}, j = 1 \dots d)$;
- (iii) the parameter array of Φ^\updownarrow is p^\updownarrow , where $p^\updownarrow := (\theta_{d-i}, \theta_i^*, i = 0 \dots d; \phi_j, \varphi_j, j = 1 \dots d)$.

3. Parameter Arrays and Bidiagonal Matrices

In this section we characterize the parameter arrays in terms of bidiagonal matrices. We will refer to the following set-up.

Definition 3.1. Let d denote a nonnegative integer and let $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a sequence of scalars taken from \mathbb{K} . We assume this sequence satisfies PA1 and PA2.

THEOREM 3.2. With reference to Definition 3.1, the following (i), (ii) are equivalent:

- (i) the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ satisfies PA3–PA5;
- (ii) there exists an invertible matrix $G \in \text{Mat}_{d+1}(\mathbb{K})$ such that both

$$G^{-1} \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix} G = \begin{pmatrix} \theta_d & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & \\ & 1 & \theta_{d-2} & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_0 \end{pmatrix}, \tag{6}$$

$$G^{-1} \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ \mathbf{0} & & & \ddots & \varphi_d \\ & & & & \theta_d^* \end{pmatrix} G = \begin{pmatrix} \theta_0^* & \phi_1 & & & \mathbf{0} \\ & \theta_1^* & \phi_2 & & \\ & & \theta_2^* & \ddots & \\ \mathbf{0} & & & \ddots & \phi_d \\ & & & & \theta_d^* \end{pmatrix}. \tag{7}$$

Proof. (i) \Rightarrow (ii) The sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ satisfies PA1–PA5 and is therefore a parameter array over \mathbb{K} . By Theorem 2.1 there exists a Leonard system over \mathbb{K} which has eigenvalue $\theta_0, \theta_1, \dots, \theta_d$, dual eigenvalue sequence $\theta_0^*, \theta_1^*, \dots, \theta_d^*$, first split sequence $\varphi_1, \varphi_2, \dots, \varphi_d$ and second split

sequence $\phi_1, \phi_2, \dots, \phi_d$. We denote this system by $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$. Let \mathcal{A} denote the ambient algebra of Φ and let V denote an irreducible left \mathcal{A} -module. Let U_0, U_1, \dots, U_d denote the decomposition of V which satisfies (4) and (5). For $0 \leq i \leq d$ let u_i denote a nonzero vector in U_i and observe u_0, u_1, \dots, u_d is a basis for V . Normalizing this basis we may assume $(A - \theta_i I)u_i = u_{i+1}$ for $0 \leq i \leq d - 1$ and $(A - \theta_d I)u_d = 0$. Since $\phi_1, \phi_2, \dots, \phi_d$ is the first split sequence of Φ we have $(A^* - \theta_i^* I)u_i = \phi_i u_{i-1}$ ($1 \leq i \leq d$), $(A^* - \theta_0^* I)u_0 = 0$. Applying these comments to Φ^\downarrow we find there exists a basis v_0, v_1, \dots, v_d for V such that $(A - \theta_{d-i} I)v_i = v_{i+1}$ ($0 \leq i \leq d - 1$) $(A - \theta_0 I)v_d = 0$ and $(A^* - \theta_i^* I)v_i = \phi_i v_{i-1}$ ($1 \leq i \leq d$), $(A^* - \theta_0^* I)v_0 = 0$. Let $G \in \text{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from the basis u_0, u_1, \dots, u_d to the basis v_0, v_1, \dots, v_d , so that $v_j = \sum_{i=0}^d G_{ij}u_i$ for $0 \leq j \leq d$. Using this and elementary linear algebra we find G is invertible and satisfies (6) and (7).

(ii) \Rightarrow (i) We apply Theorem 2.1. We show condition (ii) holds in that theorem. In order to do this we invoke some results from [19]. Consider the following matrices in $\text{Mat}_{d+1}(\mathbb{K})$:

$$A = \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & \phi_1 & & & \mathbf{0} \\ & \theta_1^* & \phi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \phi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}.$$

We observe A (resp. A^*) is multiplicity-free, with eigenvalues $\theta_0, \theta_1, \dots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \dots, \theta_d^*$). For $0 \leq i \leq d$ we let E_i (resp. E_i^*) denote the primitive idempotent for A (resp. A^*) associated with θ_i (resp. θ_i^*). By [19, Lemma 6.2, Theorem 6.3] the sequence $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system in $\text{Mat}_{d+1}(\mathbb{K})$. Let us call this system Φ . By the construction Φ has eigenvalue sequence $\theta_0, \theta_1, \dots, \theta_d$ and dual eigenvalue sequence $\theta_0^*, \theta_1^*, \dots, \theta_d^*$. From the form of A and A^* we find $\phi_1, \phi_2, \dots, \phi_d$ is the first split sequence for Φ . By the last line of [19, Theorem 6.3] we find $\phi_1, \phi_2, \dots, \phi_d$ is the second split sequence for Φ . Now Theorem 2.1(ii) holds; applying that theorem we find $(\theta_i, \theta_i^*, i = 0 \dots d; \phi_j, \phi_j, j = 1 \dots d)$ is a parameter array over \mathbb{K} . In particular $(\theta_i, \theta_i^*, i = 0 \dots d; \phi_j, \phi_j, j = 1 \dots d)$ satisfies PA3–PA5. ■

The matrix G from Theorem 3.2(ii) will be discussed further in Section 10.

4. Parameter Arrays and Polynomials

In this section we characterize the parameter arrays in terms of polynomials. We will use the following notation. Let λ denote an indeterminate, and let $\mathbb{K}[\lambda]$ denote the \mathbb{K} -algebra consisting of all polynomials in λ which have coefficients in \mathbb{K} . For the rest of this paper all polynomials which we discuss are assumed to lie in $\mathbb{K}[\lambda]$.

We view the following theorem as a variation on a theorem of Leonard [2, p. 260], [9].

THEOREM 4.1. *With reference to Definition 3.1, the following (i), (ii) are equivalent:*

- (i) the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ satisfies PA3–PA5;
- (ii) for $0 \leq i \leq d$ the polynomial

$$\sum_{n=0}^i \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\varphi_1 \varphi_2 \cdots \varphi_n} \tag{8}$$

is a scalar multiple of the polynomial

$$\sum_{n=0}^i \frac{(\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-n+1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\phi_1 \phi_2 \cdots \phi_n}. \tag{9}$$

Proof. Let us abbreviate

$$A = \begin{pmatrix} \theta_0 & & & \mathbf{0} \\ 1 & \theta_1 & & \\ & 1 & \theta_2 & \\ & & \ddots & \ddots \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & \\ & & \theta_2^* & \ddots \\ & & & \ddots & \varphi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}$$

and

$$B = \begin{pmatrix} \theta_d & & & \mathbf{0} \\ 1 & \theta_{d-1} & & \\ & 1 & \theta_{d-2} & \\ & & \ddots & \ddots \\ \mathbf{0} & & & 1 & \theta_0 \end{pmatrix}, \quad B^* = \begin{pmatrix} \theta_0^* & \phi_1 & & \mathbf{0} \\ & \theta_1^* & \phi_2 & \\ & & \theta_2^* & \ddots \\ & & & \ddots & \phi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}.$$

