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Dynamics in Linear Cournot Duopolies with Two Time Delays

Akio Matsumoto · Ferenc Szidarovszky · Hiroyuki Yoshida

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Abstract Linear differential duopolies are constructed with continuous time scales, constant coefficients and two types of information delays: fixed and continuously distributed time delays. System dynamics are considered with delays in the diagonal terms. By analyzing the associated characteristic equations, it is found that the stability is lost when the lengths of delays cross some critical values. Then it is shown that the destabilizing effect caused by the fixed delays is stronger than the destabilizing effect of the distributed delays having exponentially-declining weighting function. It is further demonstrated that the strength of the destabilizing effect is reversed if the distributed delay has a bell-shaped weighting function.

Keywords Cournot competition · Fixed time delay · Continuously distributed time delay · Two delays · Stability switches

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A. Matsumoto (✉)
Department of Economics, Chuo University, 742-1, Higasi-Nakano,
Hachioji, Tokyo 192-0393, Japan
e-mail: akiom@tamacc.chuo-u.ac.jp

F. Szidarovszky
Department of Systems and Industrial Engineering, University of Arizona, Tucson,
AZ 85721-0020, USA
e-mail: szidar@sie.arizona.edu

H. Yoshida
College of Economics, Nihon University, 1-3-2, Misaki, Chiyoda, Tokyo 101-8360, Japan
e-mail: yoshida.hiroyuki@nihon-u.ac.jp

1 Introduction

Oligopoly models play a central role in the literature of mathematical economics. Since the pioneering work of Cournot (1838), a large number of researchers discussed and examined the classical Cournot model and its variants and extensions. The existence and uniqueness of the equilibrium was the main focus of studies in the early stages and later the research turned to the dynamic analysis of oligopolistic markets. A comprehensive summary of earlier results can be found in Okuguchi (1976), and their multiproduct generalizations with several case studies are discussed in Okuguchi and Szidarovszky (1999). The attention was focused on linear and linearized models at the beginning which provided the local asymptotic properties of the equilibria. During the last two decades however an increasing attention has been given to the analysis of global dynamics. A survey of the newer results can be found in Bischi et al. (2010) which contains models with both discrete and continuous time scales. The most common dynamic processes are based either on the gradients of the profit functions or on best responses. In this paper continuous-time gradient dynamics will be considered with constant coefficients. Some applications of gradient dynamics to economic models are found in Bischi et al. (2010).

In the cases of most models discussed earlier in the literature, it was assumed that each firm has instantaneous information about its own output and also on the outputs of the competitors. This assumption has mathematical convenience, however it is unrealistic in real economics, since there are always time delays due to determining and implementing decisions. In addition to these facts, in fast changing industries the firms do not want to follow sudden market changes, they rather want to react to averaged past information. Hence there are always time delays between the times when information is obtained and the times when the decisions are implemented.

Howroyd and Russel (1984) construct two linear continuous-time dynamic oligopoly models with partial adjustment towards the best response and consider the effects caused by time delays on local stability. They show clear-cut results. In their Model I in which each firm has delayed information of its own output and its competitors' outputs, local stability can be lost when the lengths of the delays are large enough. On the other hand, in their Model II in which each firm has delayed information only in the competitors' outputs, stability is preserved regardless of the length of the delays under the plausible assumption that the coefficient matrix of the dynamic system without time delays is diagonally dominant. They, however, do not investigate the case in which each firm has delayed information on its own output and can use instantaneous information on the competitors' outputs.

This paper has three purposes. The first purpose is to complement the study of Howroyd and Russel (1984) by examining how the fixed time delays affect local dynamics in this missing case. Howroyd and Russel (1984) assume fixed time delays. There are, however, many economic situations in which the delays are uncertain or the firms are reacting to averaged past information. In such situations, continuously distributed time delays are appropriate. Thus the second purpose is to examine whether or not the results obtained under fixed time delays still hold in a dynamic oligopoly model with continuously distributed time delays. The last purpose is to compare the destabilizing effects caused by fixed and distributed time delays and to show that the relative strength between the two effects depends on the value of the shape parameter of the weighting function under distributed delay.

The paper is organized as follows. Section 2 constructs a basic duopoly model with linear price and cost functions. Section 3 assumes constant speed of adjustment and introduces fixed time delays into the basic model. Section 4 discusses the model with continuously distributed time delay. Section 5 concludes the paper.

2 The Basic Model

In this section, dynamics in a classical linear oligopoly model is briefly reviewed and its global stability is confirmed. Consider an industry of N firms producing a homogeneous good. Let $k = 1, 2, \ldots, N$ denote the firms and let x_k be the output quantity produced by firm k. The price function is assumed to be linear,

$$p = a - b \sum_{\ell=1}^{N} x_\ell \quad \text{with } a > 0 \quad \text{and} \quad b > 0.$$

Production cost is also assumed to be linear and the marginal cost of firm k is denoted by c_k. The profit function of firm k is defined by

$$\pi_k = \left(a - b \sum_{\ell=1}^{N} x_\ell \right) x_k - c_k x_k.$$

Firm k determines its output to maximize its profit with respect to x_k. Assuming interior optimal solutions of the profit maximizing problems and then solving the first-order condition for the output yield the best reply for firm k,

$$R_k(\boldsymbol{x}_{-k}) = \frac{a - c_k - b \sum_{\ell \neq k}^{N} x_\ell}{2b}$$

with

$$\boldsymbol{x}_{-k} = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N).$$