We let T, T^*, T^\downarrow denote the matrices in $\text{Mat}_{d+1}(\mathbb{K})$ which have entries $T_{ij} = \prod_{h=0}^{j-1} (\theta_i - \theta_h)$, $T_{ij}^* = \prod_{h=0}^{j-1} (\theta_i^* - \theta_h^*)$, $T_{ij}^\downarrow = \prod_{h=0}^{j-1} (\theta_{d-i} - \theta_{d-h})$ for $0 \leq i, j \leq d$. Each of T, T^*, T^\downarrow is lower triangular with diagonal entries nonzero so these matrices are invertible. Let D (resp. D^\downarrow) denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{K})$, which has ii th entry $\varphi_1 \varphi_2 \cdots \varphi_i$ (resp. $\phi_1 \phi_2 \cdots \phi_i$) for $0 \leq i \leq d$. Each of D, D^\downarrow is invertible. We let Z denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$, which has ij th entry 1 if $i + j = d$ and 0 if $i + j \neq d$, for $0 \leq i, j \leq d$. Observe $Z^2 = I$ so Z is invertible. We let H (resp. H^*) denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{K})$, which has ii th entry θ_i (resp. θ_i^*) for $0 \leq i \leq d$. One verifies $TA = HT$ so $A = T^{-1}HT$. One verifies $ZT^\downarrow B = HZT^\downarrow$ so $B = T^{\downarrow-1}ZHZT^\downarrow$. One verifies $DA^*D^{-1}T^{*t} = T^{*t}H^*$ so $A^* = D^{-1}T^{*t}H^*T^{*-1t}D$. Similarly $B^* = D^{\downarrow-1}T^{*t}H^*T^{*-1t}D^\downarrow$. For $0 \leq i \leq d$ let f_i denote the polynomial in (8). Let \mathcal{P} denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$, which has ij th entry $f_j(\theta_i)$ for $0 \leq i, j \leq d$. From the form of (8) we find $\mathcal{P} = TD^{-1}T^{*t}$. For $0 \leq i \leq d$ let f_i^\downarrow denote the polynomial in (9). Let \mathcal{P}^\downarrow denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$, which has ij th entry $f_j^\downarrow(\theta_i)$ for $0 \leq i, j \leq d$. From the form of (9) we find $\mathcal{P}^\downarrow = ZT^\downarrow D^{\downarrow-1}T^{*t}$.

(i) \Rightarrow (ii) By Theorem 3.2 there exists an invertible matrix $G \in \text{Mat}_{d+1}(\mathbb{K})$ such that $G^{-1}AG = B$ and $G^{-1}A^*G = B^*$. Evaluating $G^{-1}A^*G = B^*$ using $A^* = D^{-1}T^{*t}H^*T^{*-1t}D$ and $B^* = D^{\downarrow-1}T^{*t}H^*T^{*-1t}D^{\downarrow}$ we find $T^{*-1t}DGD^{\downarrow-1}T^{*t}$ commutes with H^* . Since H^* is diagonal with diagonal entries mutually distinct we find $T^{*-1t}DGD^{\downarrow-1}T^{*t}$ is diagonal. We denote this diagonal matrix by F and observe $G = D^{-1}T^{*t}FT^{*-1t}D^{\downarrow}$. In this product each factor is upper triangular (or diagonal) so G is upper triangular. Recall $G^{-1}AG = B$; evaluating this using $A = T^{-1}HT$ and $B = T^{\downarrow-1}ZHZT^{\downarrow}$ we find $TGT^{\downarrow-1}Z$ commutes with H . Since H is diagonal with diagonal entries mutually distinct we find $TGT^{\downarrow-1}Z$ is diagonal. We denote this diagonal matrix by Y and observe $TG = YZT^{\downarrow}$. In this equation we compute the entries in column 0. To aid in this calculation we recall G is upper triangular and observe $T_{i0} = 1, T_{i0}^{\downarrow} = 1$ for $0 \leq i \leq d$. Computing the column 0 entries in $TG = YZT^{\downarrow}$ using these facts we find $Y_{ii} = G_{00}$ for $0 \leq i \leq d$. Apparently $Y = G_{00}I$. We remark $G_{00} \neq 0$ since Y is invertible by the construction. Dividing G by G_{00} we may assume $G_{00} = 1$. Now $Y = I$ so $G = T^{-1}ZT^{\downarrow}$. Recall $T^{*-1t}DGD^{\downarrow-1}T^{*t}$ is diagonal. We evaluate this expression using $G = T^{-1}ZT^{\downarrow}$ and find $T^{*-1t}DT^{-1}ZT^{\downarrow}D^{\downarrow-1}T^{*t}$ is diagonal. But $T^{*-1t}DT^{-1}ZT^{\downarrow}D^{\downarrow-1}T^{*t} = \mathcal{P}^{-1}\mathcal{P}^{\downarrow}$ so $\mathcal{P}^{-1}\mathcal{P}^{\downarrow}$ is diagonal. Taking the inverse we find $\mathcal{P}^{\downarrow-1}\mathcal{P}$ is diagonal. For $0 \leq i \leq d$ let α_i denote the ii entry of this diagonal matrix. From the definition of \mathcal{P} and \mathcal{P}^{\downarrow} we find $f_i(\theta_j) = \alpha_i f_i^{\downarrow}(\theta_j)$ for $0 \leq i, j \leq d$. Recall $\theta_0, \theta_1, \dots, \theta_d$ are mutually distinct, and that each of f_i, f_i^{\downarrow} has degree i for $0 \leq i \leq d$. From these comments we find $f_i = \alpha_i f_i^{\downarrow}$ for $0 \leq i \leq d$.

(ii) \Rightarrow (i) We show Theorem 3.2(ii) holds. To do this we exhibit an invertible matrix $G \in \text{Mat}_{d+1}(\mathbb{K})$ such that $AG = GB$ and $A^*G = GB^*$. We define $G = T^{-1}ZT^{\downarrow}$. Observe G is invertible. The equation $AG = GB$ is routinely verified by evaluating A, B, G using $A = T^{-1}HT, B = T^{\downarrow-1}ZHZT^{\downarrow}, G = T^{-1}ZT^{\downarrow}$. We now show $A^*G = GB^*$. For $0 \leq i \leq d$ there exists $\alpha_i \in \mathbb{K}$ such that $f_i = \alpha_i f_i^{\downarrow}$. It follows $\mathcal{P} = \mathcal{P}^{\downarrow} \text{diag}(\alpha_0, \alpha_1, \dots, \alpha_d)$. From this and since H^* is diagonal we find $\mathcal{P}H^*\mathcal{P}^{-1} = \mathcal{P}^{\downarrow}H^*\mathcal{P}^{\downarrow-1}$. In this equation we multiply both sides on the left by T^{-1} and on the right by ZT^{\downarrow} to obtain $T^{-1}\mathcal{P}H^*\mathcal{P}^{-1}ZT^{\downarrow} = T^{-1}\mathcal{P}^{\downarrow}H^*\mathcal{P}^{\downarrow-1}ZT^{\downarrow}$. In this equation the left side is equal to A^*G and the right side is equal to GB^* so $A^*G = GB^*$. We have now shown G satisfies Theorem 3.2(ii). Applying that theorem we find $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ satisfies PA3–PA5. ■

We finish this section with a comment.