A Cournot equilibrium state is a vector $(x_1^c, x_2^c, \ldots, x_N^c)$ that satisfies $x_k^c = R_k(\boldsymbol{x}_{-k}^c)$ for $k = 1, 2, \ldots, N$. Thus a Cournot equilibrium output of firm k is

$$x_k^c = \frac{a - N c_k + \sum_{\ell \neq k}^{N} c_\ell}{(N+1)b}.$$

In this paper we assume that the firms continuously adjust their outputs proportionally to the change in their profits. This gradient dynamics is modeled by an N-dimensional system of ordinary differential equations of the form,

$$\dot{x}_k = \alpha_k \frac{\partial \pi_k}{\partial x_k} \quad \text{for } k = 1, 2, \ldots, N, \tag{1}$$

where α_k is a positive adjustment coefficient of firm k, and the dot over a variable means its time derivative, e.g., $\dot{x}_k = dx_k/dt$. Substituting the explicit expression of the marginal profit of firm k into (1), we obtain the continuous-time oligopoly model with gradient dynamics

$$\dot{x}_k(t) = \alpha_k \left(a - c_k - b \sum_{\ell \neq k}^{N} x_\ell(t) - 2bx_k(t) \right) \quad \text{for } k = 1, 2, \ldots, N. \tag{2}$$

Using the best reply functions, system (2) can be rewritten as

$$\dot{x}_k(t) = \bar{\alpha}_k \left(R_k(\boldsymbol{x}_{-k}(t)) - x_k(t) \right) \text{ for } k = 1, 2, \ldots, N. \tag{3}$$

where $\bar{\alpha}_k = 2b\alpha_k$. In system (3), each firm adaptively adjusts its output in such a way that the adjustment rate of the output is proportional to the difference between the profit maximizing output and the current output. That is, each firm adjusts its output into the direction toward its best reply. The transformation from (2) to (3) or vice versa implies that for the firms, the gradient adjustment of the output is the same as the adaptive adjustment toward best reply. To establish local stability of the Cournot equilibrium, it suffices to show that all eigenvalues of the coefficient matrix of either (2) or (3) have negative real parts. It is shown in Bischi et al. (2010, Theorem 2.2), that system (3) is always locally asymptotically stable. This implies that system (2) is also locally asymptotically stable. Since local stability leads to global stability in linear models, we obtain the following well-known result:

Theorem 1 *The continuous-time gradient dynamic model (2) is globally asymptotically stable.*

In the following we introduce time delays into system (2) and consider how time delays affect local stability of the Cournot equilibrium. We examine two different types of time delays: fixed or discrete time delay in Sect. 3 and continuously distributed time delay in Sect. 4. We abbreviate the first delay to a discrete-delay and the second to a continuous-delay henceforth.

3 Linear Duopolies with Two Discrete Delays

Howroyd and Russel (1984) obtain the delay oligopoly system by introducing discrete-delays into (3),

$$\dot{x}_k(t) = \bar{\alpha}_k \left(R_k(\boldsymbol{x}_{-k}(t - T_k)) - x_k(t - S_k) \right) \quad \text{for } k = 1, 2, \ldots, N,$$

where firm k experiences a time delay T_k in obtaining information about the competitors' outputs and a time delay S_k in implementing information about its own output.

The Cournot equilibrium is shown to be stable in their Model I with $T_k = S_k = r_k > 0$ if $2\bar{\alpha}_k r_k \leq 1$ and $(N-1)b < 1$ hold. It is also shown to be always stable regardless of the lengths of delays in their Model II with $T_k > 0$ and $S_k = 0$.[1] In the following, we consider the opposite case with $S_k > 0$ and $T_k = 0$ in the gradient dynamic model (2). Since it is, however, difficult to examine the general N-firm oligopoly model, we confine our analysis to delay duopoly dynamics (i.e., $N = 2$).

For the sake of notational simplicity, we use the followings: $x = x_1,\ y = x_2,\ \alpha = \alpha_1,\ \beta = \alpha_2,\ c_x = c_1$ and $c_y = c_2$. Then a duopoly version of the gradient dynamic model is presented by

$$\begin{aligned}\dot{x}(t) &= \alpha(a - c_x - 2bx(t - S_x) - by(t - T_x)),\\ \dot{y}(t) &= \beta(a - c_y - bx(t - T_y) - 2by(t - S_y)).\end{aligned} \tag{4}$$

A steady state (i.e., Cournot outputs) of the delay model (4) is obtained by substituting 2 for N in x_k^c,

$$x^c = \frac{a - 2c_x + c_y}{3b} \quad \text{and} \quad y^c = \frac{a - 2c_y + c_x}{3b},$$

which are independent of the length of delays. Since the discrete-delays in the off-diagonal terms are harmless to stability, we assume the following:

Assumption 1 $T_x = T_y = 0$.