LEMMA 4.2. Referring to Theorem 4.1, assume the equivalent conditions (i), (ii) from that theorem hold. Then for $0 \leq i \leq d$ the scalar referred to in condition (ii) is equal to

$$\frac{\phi_1\phi_2 \cdots \phi_i}{\varphi_1\varphi_2 \cdots \varphi_i}.$$

Proof. Compare the coefficient of λ^i in (8) and (9). ■

5. The Parameter Arrays

In this section we display all the parameter arrays over \mathbb{K} . We will use the following notation.

Definition 5.1. Let $p = (\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} . By a *base* for p , we mean a nonzero scalar q in the algebraic closure of \mathbb{K} such that $q + q^{-1} + 1$ is equal to the common value of (1) for $2 \leq i \leq d - 1$. We remark on the uniqueness of the base. Suppose $d \geq 3$. If q is a base for p then so is q^{-1} and p has no other base. Suppose $d < 3$. Then any nonzero scalar in the algebraic closure of \mathbb{K} is a base for p .

Definition 5.2. Let $p = (\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} . For $0 \leq i \leq d$ we let f_i denote the following polynomial in $\mathbb{K}[\lambda]$:

$$f_i = \sum_{n=0}^i \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\varphi_1 \varphi_2 \cdots \varphi_n}. \tag{10}$$

We call f_0, f_1, \dots, f_d the *polynomials which correspond to p* .

We now display all the parameter arrays over \mathbb{K} . For each displayed array $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ we give a base and present $f_i(\theta_j)$ for $0 \leq i, j \leq d$, where f_0, f_1, \dots, f_d are the corresponding polynomials. Our presentation is organized as follows. In each of Example 5.3–5.15 we give a family of parameter arrays over \mathbb{K} . In Theorem 5.16 we show every parameter array over \mathbb{K} is contained in at least one of these families.

In each of Example 5.3–5.15 the following implicit assumptions apply: d denotes a nonnegative integer, the scalars $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ are contained in \mathbb{K} , and the scalars $q, h, h^* \dots$ are contained in the algebraic closure of \mathbb{K} .

EXAMPLE 5.3 (*q-Racah*). *Assume*

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \tag{11}$$

$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i} \tag{12}$$

for $0 \leq i \leq d$ and

$$\varphi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - r_1q^i)(1 - r_2q^i), \tag{13}$$

$$\phi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^* \tag{14}$$

for $1 \leq i \leq d$. *Assume $h, h^*, q, s, s^*, r_1, r_2$ are nonzero and $r_1r_2 = ss^*q^{d+1}$. Assume none of $q^i, r_1q^i, r_2q^i, s^*q^i/r_1, s^*q^i/r_2$ is equal to 1 for $1 \leq i \leq d$ and that neither of sq^i, s^*q^i is equal to 1 for $2 \leq i \leq 2d$. Then $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ is a parameter array over \mathbb{K} which has base q . The corresponding polynomials f_i satisfy*

$$f_i(\theta_j) = {}_4\phi_3 \left(\begin{matrix} q^{-i}, s^*q^{i+1}, q^{-j}, sq^{j+1} \\ r_1q, r_2q, q^{-d} \end{matrix} \middle| q, q \right)$$

for $0 \leq i, j \leq d$. These f_i are the q -Racah polynomials.

EXAMPLE 5.4 (q -Hahn). Assume

$$\begin{aligned}\theta_i &= \theta_0 + h(1 - q^i)q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - rq^i), \\ \phi_i &= -hh^*q^{1-i}(1 - q^i)(1 - q^{i-d-1})(r - s^*q^i)\end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, s^*, r are nonzero. Assume none of $q^i, rq^i, s^*q^i/r$ is equal to 1 for $1 \leq i \leq d$ and that $s^*q^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ is a parameter array over \mathbb{K} which has base q . The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i}, s^*q^{i+1}, q^{-j} \\ rq, q^{-d} \end{matrix} \middle| q, q\right)$$

for $0 \leq i, j \leq d$. These f_i are the q -Hahn polynomials.

EXAMPLE 5.5 (dual q -Hahn). Assume

$$\begin{aligned}\theta_i &= \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - rq^i), \\ \phi_i &= hh^*q^{d+2-2i}(1 - q^i)(1 - q^{i-d-1})(s - rq^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, r, s are nonzero. Assume none of $q^i, rq^i, sq^i/r$ is equal to 1 for $1 \leq i \leq d$ and that $sq^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ is a parameter array over \mathbb{K} which has base q . The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i}, q^{-j}, sq^{j+1} \\ rq, q^{-d} \end{matrix} \middle| q, q\right)$$

for $0 \leq i, j \leq d$. These f_i are dual the q -Hahn polynomials.

EXAMPLE 5.6 (quantum q -Krawtchouk). Assume

$$\begin{aligned}\theta_i &= \theta_0 - sq(1 - q^i), \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= -rh^*q^{1-i}(1-q^i)(1-q^{i-d-1}), \\ \phi_i &= h^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$. Assume h^*, q, r, s are nonzero. Assume neither of $q^i, sq^i/r$ is equal to 1 for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ is a parameter array over \mathbb{K} which has base q . The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_2\phi_1\left(\begin{matrix} q^{-i}, q^{-j} \\ q^{-d} \end{matrix} \middle| q, sr^{-1}q^{j+1}\right)$$

for $0 \leq i, j \leq d$. These f_i are the quantum q -Krawtchouk polynomials.

EXAMPLE 5.7 (q -Krawtchouk). Assume

$$\begin{aligned}\theta_i &= \theta_0 + h(1-q^i)q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1}), \\ \phi_i &= hh^*s^*q(1-q^i)(1-q^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, s^* are nonzero. Assume $q^i \neq 1$ for $1 \leq i \leq d$ and that $s^*q^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ is a parameter array over \mathbb{K} which has base q . The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i}, s^*q^{i+1}, q^{-j} \\ 0, q^{-d} \end{matrix} \middle| q, q\right)$$

for $0 \leq i, j \leq d$. These f_i are the q -Krawtchouk polynomials.

EXAMPLE 5.8 (affine q -Krawtchouk). Assume

$$\begin{aligned}\theta_i &= \theta_0 + h(1-q^i)q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1-q^i)q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i), \\ \phi_i &= -hh^*rq^{1-i}(1-q^i)(1-q^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, r are nonzero. Assume neither of q^i, rq^i is equal to 1 for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ is a parameter array over \mathbb{K} which has base q . The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i}, 0, q^{-j} \\ rq, q^{-d} \end{matrix} \middle| q, q\right)$$

for $0 \leq i, j \leq d$. These f_i are the affine q -Krawtchouk polynomials.

EXAMPLE 5.9 (dual q -Krawtchouk). Assume

$$\begin{aligned}\theta_i &= \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)q^{-i}\end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned}\varphi_i &= hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1}), \\ \phi_i &= hh^*sq^{d+2-2i}(1 - q^i)(1 - q^{i-d-1})\end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, s are nonzero. Assume $q^i \neq 1$ for $1 \leq i \leq d$ and $sq^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ is a parameter array over \mathbb{K} which has base q . The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i}, q^{-j}, sq^{j+1} \\ 0, q^{-d} \end{matrix} \middle| q, q\right)$$

for $0 \leq i, j \leq d$. These f_i are the dual q -Krawtchouk polynomials.