We start from the local stability analysis. For this purpose we consider the corresponding homogenous system of (4):

$$\begin{aligned}\dot{x}_\delta(t) &= \alpha(-2bx_\delta(t - S_x) - by_\delta(t)),\\ \dot{y}_\delta(t) &= \beta(-bx_\delta(t) - 2by_\delta(t - S_y)),\end{aligned}$$

where $x_\delta(t) = x(t) - x^c$ and $y_\delta(t) = y(t) - y^c$ are deviations from the corresponding equilibrium values. The characteristic equation of the linearized system can be obtained by looking for the solutions in exponential forms,

$$x_\delta(t) = e^{\lambda t}u \quad \text{and} \quad y_\delta(t) = e^{\lambda t}v$$

and substituting them into the homogeneous equations. The resultant system can be written in the matrix form:

$$\begin{pmatrix} \lambda + 2\alpha b e^{-\lambda S_x} & \alpha b \\ \beta b & \lambda + 2\beta b e^{-\lambda S_y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

[1] More general result, asymptotic stability for linear differential system with general delays $T_{kj} \geq 0$ for $1 \leq k \neq j \leq N$ and $S_{kk} = 0$ for $k = 1, 2, \ldots, N$, is shown in Hofbauer and So (2000).

A nontrivial solution exists if and only if the determinant of the coefficient matrix is zero, which provides a mixed exponential-polynomial equation for λ:

$$\left(\lambda + 2\alpha b e^{-\lambda S_x}\right)\left(\lambda + 2\beta b e^{-\lambda S_y}\right) - b^2\alpha\beta = 0. \tag{5}$$

Expanding (5), we obtain the characteristic equation of the form

$$\lambda^2 - \alpha\beta b^2 + 2\alpha b\lambda e^{-\lambda S_x} + 2\beta b\lambda e^{-\lambda S_y} + 4\alpha\beta b^2 e^{-\lambda(S_x+S_y)} = 0. \tag{6}$$

Due to Theorem 1, the duopoly system (4) is locally asymptotically stable in the absence of information delays. Although we assume away information delays in the competitors' outputs and reduce the N-firm model to the duopoly model, the dynamic analysis of system (4) is still complicated in the case of multiple distinctive delays, $S_x > 0,\ S_y > 0$ and $S_x \neq S_y$. Following Li and Wei (2009), we take the two-step procedure. At the first step, setting $S_x = 0$, we reduce (6) to the equation with single discrete-delay S_y. We first find a stable interval of S_y in which all roots of the characteristic equation (6) have negative real parts. Then we proceed to the question of stability switching. The stability analysis concerns with whether all roots lie in the left half of the complex plain. The analysis of stability switching concerns with whether the roots cross the imaginary axis when the delay changes. At the second step, selecting a value of S_y from the stable interval, we treat S_y as a parameter and find a stable interval of S_x, repeating the same procedure taken at the first step. The stable interval of S_x depends on the value of S_y. Moving the value of S_y from one extreme of its stable interval to the other extreme generates the S_y-dependent corresponding interval of S_x. Putting these intervals together constructs the region of S_x and S_y in which the Cournot equilibrium is stable.

First Step $S_x = 0$ and $S_y > 0$.

The condition $S_x = 0$ reduces the characteristic equation (6) to the form,

$$\lambda^2 + 2\alpha b\lambda - \alpha\beta b^2 + 2\beta b(\lambda + 2\alpha b)e^{-\lambda S_y} = 0. \tag{7}$$

It can be checked that $\lambda = 0$ is not a solution of the above equation. Let $\lambda = i\omega$ with $\omega > 0$ be a solution. Substituting it into (7) and separating the real and imaginary parts, we have

$$\begin{aligned} 2\beta b(2\alpha b\cos(\omega S_y) + \omega\sin(\omega S_y)) &= \omega^2 + \alpha\beta b^2 \\ 2\beta b(\omega\cos(\omega S_y) - 2\alpha b\sin(\omega S_y)) &= -2\alpha b\omega. \end{aligned} \tag{8}$$

The sum of the squares of these two equations yields the fourth-order equation in ω

$$\omega^4 - 2b^2(2\beta^2 - 2\alpha^2 - \alpha\beta)\omega^2 - 15\alpha^2\beta^2 b^4 = 0 \tag{9}$$

implying that

$$\omega_\pm^2 = b^2(2\beta^2 - 2\alpha^2 - \alpha\beta \pm \sqrt{D})$$

where D is the discriminant and has the form

$$(2\beta^2 - 2\alpha^2 - \alpha\beta)^2 + 15\alpha^2\beta^2 > 0.$$

Clearly $\omega_+^2 > 0$ and $\omega_-^2 < 0$. Substituting the positive root ω_+ into (8) and solving the resultant equations we have

$$\cos\left(\omega_+ S_y\right) = \frac{\alpha^2 b^2}{4\alpha^2 b^2 + \omega_+^2}$$

and

$$\sin\left(\omega_+ S_y\right) = \frac{\omega_+(\omega_+^2 + \alpha\beta b^2 + 4\alpha^2 b^2)}{2\beta b(4\alpha^2 b^2 + \omega_+^2)}.$$

Hence there is a unique $\theta = \omega_+ S_y \in (0, 2\pi]$ that makes both equations hold. Solving either of the last two equations for S_y yields the threshold value of time delay S_y, given $S_x = 0$,

$$S_y^{D0} = \frac{1}{\omega_+} \cos^{-1}\left[\frac{\alpha^2 b^2}{4\alpha^2 b^2 + \omega_+^2}\right]$$

or

$$S_y^{D0} = \frac{1}{\omega_+} \sin^{-1}\left[\frac{\omega_+(\omega_+^2 + \alpha\beta b^2 + 4\alpha^2 b^2)}{2\beta b(4\alpha^2 b^2 + \omega_+^2)}\right]$$

where the values of the right hand sides of these expressions are the same.