EXAMPLE 5.10 (Racah). Assume

$$\theta_i = \theta_0 + hi(i + 1 + s), \tag{15}$$

$$\theta_i^* = \theta_0^* + h^*i(i + 1 + s^*) \tag{16}$$

for $0 \leq i \leq d$ and

$$\varphi_i = hh^*i(i - d - 1)(i + r_1)(i + r_2), \tag{17}$$

$$\phi_i = hh^*i(i - d - 1)(i + s^* - r_1)(i + s^* - r_2) \tag{18}$$

for $1 \leq i \leq d$. Assume h, h^* are nonzero and that $r_1 + r_2 = s + s^* + d + 1$. Assume the characteristic of \mathbb{K} is 0 or a prime greater than d . Assume none of $r_1, r_2, s^* - r_1, s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d$ and that neither of s, s^* is equal to $-i$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ is a parameter array over \mathbb{K} which has base 1. The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_4F_3\left(\begin{matrix} -i, i + 1 + s^*, -j, j + 1 + s \\ r_1 + 1, r_2 + 1, -d \end{matrix} \middle| 1\right)$$

for $0 \leq i, j \leq d$. These f_i are the Racah polynomials.

EXAMPLE 5.11 (Hahn). Assume

$$\theta_i = \theta_0 + si,$$

$$\theta_i^* = \theta_0^* + h^*i(i + 1 + s^*)$$

for $0 \leq i \leq d$ and

$$\varphi_i = h^*si(i - d - 1)(i + r),$$

$$\phi_i = -h^*si(i - d - 1)(i + s^* - r)$$

for $1 \leq i \leq d$. Assume h^*, s are nonzero. Assume the characteristic of \mathbb{K} is 0 or a prime greater than d . Assume neither of $r, s^* - r$ is equal to $-i$ for $1 \leq i \leq d$ and that $s^* \neq -i$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ is a parameter array over \mathbb{K} which has base 1. The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_3F_2 \left(\begin{matrix} -i, i+1+s^*, -j \\ r+1, -d \end{matrix} \middle| 1 \right)$$

for $0 \leq i, j \leq d$. These f_i are the Hahn polynomials.

EXAMPLE 5.12 (dual Hahn). Assume

$$\begin{aligned} \theta_i &= \theta_0 + hi(i+1+s), \\ \theta_i^* &= \theta_0^* + s^*i \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= hs^*i(i-d-1)(i+r), \\ \phi_i &= hs^*i(i-d-1)(i+r-s-d-1) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, s^* are nonzero. Assume the characteristic of \mathbb{K} is 0 or a prime greater than d . Assume neither of $r, s-r$ is equal to $-i$ for $1 \leq i \leq d$ and that $s \neq -i$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ is a parameter array over \mathbb{K} which has base 1. The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_3F_2 \left(\begin{matrix} -i, -j, j+1+s \\ r+1, -d \end{matrix} \middle| 1 \right)$$

for $0 \leq i, j \leq d$. These f_i are the dual Hahn polynomials.

EXAMPLE 5.13 (Krawtchouk). Assume

$$\begin{aligned} \theta_i &= \theta_0 + si, \\ \theta_i^* &= \theta_0^* + s^*i \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= ri(i-d-1) \\ \phi_i &= (r-ss^*)i(i-d-1) \end{aligned}$$

for $1 \leq i \leq d$. Assume r, s, s^* are nonzero. Assume the characteristic of \mathbb{K} is 0 or a prime greater than d . Assume $r \neq ss^*$. Then the sequence $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ is a parameter array over \mathbb{K} which has base 1. The corresponding polynomials f_i satisfy

$$f_i(\theta_j) = {}_2F_1 \left(\begin{matrix} -i, -j \\ -d \end{matrix} \middle| r^{-1}ss^* \right)$$

for $0 \leq i, j \leq d$. These f_i are the Krawtchouk polynomials.

EXAMPLE 5.14 (Bannai/Ito). Assume

$$\theta_i = \theta_0 + h(s - 1 + (1 - s + 2i)(-1)^i), \tag{19}$$

$$\theta_i^* = \theta_0^* + h^*(s^* - 1 + (1 - s^* + 2i)(-1)^i) \tag{20}$$

for $0 \leq i \leq d$ and

$$\varphi_i = \begin{cases} -4hh^*i(i+r_1) & \text{if } i \text{ even, } d \text{ even,} \\ -4hh^*(i-d-1)(i+r_2) & \text{if } i \text{ odd, } d \text{ even,} \\ -4hh^*i(i-d-1) & \text{if } i \text{ even, } d \text{ odd,} \\ -4hh^*(i+r_1)(i+r_2) & \text{if } i \text{ odd, } d \text{ odd,} \end{cases} \tag{21}$$

$$\phi_i = \begin{cases} 4hh^*i(i-s^*-r_1) & \text{if } i \text{ even, } d \text{ even,} \\ 4hh^*(i-d-1)(i-s^*-r_2) & \text{if } i \text{ odd, } d \text{ even,} \\ -4hh^*i(i-d-1) & \text{if } i \text{ even, } d \text{ odd,} \\ -4hh^*(i-s^*-r_1)(i-s^*-r_2) & \text{if } i \text{ odd, } d \text{ odd} \end{cases} \tag{22}$$

for $1 \leq i \leq d$. Assume h, h^* are nonzero and that $r_1 + r_2 = -s - s^* + d + 1$. Assume the characteristic of \mathbb{K} is either 0 or an odd prime greater than $d/2$. Assume neither of $r_1, -s^* - r_1$ is equal to $-i$ for $1 \leq i \leq d, d - i$ even. Assume neither of $r_2, -s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d, i$ odd. Assume neither of s, s^* is equal to $2i$ for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ is a parameter array over \mathbb{K} which has base -1 . We call the corresponding polynomials from Definition 5.2 the Bannai/Ito polynomials [2, p. 260].

EXAMPLE 5.15 (Orphan). For this example assume \mathbb{K} has characteristic 2. For notational convenience we define some scalars $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in \mathbb{K} . We define $\gamma_i = 0$ for $i \in \{0, 3\}$ and $\gamma_i = 1$ for $i \in \{1, 2\}$. Assume

$$\theta_i = \theta_0 + h(si + \gamma_i), \tag{23}$$

$$\theta_i^* = \theta_0^* + h^*(s^*i + \gamma_i) \tag{24}$$

for $0 \leq i \leq 3$. Assume $\varphi_1 = hh^*r, \varphi_2 = hh^*, \varphi_3 = hh^*(r + s + s^*)$ and $\phi_1 = hh^*(r + s(1 + s^*)), \phi_2 = hh^*, \phi_3 = hh^*(r + s^*(1 + s))$. Assume each of h, h^*, s, s^*, r is nonzero. Assume neither of s, s^* is equal to 1 and that r is equal to none of $s + s^*, s(1 + s^*), s^*(1 + s)$. Then the sequence $(\theta_i, \theta_i^*, i = 0 \dots 3; \varphi_j, \phi_j, j = 1 \dots 3)$ is a parameter array over \mathbb{K} which has diameter 3 and base 1. We call the corresponding polynomials from Definition 5.2 the Orphan polynomials.

THEOREM 5.16. Every parameter array over \mathbb{K} is listed in at least one of the Examples 5.3–5.15.