To verify stability switching, we need to determine the sign of the derivative of $\mathrm{Re}\left[\lambda(S_y)\right]$ at the point where $\lambda(S_y)$ is purely imaginary. By differentiating (7) with respect to S_y, we have

$$\left\{2(\lambda + \alpha b + \beta b e^{-\lambda S_y}) - 2\beta b(\lambda + 2\alpha b)e^{-\lambda S_y} S_y\right\} \frac{d\lambda}{dS_y} = 2\beta b(\lambda + 2\alpha b)\lambda e^{-\lambda S_y}.$$

For convenience we check the sign of $(d\lambda/dS_y)^{-1}$ that is written as

$$\left(\frac{d\lambda}{dS_y}\right)^{-1} = \frac{(\lambda + \alpha b)e^{\lambda S_y} + \beta b}{\beta b \lambda(\lambda + 2\alpha b)} - \frac{S_y}{\lambda}$$

where, from (7), we have

$$e^{\lambda S_y} = -\frac{2\beta b(\lambda + 2\alpha b)}{\lambda^2 + 2\alpha b\lambda - \alpha\beta b^2}.$$

Therefore

$$\begin{aligned}
&\operatorname{sign}\left[\frac{d(\operatorname{Re}\lambda)}{dS_y}\right]_{\lambda=i\omega_+}\\
&= \operatorname{sign}\left[\operatorname{Re}\left(\frac{d\lambda}{dS_y}\right)^{-1}\right]_{\lambda=i\omega_+}\\
&= \operatorname{sign}\left[\operatorname{Re}\left(-\frac{2(\lambda+\alpha b)}{\lambda(\lambda^2+2\alpha b\lambda-\alpha\beta b^2)}+\frac{1}{\lambda(\lambda+2\alpha b)}\right)\right]_{\lambda=i\omega_+}\\
&= \operatorname{sign}\left[\frac{2(\omega_+^2+\alpha\beta b^2+2\alpha^2 b^2)}{(\omega_+^2+\alpha\beta b^2)^2+(2\alpha b\omega_+)^2}-\frac{1}{4\alpha^2 b^2+\omega_+^2}\right]\\
&= \operatorname{sign}\left[2(\omega_+^2+\alpha\beta b^2+2\alpha^2 b^2)(4\alpha^2 b^2+\omega_+^2)-(\omega_+^2+\alpha\beta b^2)^2-(2\alpha b\omega_+)^2\right]\\
&= \operatorname{sign}\left[15\alpha^2\beta^2 b^4+\omega_+^4+8\alpha^2 b^4\sqrt{D}\right]>0,
\end{aligned}$$

where the definition of ω_+^2 was used in the last step. This result implies that the crossing of the imaginary axis is from the left to the right as S_y increases and thus leads to the loss of stability. It can be noticed that the firms are symmetric in the adjustment coefficients and discrete-delays. We can get the threshold value of S_x by interchanging α and β. Concerning stability of the Cournot equilibrium, the following conclusion holds at the first step:

Theorem 2 *The discrete-delay duopoly model with $S_i > 0$ and $S_j = 0$ $(i, j = x, y, i \neq j)$ is locally asymptotically stable for $S_i < S_i^{D0}$ and unstable for $S_i > S_i^{D0}$ where the critical level of discrete-delay S_i^{D0} is defined by*

$$S_i^{D0} = \frac{1}{\omega_{i+}}\cos^{-1}\left[\frac{\boldsymbol{b}^2 b^2}{4\boldsymbol{b}^2 b^2+\omega_{i+}^2}\right]$$

with

$$\omega_{i+} = b\sqrt{2\boldsymbol{a}^2-2\boldsymbol{b}^2-\alpha\beta+\sqrt{D_i}}$$

and

$$D_i = (2\boldsymbol{a}^2-2\boldsymbol{b}^2-\alpha\beta)^2+15\alpha^2\beta^2$$

where

$$\boldsymbol{a}=\alpha \quad \text{and} \quad \boldsymbol{b}=\beta \quad \text{if } i=x \quad \text{and} \quad \boldsymbol{a}=\beta \quad \text{and} \quad \boldsymbol{b}=\alpha \quad \text{if } i=y.$$

Second Step $S_x > 0$ and $S_y > 0$.

Suppose that the characteristic equation (6) has a value of S_y selected from the stability interval $[0, S_y^{D0})$ obtained at the first step. By the same token, let $\lambda = iw,\ w > 0$ be a solution of the characteristic equation. Substituting it into (6) we have the real and imaginary parts in the following forms:

$$\begin{aligned} &2\alpha bw \sin(wS_x) + 4\alpha\beta b^2 \cos\left(w(S_x + S_y)\right) = w^2 + \alpha\beta b^2 - 2\beta bw \sin\left(wS_y\right), \\ &2\alpha bw \cos(wS_x) - 4\alpha\beta b^2 \sin\left(w(S_x + S_y)\right) = -2\beta bw \cos\left(wS_y\right). \end{aligned} \tag{10}$$

Hence the sum of the squares of the two equations in (10) is arranged to be

$$\begin{aligned} &4\alpha^2 b^2 w^2 + 16\alpha^2\beta^2 b^4 - 16\alpha^2\beta b^3 w \sin\left(wS_y\right) \\ &\quad = (w^2 + \alpha\beta b^2)^2 + 4\beta^2 b^2 w^2 - 4\beta b(w^2 + \alpha\beta b^2) w \sin\left(wS_y\right) \end{aligned} \tag{11}$$

which is written as a fourth-order equation in w,

$$\begin{aligned} &w^4 + 2b^2(2\beta^2 - 2\alpha^2 + \alpha\beta)w^2 - 15\alpha^2\beta^2 b^4 \\ &\quad +4\beta bw(4\alpha^2 b^2 - w^2 - \alpha\beta b^2)\sin\left(wS_y\right) = 0. \end{aligned}$$

It is not easy to solve this equation analytically for w. However, under a specific condition such as $\alpha = \beta = k$, the equation is reduced to

$$(w^2 - 3k^2b^2)(w^2 + 5k^2b^2 - 4kbw\sin(wS_y)) = 0.$$

Since the second factor is always positive, a positive solution of w is uniquely determined to be $w_+ = \sqrt{3}kb$ which is independent of the specific value of S_y.

Using the addition theorem for trigonometric functions, we solve (10) to obtain

$$\sin(wS_x) = \frac{(w^2 + \alpha\beta b^2) + 4\beta^2 b^2 w - 2\beta b(\alpha\beta b^2 + 2w^2)\sin\left(wS_y\right)}{2\alpha b(w^2 + 4\beta^2 b^2 - 4\beta b \sin\left(wS_y\right))},$$

and

$$\cos(wS_x) = \frac{\beta^2 b^2 \cos\left(wS_y\right)}{w^2 + 4\beta^2 b^2 - 4\beta b \sin\left(wS_y\right)}.$$

Again, we need to determine the sign of the derivative of $\mathrm{Re}[\lambda(S_x)]$ at the point where $\lambda(S_x)$ is purely imaginary. From (6), we have

$$\left(\frac{d\lambda}{dS_x}\right)^{-1} = \frac{(2\lambda + 2\beta b(1 - \lambda S_y)e^{-\lambda S_y})e^{\lambda S_x}}{\lambda(2\alpha b\lambda + 4\alpha\beta b^2 e^{-\lambda S_y})} + \frac{2\alpha b - 4\alpha\beta b^2 e^{-\lambda S_y}}{\lambda(2\alpha b\lambda + 4\alpha\beta b^2 e^{-\lambda S_y})} - \frac{S_x}{\lambda}$$

where from (6)

$$e^{\lambda S_x} = -\frac{2\alpha b\lambda + 4\alpha\beta b^2 e^{-\lambda S_y}}{\lambda^2 - \alpha\beta b^2 + 2\beta b e^{-\lambda S_y}}.$$

Therefore

$$\begin{aligned}
&\operatorname{sign}\left[\frac{d(\operatorname{Re}\lambda)}{dS_x}\right]_{\lambda=iw_+}\\
&=\operatorname{sign}\left[\operatorname{Re}\left(\frac{d\lambda}{dS_x}\right)^{-1}\right]_{\lambda=iw_+}\\
&=\operatorname{sign}\left[\operatorname{Re}\left(-\frac{2\lambda+2\beta b(1-\lambda S_y)e^{-\lambda S_y}}{\lambda(\lambda^2-\alpha\beta b^2+2\beta b\lambda e^{-\lambda S_y})}+\frac{2\alpha b-4\alpha\beta b^2 S_y e^{-\lambda S_y}}{\lambda(2\alpha b\lambda+4\alpha\beta b^2 e^{-\lambda S_y})}\right)\right]_{\lambda=iw_+}\\
&=\operatorname{sign}\left[-\frac{-4\beta^2b^2w_+-2(\alpha\beta b^2+w_+^2)+2\beta b(3w_+^2+\alpha\beta b^2)\sin\left(w_+S_y\right)+2\beta bw_+(\alpha\beta b^2+w_+^2)S_y\cos\left(w_+S_y\right)}{w_+(4\beta^2b^2w_+^2+(\alpha\beta b^2+w_+^2)^2-4\beta b(\alpha\beta b+w_+^2)w_+\sin\left(w_+S_y\right))}\right.\\
&\quad\left.+\frac{8\alpha^2\beta b^3(\sin\left(w_+S_y\right)+w_+S_y\cos\left(w_+S_y\right))-4\alpha^2b^2w_+}{w_+(4\alpha^2b^2w_+^2+16\alpha^2\beta^2b^4-8\alpha^2\beta b^3w_+\sin\left(w_+S_y\right))}\right]\\
&=\operatorname{sign}\left[4b^2(\beta^2-\alpha^2)w_++2(\alpha\beta b^2+w_+^2)w_++2\beta b(4\alpha^2b^2-\alpha\beta b^2-3w_+^2)\sin\left(w_+S_y\right)\right.\\
&\quad\left.+2\beta b(4\alpha^2b^2-\alpha\beta b^2-w_+^2)w_+S_y\cos\left(w_+S_y\right)\right]
\end{aligned}$$

where we use (11) in the last step. Under the specific condition $\alpha = \beta = k$, the expression in the last brackets is simplified as

$$2w_+(k^2b^2+w_+^2)-4kbw_+^2\sin\left(w_+S_y\right) > (8\sqrt{3}-12)k^2b^3 > 0.$$

The first inequality is due to $\left|\sin w_+S_y\right| < 1$ and $w_+ = \sqrt{3}kb$. Therefore we obtain

$$\left.\frac{d(\operatorname{Re}\lambda)}{dS_x}\right|_{\lambda=iw_+} > 0$$

implying that all the roots cross the imaginary axis from left to right as S_x increases when S_y is chosen from its stability interval. The result is summarized in the following theorem:

Theorem 3 *Given $\alpha = \beta = k$, the discrete-delay duopoly model with $S_x > 0$ and $S_y \in (0, S_y^{D0}]$ is locally asymptotically stable for $S_x < \bar{S}_x$ and unstable for $S_x > \bar{S}_x$ where $\bar{S}_x = \bar{S}_x(S_y)$ is defined by*

$$\bar{S}_x(S_y) = \frac{1}{w_+}\cos^{-1}\left[\frac{\beta^2b^2\cos\left(w_+S_y\right)}{w_+^2+4\beta^2b^2-4\beta b\sin\left(w_+S_y\right)}\right]$$

with

$$w_+ = \sqrt{3}kb.$$

Notice that $\bar{S}_x(S_y)$ converges to the threshold value S_x^{D0} obtained in Theorem 2, when S_y approaches 0. The locus of $\bar{S}_x = \bar{S}_x(S_y)$ for $S_y \in (0, S_y^{D0}]$ is the partition curve dividing the discrete-delays (S_x, S_y) space into stable and unstable regions and is depicted as a thicker concave curve in Fig. 1b.

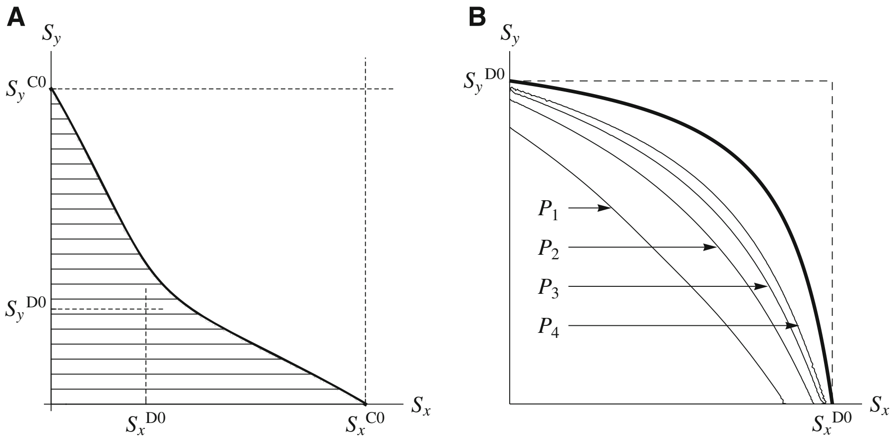

Fig. 1 Partition lines of the discrete- and continuous-delay models. **A** $n_k = 0$ for $k = x, y$ and **B** $n_k = 1, 2, 3, 4$ for $k = x, y$

4 Linear Duopolies with Two Continuous-Delays

In real economic situations, the delays are usually uncertain and can be considered to be fixed only under special circumstances. Therefore, we will re-model time delays in a continuously distributed manner and compare the discrete-delay effects with the continuous-delay effects. As in the discrete-delay model, we examine the case where the firms have time delays only in implementing information about their own outputs.[2] State variables $x(t)$ and $y(t)$ in the diagonal terms are replaced by certain averages of past values. The continuous-delay dynamic system is frequently modeled with Volterra type integro-differential equations

$$\begin{aligned} \dot{x}(t) &= \alpha(a - c_x - 2bx^e(t) - by(t)), \\ \dot{y}(t) &= \beta(a - c_y - bx(t) - 2by^e(t)), \end{aligned} \tag{12}$$

with expectations on its own outputs,

$$x^e(t) = \int_0^t \phi(t - s, S_x, n_x)x(s)ds$$

and

$$y^e(t) = \int_0^t \phi(t - s, S_y, n_y)y(s)ds.$$

[2] It is possible to show that time delays in the competitor's output are harmless to stability even in the continuous-delay model. See Matsumoto and Szidarovszky (2009).

The weighting function ϕ is assumed to have the form

$$\phi(t-s,\Gamma,\ell)=\begin{cases}\dfrac{1}{\Gamma}e^{-\frac{t-s}{\Gamma}} & \text{if } \ell=0,\\[2ex] \dfrac{1}{\ell!}\left(\dfrac{\ell}{\Gamma}\right)^{\ell+1}(t-s)^{\ell}e^{-\frac{\ell(t-s)}{\Gamma}} & \text{if } \ell\geq 1.\end{cases} \tag{13}$$

Here we assume that $\Gamma > 0$ and ℓ is a nonnegative integer. For $\ell = 0$, weights are exponentially declining with most weight given to the most current data. For $\ell > 0$, the weighting function has a bell-shaped profile indicating that zero weight is given to the most recent data, rising to maximum at $s = t - \Gamma$ and declining thereafter. As ℓ increases, the weighting function becomes more peaked around Γ and as ℓ goes to infinity, the weighting function converges to the Dirac delta function.[3]

Substituting (13) into the above expectation formations and the resulting expressions into (12) yield a system of integro-differential equations. A steady state of system (12) is identical with the Cournot equilibrium, x^c and y^c. In order to analyze the dynamic behavior of the system, we consider the corresponding homogeneous system. Letting x_δ and y_δ, as before, denote the deviations of x and y from their Cournot output levels, x^c and y^c, the homogeneous system can be formulated as follows:

$$\begin{aligned}\dot{x}_\delta &= \alpha\left\{-2b\int_0^t \phi(t-s,S_x,n_x)x_\delta(s)ds - by(t)\right\},\\ \dot{y}_\delta &= \beta\left\{-bx(t) - 2b\int_0^t \phi(t-s,S_y,n_y)y_\delta(s)ds\right\}.\end{aligned} \tag{14}$$

We seek the solutions in the exponential forms

$$x_\delta(t)=e^{\lambda t}u \quad \text{and} \quad y_\delta(t)=e^{\lambda t}v,$$

and substituting these solutions into Eq. 14 and arranging terms yield

$$\begin{aligned}&\left(\lambda+2\alpha b\int_0^t \phi(t-s,S_x,n_x)e^{-\lambda(t-s)}ds\right)u+\alpha b v=0,\\ &\beta b u+\left(\lambda+2\beta b\int_0^t \phi(t-s,S_y,n_y)e^{-\lambda(t-s)}ds\right)v=0.\end{aligned} \tag{15}$$

[3] See Bischi et al. (2010) for more properties of the weighting function, $\phi(t-s,\Gamma,\ell)$.