Proof. Let $p := (\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} . We show this array is given in at least one of the Examples 5.3–5.15. We assume $d \geq 1$; otherwise the result is trivial. For notational convenience let \tilde{K} denote the

algebraic closure of \mathbb{K} . Let q denote a base for p as in Definition 5.1. For $d < 3$ we may assume $q \neq 1$ and $q \neq -1$ in view of the remark in Definition 5.1. By PA5 and Definition 5.1 both

$$\theta_{i-2} - [3]_q \theta_{i-1} + [3]_q \theta_i - \theta_{i+1} = 0, \tag{25}$$

$$\theta_{i-2}^* - [3]_q \theta_{i-1}^* + [3]_q \theta_i^* - \theta_{i+1}^* = 0 \tag{26}$$

for $2 \leq i \leq d-1$, where $[3]_q := q + q^{-1} + 1$. We divide the argument into Case I–IV. (I) $q \neq 1, q \neq -1$; (II) $q = 1$ and $\text{char}(\mathbb{K}) \neq 2$; (III) $q = -1$ and $\text{char}(\mathbb{K}) \neq 2$; (IV) $q = 1$ and $\text{char}(\mathbb{K}) = 2$.

Case I. $q \neq 1, q \neq -1$.

By (25) there exist scalars η, μ, h in $\tilde{\mathbb{K}}$ such that

$$\theta_i = \eta + \mu q^i + h q^{-i}, \quad 0 \leq i \leq d. \tag{27}$$

By (26) there exist scalars η^*, μ^*, h^* in $\tilde{\mathbb{K}}$ such that

$$\theta_i^* = \eta^* + \mu^* q^i + h^* q^{-i}, \quad 0 \leq i \leq d. \tag{28}$$

Observe μ, h are not both 0; otherwise $\theta_1 = \theta_0$ by (27). Similarly μ^*, h^* are not both 0. For $1 \leq i \leq d$ we have $q^i \neq 1$; otherwise $\theta_i = \theta_0$ by (27). Setting $i = 0$ in (27) and (28) we obtain

$$\theta_0 = \eta + \mu + h, \tag{29}$$

$$\theta_0^* = \eta^* + \mu^* + h^*. \tag{30}$$

We claim there exists $\tau \in \tilde{\mathbb{K}}$ such that both

$$\varphi_i = (q^i - 1)(q^{d-i+1} - 1)(\tau - \mu \mu^* q^{i-1} - h h^* q^{-i-d}), \tag{31}$$

$$\phi_i = (q^i - 1)(q^{d-i+1} - 1)(\tau - h \mu^* q^{i-d-1} - \mu h^* q^{-i}) \tag{32}$$

for $1 \leq i \leq d$. Since $q \neq 1$ and $q^d \neq 1$ there exists $\tau \in \tilde{\mathbb{K}}$ such that (31) holds for $i = 1$. In the equation of PA4, we eliminate φ_1 using (31) at $i = 1$, and evaluate the result using (27), (28), and [13, Lemma 10.2] in order to obtain (32) for $1 \leq i \leq d$. In the equation of PA3, we eliminate ϕ_1 using (32) at $i = 1$, and evaluate the result using (27), (28), and [13, Lemma 10.2] in order to obtain (31) for $1 \leq i \leq d$. We have now proved the claim. We now break the argument into subcases. For each subcase our argument is similar. We will discuss the first subcase in detail in order to give the idea; for the remaining subcases we give the essentials only.

Subcase q -Racah: $\mu \neq 0, \mu^* \neq 0, h \neq 0, h^* \neq 0$. We show p is listed in Example 5.3. Define

$$s := \mu h^{-1} q^{-1}, \quad s^* := \mu^* h^{*-1} q^{-1}. \tag{33}$$

Eliminating η in (27) using (29) and eliminating μ in the result using the equation on the left in (33), we obtain (11) for $0 \leq i \leq d$. Similarly we obtain (12) for $0 \leq i \leq d$. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains scalars r_1, r_2 such that both

$$r_1 r_2 = s s^* q^{d+1}, \quad r_1 + r_2 = \tau h^{-1} h^{*-1} q^d. \tag{34}$$

Eliminating μ, μ^*, τ in (31), (32) using (33) and the equation on the right in (34), and evaluating the result using the equation on the left in (34), we obtain (13), (14) for $1 \leq i \leq d$. By the construction each of h, h^*, q, s, s^* is nonzero. Each of r_1, r_2 is nonzero by the equation on the left in (34). The remaining inequalities mentioned below (14) follow from PA1, PA2 and (11)–(14). We have now shown p is listed in Example 5.3.

We now give the remaining subcases of Case I. We list the essentials only.

Subcase q -Hahn: $\mu = 0, \mu^* \neq 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definitions.

$$s^* := \mu^* h^{*-1} q^{-1}, \quad r := \tau h^{-1} h^{*-1} q^d.$$

Subcase dual q -Hahn: $\mu \neq 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definitions.

$$s := \mu h^{-1} q^{-1}, \quad r := \tau h^{-1} h^{*-1} q^d.$$

Subcase quantum q -Krawtchouk: $\mu \neq 0, \mu^* = 0, h = 0, h^* \neq 0, \tau \neq 0$. Definitions.

$$s := \mu q^{-1}, \quad r := \tau h^{*-1} q^d.$$

Subcase q -Krawtchouk: $\mu = 0, \mu^* \neq 0, h \neq 0, h^* \neq 0, \tau = 0$. Definition.

$$s^* := \mu^* h^{*-1} q^{-1}.$$

Subcase affine q -Krawtchouk: $\mu = 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definition.

$$r := \tau h^{-1} h^{*-1} q^d.$$

Subcase dual q -Krawtchouk: $\mu \neq 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau = 0$. Definition.

$$s := \mu h^{-1} q^{-1}.$$

We have a few more comments concerning Case I. Earlier we mentioned that μ, h are not both 0 and that μ^*, h^* are not both 0. Suppose one of μ, h is 0 and one of μ^*, h^* is 0. Then $\tau \neq 0$; otherwise $\varphi_1 = 0$ by (31) or $\phi_1 = 0$ by (32). Suppose $\mu^* \neq 0, h^* = 0$. Replacing q by q^{-1} we obtain $\mu^* = 0, h^* \neq 0$. Suppose $\mu^* \neq 0, h^* \neq 0, \mu \neq 0, h = 0$. Replacing q by q^{-1} we obtain $\mu^* \neq 0, h^* \neq 0, \mu = 0, h \neq 0$. By these comments we find that after replacing q by q^{-1} if necessary, one of the above subcases holds. This completes our argument for Case I.

Case II. $q = 1$ and $\text{char}(\mathbb{K}) \neq 2$.