Introducing a new variable $z = t - s$, we can simplify the integral terms by noticing that

$$\int_0^t \phi(t-s, \Gamma, \ell) e^{-\lambda(t-s)} ds = \int_0^t \phi(z, \Gamma, \ell) e^{-\lambda z} dz.$$

Since we are interested in the asymptotic behavior of the system, we let $t \to \infty$ to have

$$\lim_{t\to\infty} \int_0^t \phi(z, \Gamma, \ell) e^{-\lambda z} dz = \left(1 + \frac{\lambda\Gamma}{q}\right)^{-(\ell+1)}$$

with

$$q = \begin{cases} 1 & \text{if } \ell = 0, \\ \ell & \text{if } \ell \geq 1. \end{cases}$$

Then Eq. (15) can be written in the matrix form

$$\begin{pmatrix} A_x(\lambda) & B_x(\lambda) \\ B_y(\lambda) & A_y(\lambda) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$A_x(\lambda) = \left[\lambda \left(1 + \frac{\lambda S_x}{q_x}\right)^{n_x+1} + 2\alpha b\right],$$

$$A_y(\lambda) = \left[\lambda \left(1 + \frac{\lambda S_y}{q_y}\right)^{n_y+1} + 2\beta b\right],$$

$$B_x(\lambda) = \alpha b \left(1 + \frac{\lambda S_x}{q_x}\right)^{n_x+1}$$

and

$$B_y(\lambda) = \beta b \left(1 + \frac{\lambda S_y}{q_y}\right)^{n_y+1}.$$

A non-trivial solution exists if and only if

$$A_x(\lambda) A_y(\lambda) - B_x(\lambda) B_y(\lambda) = 0 \tag{16}$$

which is called the characteristic polynomial of system (14). Since it is difficult to check whether the real parts of the roots of (16) are negative or positive, we will specialize the from of the density function to obtain some analytical results.

We have already assumed that both firms have information delays about their own outputs, $S_x > 0,\ S_y > 0$ and $S_x \neq S_y$. In addition to this, we further assume that the firms have exponentially declining weighting functions, that is, $n_x = n_y = 0$. Then equation (16) under these assumptions becomes a fourth degree polynomial equation:

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

where the coefficients are defined as

$$\begin{aligned}
a_0 &= S_x S_y > 0,\\
a_1 &= S_x + S_y > 0,\\
a_2 &= 1 + 2b(\alpha S_y + \beta S_x) - b^2\alpha\beta S_x S_y \gtreqless 0,\\
a_3 &= 2b(\alpha + \beta) - b^2\alpha\beta(S_x + S_y) \gtreqless 0,\\
a_4 &= 3b^2\alpha\beta > 0.
\end{aligned}$$

The Routh-Hurwitz stability theorem implies that the roots of the characteristic equation have negative real parts if and only if all coefficients are positive and the following determinants are positive:

$$J_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} \quad \text{and} \quad J_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix}.$$

Notice that

$$J_2 = S_x + S_y + 2b(\alpha S_y^2 + \beta S_x^2) > 0$$

and

$$J_3 = a_3 J_2 - a_1^2 a_4.$$

Since $J_2,\ a_1$ and a_4 are positive, $J_3 > 0$ implies $a_3 > 0$. Then relation $J_2 = a_1a_2 - a_0a_3 > 0$ implies that a_2 also have to be positive. It is thus sufficient for our purpose to assume that $J_3 > 0$, which implies that all roots have negative real parts.

Theorem 4 *If $S_x > 0$ with $n_x = 0$ and $S_y > 0$ with $n_y = 0$, then the continuous-delay system (12) is globally asymptotically stable for (S_x, S_y) below the partition curve, $J_3 = 0$, and unstable for (S_x, S_y) above the curve where*

$$J_3 = b\left\{\left[S_x+S_y+2b(\alpha S_y^2+\beta S_x^2)\right]\left[2(\alpha+\beta)-b\alpha\beta(S_x+S_y)\right]-3b\alpha\beta(S_x+S_y)^2\right\}.$$

The graphical representation of Theorem 4 is given in Fig. 1a where the downward sloping curve is the locus of $J_3 = 0$ with $\alpha = \beta = b = 1$. In the horizontally-shaded region under the locus, we have $J_3 > 0$ and thus the Cournot equilibrium is asymptotically stable. Figure 1b is an enlargement of the square region surrounded by the two loci, $S_x = S_x^{D0}$ and $S_y = S_y^{D0}$, in the lower-left part of Fig. 1a. The downward-sloping thicker curve is the partition curve under the two discrete delays being within the horizontally-shaded region of Fig. 1a. Therefore the stability region with the discrete-delay is smaller than the stability region with the continuous delay with $n_x = n_y = 0$.