By (25) and since $\text{char}(\mathbb{K}) \neq 2$, there exist scalars η, μ, h in $\tilde{\mathbb{K}}$ such that

$$\theta_i = \eta + (\mu + h)i + hi^2, \quad 0 \leq i \leq d. \tag{35}$$

Similarly there exist scalars η^*, μ^*, h^* in $\tilde{\mathbb{K}}$ such that

$$\theta_i^* = \eta^* + (\mu^* + h^*)i + h^*i^2, \quad 0 \leq i \leq d. \tag{36}$$

Observe μ, h are not both 0; otherwise $\theta_1 = \theta_0$. Similarly μ^*, h^* are not both 0. For any prime i such that $i \leq d$ we have $\text{char}(\mathbb{K}) \neq i$; otherwise $\theta_i = \theta_0$ by (35). Therefore, $\text{char}(\mathbb{K})$ is 0 or a prime greater than d . Setting $i = 0$ in (35), (36) we obtain

$$\theta_0 = \eta, \quad \theta_0^* = \eta^*. \tag{37}$$

We claim there exists $\tau \in \tilde{\mathbb{K}}$ such that both

$$\varphi_i = i(d - i + 1)(\tau - (\mu h^* + h \mu^*)i - h h^* i(i + d + 1)), \tag{38}$$

$$\phi_i = i(d - i + 1)(\tau + \mu \mu^* + h \mu^*(1 + d) + (\mu h^* - h \mu^*)i + h h^* i(d - i + 1)) \tag{39}$$

for $1 \leq i \leq d$. There exists $\tau \in \tilde{\mathbb{K}}$ such that (38) holds for $i = 1$. In the equation of PA4, we eliminate φ_1 using (38) at $i = 1$, and evaluate the result using (35), (36), and [13, Lemma 10.2] in order to obtain (39) for $1 \leq i \leq d$. In the equation of PA3, we eliminate ϕ_1 using (39) at $i = 1$, and evaluate the result using (35), (36), and [13, Lemma 10.2] in order to obtain (38) for $1 \leq i \leq d$. We have now proved the claim. We now break the argument into subcases.

Subcase Racah: $h \neq 0, h^* \neq 0$. We show p is listed in Example 5.10. Define

$$s := \mu h^{-1}, \quad s^* := \mu^* h^{*-1}. \tag{40}$$

Eliminating η, μ in (35) using (37), (40) we obtain (15) for $0 \leq i \leq d$. Eliminating η^*, μ^* in (36) using (37), (40) we obtain (16) for $0 \leq i \leq d$. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains scalars r_1, r_2 such that both

$$r_1 r_2 = -\tau h^{-1} h^{*-1}, \quad r_1 + r_2 = s + s^* + d + 1. \tag{41}$$

Eliminating μ, μ^*, τ in (38), (39) using (40) and the equation on the left in (41) we obtain (17), (18) for $1 \leq i \leq d$. By the construction each of h, h^* is nonzero. The remaining inequalities mentioned below (18) follow from PA1, PA2 and (15)–(18). We have now shown p is listed in Example 5.10.

We now give the remaining subcases of Case II. We list the essentials only.

Subcase Hahn: $h = 0, h^* \neq 0$. Definitions.

$$s = \mu, \quad s^* := \mu^* h^{*-1}, \quad r := -\tau \mu^{-1} h^{*-1}.$$

Subcase dual Hahn: $h \neq 0, h^* = 0$. Definitions.

$$s := \mu h^{-1}, \quad s^* = \mu^*, \quad r := -\tau h^{-1} \mu^{*-1}.$$

Subcase Krawtchouk: $h = 0, h^* = 0$. Definitions.

$$s := \mu, \quad s^* := \mu^*, \quad r := -\tau.$$

Case III. $q = -1$ and $\text{char}(\mathbb{K}) \neq 2$.

We show p is listed in Example 5.14. By (25) and since $\text{char}(\mathbb{K}) \neq 2$, there exist scalars η, μ, h in $\tilde{\mathbb{K}}$ such that

$$\theta_i = \eta + \mu(-1)^i + 2hi(-1)^i, \quad 0 \leq i \leq d. \tag{42}$$

Similarly there exist scalars η^*, μ^*, h^* in $\tilde{\mathbb{K}}$ such that

$$\theta_i^* = \eta^* + \mu^*(-1)^i + 2h^*i(-1)^i, \quad 0 \leq i \leq d. \tag{43}$$

Observe $h \neq 0$; otherwise $\theta_2 = \theta_0$ by (42). Similarly $h^* \neq 0$. For any prime i such that $i \leq d/2$ we have $\text{char}(\mathbb{K}) \neq i$; otherwise $\theta_{2i} = \theta_0$ by (42). By this and since $\text{char}(\mathbb{K}) \neq 2$ we find $\text{char}(\mathbb{K})$ is either 0 or an odd prime greater than $d/2$. Setting $i = 0$ in (42), (43) we obtain

$$\theta_0 = \eta + \mu, \quad \theta_0^* = \eta^* + \mu^*. \tag{44}$$

We define

$$s := 1 - \mu h^{-1}, \quad s^* = 1 - \mu^* h^{*-1}. \tag{45}$$

Eliminating η in (42) using (44) and eliminating μ in the result using (45) we find (19) holds for $0 \leq i \leq d$. Similarly we find (20) holds for $0 \leq i \leq d$. We now define r_1, r_2 . First assume d is odd. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains r_1, r_2 such that

$$r_1 + r_2 = -s - s^* + d + 1 \tag{46}$$

and such that

$$4hh^*(1+r_1)(1+r_2) = -\varphi_1. \tag{47}$$

Next assume d is even. Define

$$r_2 := -1 + \frac{\varphi_1}{4hh^*d} \tag{48}$$

and define r_1 so that (46) holds. We have now defined r_1, r_2 for either parity of d . In the equation of PA4, we eliminate φ_1 using (47) or (48), and evaluate the result using (19), (20), and [13, Lemma 10.2] in order to obtain (22) for $1 \leq i \leq d$. In the equation of PA3, we eliminate ϕ_1 using (22) at $i = 1$, and evaluate the result using (19), (20), and [13, Lemma 10.2] in order to obtain (21) for $1 \leq i \leq d$. We mentioned each of h, h^* is nonzero. The remaining inequalities mentioned below (22) follow from PA1, PA2 and (19)–(22). We have now shown p is listed in Example 5.14.

Case IV. $q = 1$ and $\text{char}(\mathbb{K}) = 2$.

We show p is listed in Example 5.15. We first show $d = 3$. Recall $d \geq 3$ since $q = 1$. Suppose $d \geq 4$. By (25) we have $\sum_{j=0}^3 \theta_j = 0$ and $\sum_{j=1}^4 \theta_j = 0$. Adding these sums we find $\theta_0 = \theta_4$, which contradicts PA1. Therefore $d = 3$. We claim there exist nonzero scalars h, s in \mathbb{K} such that (23) holds for $0 \leq i \leq 3$. Define $h = \theta_0 + \theta_2$. Observe $h \neq 0$; otherwise $\theta_0 = \theta_2$. Define $s = (\theta_0 + \theta_3)h^{-1}$. Observe $s \neq 0$; otherwise $\theta_0 = \theta_3$. Using these values for h, s we find (23) holds for $i = 0, 2, 3$. By this and $\sum_{j=0}^3 \theta_j = 0$ we find (23) holds for $i = 1$. We have now proved our claim. Similarly there exist nonzero scalars h^*, s^* in \mathbb{K} such that (24) holds for $0 \leq i \leq 3$. Define

$r := \varphi_1 h^{-1} h^{*-1}$. Observe $r \neq 0$ and that $\varphi_1 = hh^*r$. In the equation of PA4, we eliminate φ_1 using $\varphi_1 = hh^*r$ and evaluate the result using (23), (24) and [13, Lemma 10.2] in order to obtain $\phi_1 = hh^*(r + s(1 + s^*))$, $\phi_2 = hh^*$, $\phi_3 = hh^*(r + s^*(1 + s))$. In the equation of PA3, we eliminate ϕ_1 using $\phi_1 = hh^*(r + s(1 + s^*))$ and evaluate the result using (23), (24) and [13, Lemma 10.2] in order to obtain $\varphi_2 = hh^*$, $\varphi_3 = hh^*(r + s + s^*)$. We mentioned each of h, h^*, s, s^*, r is nonzero. Observe $s \neq 1$; otherwise $\theta_1 = \theta_0$. Similarly $s^* \neq 1$. Observe $r \neq s + s^*$; otherwise $\varphi_3 = 0$. Observe $r \neq s(1 + s^*)$; otherwise $\phi_1 = 0$. Observe $r \neq s^*(1 + s)$; otherwise $\phi_3 = 0$. We have now shown p is listed in Example 5.15. We are done with Case IV and the proof is complete. ■

6. The Orthogonality Relation in Terms of the Parameter Array

Some facts about the polynomials in Examples 5.3–5.15 can be expressed in a uniform and attractive manner by writing things in terms of the associated parameter array. We illustrate this by giving the orthogonality relation, the three-term recurrence, and the difference equation in terms of the parameter array. We start with the orthogonality relation. In order to state the result we define some scalars.

Definition 6.1. Let $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} . For $0 \leq i \leq d$ we let k_i equal

$$\frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\phi_1 \phi_2 \cdots \phi_i}$$

times

$$\frac{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{(\theta_i^* - \theta_0^*) \cdots (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) \cdots (\theta_i^* - \theta_d^*)}.$$

For $0 \leq i \leq d$ we let k_i^* equal

$$\frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\phi_d \phi_{d-1} \cdots \phi_{d-i+1}}$$

times

$$\frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)}{(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d)}.$$

We observe $k_0 = 1$, $k_0^* = 1$. We define

$$v = \frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{\phi_1 \phi_2 \cdots \phi_d}.$$

THEOREM 6.2 [15, Lines (128), (129)]. *Let $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} and let f_0, f_1, \dots, f_d denote the corresponding polynomials*

from Definition 5.2. Then both

$$\sum_{r=0}^d f_i(\theta_r) f_j(\theta_r) k_r^* = \delta_{ij} v k_i^{-1}, \quad 0 \leq i, j \leq d, \quad (49)$$

$$\sum_{r=0}^d f_r(\theta_i) f_r(\theta_j) k_r = \delta_{ij} v k_i^{*-1}, \quad 0 \leq i, j \leq d. \quad (50)$$

The scalars k_i, k_i^*, v are from Definition 6.1.

We have a comment.

LEMMA 6.3. With reference to Definition 6.1, both

$$v = \sum_{r=0}^d k_r, \quad v = \sum_{r=0}^d k_r^*. \quad (51)$$

Proof. To get the equation on the left in (51) set $i=0, j=0$ in (50) and observe $f_r(\theta_0)=1$ for $0 \leq r \leq d$. To get the equation on the right in (51) set $i=0, j=0$ in (49) and observe $f_0=1$. ■

7. The Three-term Recurrence in Terms of the Parameter Array

In this section we give a three-term recurrence satisfied by the polynomials in Example 5.3–5.15. We express the result in terms of the associated parameter array. In order to state the result we define some scalars.

Definition 7.1. Let $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ denote a parameter array over \mathbb{K} . We define

$$b_i = \varphi_{i+1} \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{i-1}^*)}{(\theta_{i+1}^* - \theta_0^*)(\theta_{i+1}^* - \theta_1^*) \cdots (\theta_{i+1}^* - \theta_i^*)}, \quad 0 \leq i \leq d-1 \quad (52)$$

and $b_d=0$. We define

$$c_i = \phi_i \frac{(\theta_i^* - \theta_d^*)(\theta_i^* - \theta_{d-1}^*) \cdots (\theta_i^* - \theta_{i+1}^*)}{(\theta_{i-1}^* - \theta_d^*)(\theta_{i-1}^* - \theta_{d-1}^*) \cdots (\theta_{i-1}^* - \theta_i^*)}, \quad 1 \leq i \leq d \quad (53)$$

and $c_0=0$. We define

$$a_i = \theta_0 - c_i - b_i, \quad 0 \leq i \leq d. \quad (54)$$

THEOREM 7.2. Let $(\theta_i, \theta_i^*, i=0\dots d; \varphi_j, \phi_j, j=1\dots d)$ denote a parameter array over \mathbb{K} and let f_0, f_1, \dots, f_d denote the corresponding polynomials from Definition 5.2. For $0 \leq i, j \leq d$ we have

$$\theta_j f_i(\theta_j) = c_i f_{i-1}(\theta_j) + a_i f_i(\theta_j) + b_i f_{i+1}(\theta_j), \quad (55)$$

where f_{-1}, f_{d+1} are indeterminates and where the a_i, b_i, c_i are from Definition 7.1.

Proof. Let \mathcal{P} denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$, which has ij th entry $f_j(\theta_i)$ for $0 \leq i, j \leq d$. Let K^* denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{K})$, which has entries $K_{ii}^* = k_i^*$ for $0 \leq i \leq d$, where the k_i^* are from Definition 6.1. Let H denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{K})$, which has entries $H_{ii} = \theta_i$ for $0 \leq i \leq d$. Let C denote the following matrix in $\text{Mat}_{d+1}(\mathbb{K})$.

$$C = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \\ & & & \cdot & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{pmatrix}.$$

We define $P^* = \mathcal{P}^t K^*$. By [15, Line (118)] we have $CP^* = P^*H$. By this and since K^*, H are diagonal we find $C\mathcal{P}^t = \mathcal{P}^t H$. In this equation we expand each side using matrix multiplication and routinely obtain (55). ■

We finish this section with a comment.

LEMMA 7.3. *With reference to Definitions 6.1 and 7.1,*

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}, \quad 0 \leq i \leq d.$$

Proof. Compare the formulae for the k_i, b_i, c_i given in Definitions 6.1 and 7.1. ■

8. The Difference Equation in Terms of the Parameter Array

In this section we give a difference equation satisfied by the polynomials in Example 5.3–5.15. We express the result in terms of the associated parameter array. In order to state the result we define some scalars.

Definition 8.1. Let $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} . We define

$$b_i^* = \varphi_{i+1} \frac{(\theta_i - \theta_0)(\theta_i - \theta_1) \cdots (\theta_i - \theta_{i-1})}{(\theta_{i+1} - \theta_0)(\theta_{i+1} - \theta_1) \cdots (\theta_{i+1} - \theta_i)}, \quad 0 \leq i \leq d - 1$$

and $b_d^* = 0$. We define

$$c_i^* = \phi_{d-i+1} \frac{(\theta_i - \theta_d)(\theta_i - \theta_{d-1}) \cdots (\theta_i - \theta_{i+1})}{(\theta_{i-1} - \theta_d)(\theta_{i-1} - \theta_{d-1}) \cdots (\theta_{i-1} - \theta_i)}, \quad 1 \leq i \leq d$$

and $c_0^* = 0$. We define

$$a_i^* = \theta_0^* - c_i^* - b_i^*, \quad 0 \leq i \leq d.$$

THEOREM 8.2. *Let $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} and let f_0, f_1, \dots, f_d denote the corresponding polynomials from Definition 5.2. For $0 \leq i, j \leq d$ we have*

$$\theta_i^* f_i(\theta_j) = c_j^* f_i(\theta_{j-1}) + a_j^* f_i(\theta_j) + b_j^* f_i(\theta_{j+1}), \tag{56}$$

where $\theta_{-1}, \theta_{d+1}$ are indeterminates and the a_j^*, b_j^*, c_j^* are from Definition 8.1.