Furthermore, if we solve $J_3 = 0$, taking $S_i > 0$ and $S_j = 0$, $(i, j = x, y, \ i \neq j)$, then we obtain the threshold value of the continuous-delay of firm i in the case where firm i has continuous-delay information on its own output while firm j has instantaneous information about its own output,

$$S_i^{C0} = \frac{\boldsymbol{c} + \sqrt{\alpha^2 + \alpha\beta + \beta^2}}{\alpha\beta b}$$

with

$$\boldsymbol{c} = \beta \quad \text{if } i = x \quad \text{and} \quad \boldsymbol{c} = \alpha \quad \text{if } i = y.$$

The continuous-delay system with $S_i > 0$ and $S_j = 0$ is asymptotically stable if $0 < S_i < S_i^{C0}$ and unstable otherwise. This result is also visualized in Fig. 1a where the outer dotted horizontal or vertical line is the locus of $S_i = S_i^{C0}$. Under the horizontal line or in the left of the vertical line, the Cournot outputs are asymptotically stable when only one firm experiences a continuous-delay with exponentially declining weighting function.

It can be also observed in Fig. 1a that the stability region with two continuous-delays is smaller than the stability region with a single continuous-time delay. The partition line under the single discrete-delay, $S_i = S_i^{D0}$, is also depicted in the shaded region as the inner horizontal or vertical dotted line. The partition curve with two discrete-delays is depicted as the outermost concave-shaped curve in Fig. 1b where the stability region with two discrete-delays is smaller than the stability region with a single discrete-delay. Locations of these dotted lines indicate that the destabilizing effect of the single discrete-delay is stronger than the destabilizing effect of the single continuous-delay in the sense that the stability region with the discrete-delay is smaller than the stability region with the continuous delay.

It is possible to derive an analytic form of the partition curve with larger values of n_x and n_y. It becomes, however, clumsy and much more complicated. Therefore we numerically check the shapes of the partition curves. Taking $\alpha = \beta = b = 1$ again and repeating the same procedure with increasing values of $n_x = n_y = 1, 2, 3, 4$, we obtain the four partition curves illustrated in Fig. 1b in which P_i means the partition curve when $n_x = n_y = i$. It can be seen that the continuous-delay partition curve moves outward as the value of $n_x = n_y$ increases. In other word, the stability region under the continuous-delays expands and get closer to the stability region under the discrete-delays.

There is an interesting relation between the characteristic polynomials (6) and (16) of the systems with discrete- and continuous-delays. Assume that n_x and n_y converge to infinity. Since $q_i = n_i$ for $i = x, y$, we have the limits

$$A_x(\lambda) \to \left(\lambda e^{\lambda S_x} + 2\alpha b\right),$$
$$A_y(\lambda) \to \left(\lambda e^{\lambda S_y} + 2\beta b\right),$$
$$B_x(\lambda) \to \alpha b e^{\lambda S_x}$$

and

$$B_y(\lambda) \to \beta b e^{\lambda S_y}.$$

Multiplying both sides of the characteristic equation (16) by $e^{-\lambda S_x} e^{-\lambda S_y}$ and arranging terms, we get the characteristic equation (5),

$$\left(\lambda + 2\alpha b e^{-\lambda S_x}\right)\left(\lambda + 2\beta b e^{-\lambda S_y}\right) - \alpha\beta b^2 = 0.$$

Therefore in the limiting case, the characteristic equation of the continuous-delay model converges to the characteristic equation of the discrete-delay model. The former model can approximate dynamics generated by the latter model if the value of the shape parameters (i.e., n_x and n_y) of the weighting functions is large enough.

In short, we find that increasing the shape parameters of the weighting function has a stabilizing effect in the sense that the stability region in the parameter space enlarges as the values of n_x and n_y increase.

5 Concluding Remarks

It is well-known that a continuous-time linear oligopoly is globally asymptotically stable. In this paper, we introduce two types of a time delay, a discrete-delay and continuous-delay, and consider the destabilizing effects caused by the time delay in the duopoly framework. We draw attention to the case where each firm has delay information only on its own output and obtain the following results.

(1) The analytical form of the partition curve dividing the (S_x, S_y) space into the stability and instability regions is explicitly derived as summarized in Theorems 2, 3 and 4.

(2) Our numerical study indicates that the destabilizing effect caused by the discrete-delay is stronger than the destabilizing effect by the continuous-delay having the exponentially declining weighting function (i.e., $n_x = n_y = 0$) in the sense that the stability region with the former is smaller than the stability region with the latter as depicted in Fig. 1a.

(3) The strength of the destabilizing effects is reversed if the continuous-delay has the bell-shaped weighting function (i.e., $n_x = n_y > 0$) as shown in Fig. 1b.

(4) The stability region with continuous-delay expands as the shape parameter of the weighting function increases and converges to the stability region of discrete-delay when the shape parameter tends to infinity.

The main result in this paper is that the existence of time delays in the adjustment process has the destabilizing effect on the economy. The intuitive reason for this result is as follows. As pointed out earlier, many types of time delays, such as the decision lag and the operational lag, arise in the real world. In such a circumstance, firms inevitably rely on inaccurate and untimely information about the economic situation. They are accordingly liable to misuse and mistreat the data in determining their own levels of output. This leads to instability of the economy when the lengths of delays are large enough. In this sense, our analysis shows that the Cournot adjustment process may fail to establish the Nash equilibrium.

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