Proof. By Lemma 2.2(i) the sequence $(\theta_i^*, \theta_i, i = 0 \dots d; \varphi_j, \phi_{d-j+1}, j = 1 \dots d)$ is a parameter array over \mathbb{K} . Let $f_0^*, f_1^*, \dots, f_d^*$ denote the corresponding polynomials from Definition 5.2, so that

$$f_i^* = \sum_{n=0}^i \frac{(\lambda - \theta_0^*)(\lambda - \theta_1^*) \dots (\lambda - \theta_{n-1}^*)(\theta_i - \theta_0)(\theta_i - \theta_1) \dots (\theta_i - \theta_{n-1})}{\varphi_1 \varphi_2 \dots \varphi_n} \tag{57}$$

for $0 \leq i \leq d$. Applying Theorem 7.2 to $(\theta_i^*, \theta_i, i = 0 \dots d; \varphi_j, \phi_{d-j+1}, j = 1 \dots d)$ and $f_0^*, f_1^*, \dots, f_d^*$ we find that for $0 \leq i, j \leq d$,

$$\theta_j^* f_i^*(\theta_j^*) = c_i^* f_{i-1}^*(\theta_j^*) + a_i^* f_i^*(\theta_j^*) + b_i^* f_{i+1}^*(\theta_j^*), \tag{58}$$

where f_{-1}^*, f_{d+1}^* are indeterminates. Comparing (10) and (57) we find

$$f_i(\theta_j) = f_j^*(\theta_i^*), \quad 0 \leq i, j \leq d. \tag{59}$$

Evaluating (58) using (59) and reindexing the result we obtain (56). ■
We finish this section with a comment.

LEMMA 8.3. *With reference to Definitions 6.1 and 8.1,*

$$k_i^* = \frac{b_0^* b_1^* \dots b_{i-1}^*}{c_1^* c_2^* \dots c_i^*}, \quad 0 \leq i \leq d.$$

Proof. Similar to the proof of Lemma 7.3. ■

9. Some Useful Formulae

In this section we give alternative formulae for the scalars a_i, b_i, c_i from Definition 7.1. To avoid trivialities we assume the diameter $d \geq 1$. We begin with the a_i .

THEOREM 9.1. [13, Lemma 5.1]. *With reference to Definition 7.1, let us assume $d \geq 1$. Then*

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1^*}, \tag{60}$$

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*}, \quad 1 \leq i \leq d-1, \tag{61}$$

$$a_d = \theta_d + \frac{\varphi_d}{\theta_d^* - \theta_{d-1}^*}. \tag{62}$$

LEMMA 9.2. *With reference to Definition 7.1, assume $d \geq 1$. Then*

$$c_i(\theta_{i-1}^* - \theta_i^*) - b_i(\theta_i^* - \theta_{i+1}^*) = (\theta_1 - \theta_0)(\theta_i^* - \theta_0^*) + \varphi_1 \tag{63}$$

for $0 \leq i \leq d$, where $\theta_{-1}^*, \theta_{d+1}^*$ denote indeterminates.

Proof. Setting $\lambda = \theta_1$ in (10) we find

$$f_i(\theta_1) = 1 + \frac{(\theta_1 - \theta_0)(\theta_i^* - \theta_0^*)}{\varphi_1}, \quad 0 \leq i \leq d. \tag{64}$$

Setting $j = 1$ in (55) and evaluating the result using (54), (64) we obtain (63). ■

THEOREM 9.3. *With reference to Definition 7.1, assume $d \geq 1$. Then*

$$b_0 = \frac{\varphi_1}{\theta_1^* - \theta_0^*}, \tag{65}$$

$$b_i = \frac{(\theta_0 - a_i)(\theta_i^* - \theta_{i-1}^*) + (\theta_0 - \theta_1)(\theta_0^* - \theta_i^*) + \varphi_1}{\theta_{i+1}^* - \theta_{i-1}^*}, \quad 1 \leq i \leq d-1, \tag{66}$$

$$b_d = 0, \tag{67}$$

$$c_0 = 0, \tag{68}$$

$$c_i = \frac{(\theta_0 - a_i)(\theta_i^* - \theta_{i+1}^*) + (\theta_0 - \theta_1)(\theta_0^* - \theta_i^*) + \varphi_1}{\theta_{i-1}^* - \theta_{i+1}^*}, \quad 1 \leq i \leq d-1, \tag{69}$$

$$c_d = \frac{\phi_d}{\theta_{d-1}^* - \theta_d^*}. \tag{70}$$

Proof. Lines (67) and (68) are clear. To get (65) set $i = 0$ in (52). To get (70) set $i = d$ in (53). To get (66) and (69), solve the linear system (54) and (63) for b_i, c_i . ■

Results similar to Theorem 9.1, Lemma 9.2, and Theorem 9.3 hold for the a_i^*, b_i^*, c_i^* .

10. Remarks

We conclude this paper with a few remarks.

Let $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} and let f_0, f_1, \dots, f_d denote the corresponding polynomials from Definition 5.2. Applying Theorem 4.1 with $\lambda = \theta_d$ and using Lemma 4.2 we find

$$f_i(\theta_d) = \frac{\phi_1 \phi_2 \cdots \phi_i}{\varphi_1 \varphi_2 \cdots \varphi_i}, \quad 0 \leq i \leq d. \tag{71}$$

Let the scalars k_i be as in Definition 6.1. Comparing (71) with the formulae for k_i given in Definition 6.1 we find

$$k_i f_i(\theta_d) = \frac{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{(\theta_i^* - \theta_0^*) \cdots (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) \cdots (\theta_i^* - \theta_d^*)}$$

for $0 \leq i \leq d$.

We describe the matrix G from Theorem 3.2. We use the following notation. Let $(\theta_i, \theta_i^*, i = 0 \dots d; \varphi_j, \phi_j, j = 1 \dots d)$ denote a parameter array over \mathbb{K} and let q denote a base for this array. To keep things simple we assume $q \neq 1, q \neq -1$. For nonnegative integers r, s, t such that $r + s + t \leq d$ we define

$$[r, s, t]_q := \frac{(q; q)_{r+s}(q; q)_{r+t}(q; q)_{s+t}}{(q; q)_r(q; q)_s(q; q)_t(q; q)_{r+s+t}},$$

where

$$(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n = 0, 1, 2, \dots$$

We comment $[r, s, t]_q \in \mathbb{K}$ [15, Definition 13.1]. Let S denote the upper triangular matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which has entries

$$S_{ij} = (\theta_0 - \theta_d)(\theta_0 - \theta_{d-1}) \cdots (\theta_0 - \theta_{d-j+i+1})[i, j - i, d - j]_q$$

for $0 \leq i \leq j \leq d$. Then for $G \in \text{Mat}_{d+1}(\mathbb{K})$, G satisfies Theorem 3.2(ii) if and only if there exists a nonzero $\alpha \in \mathbb{K}$ such that $G = \alpha S$ [15, Theorem 15.2]. Similar results hold for $q = 1$ and $q = -1$ [15, Lemma 13.2].

11. Open Problems

PROBLEM 11.1. *Generalize Theorem 4.1 so that it applies to polynomial sequences of infinite length. Use this result to characterize the polynomials of the Askey scheme.*

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