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An R&D Investment Game under Uncertainty in Real Option Analysis

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Abstract One of the problems of using the financial options methodology to analyse investment decisions is that strategic considerations become extremely important. So, the theory of real option games combines two successful theories, namely real options and game theory. The investment opportunity and the value of flexibility can be valued as a real option while the competition can be analyzed with game theory. In our model we develop an interaction between two firms that invest in R&D. The firm that invests first, defined as the Leader, acquires a first mover advantage that we assume as a higher share of market. But, several R&D investments present positive externalities and so, the option exercise by the Leader generates an "Information-Revelation", that benefits the Follower.

Keywords Real Options · Exchange Options · Option games · Information Revelation

JEL Classification G13 · C72 · D80

1 Introduction

The theory of option games is the combination of two successful theories, namely real options and game theory. As it is widely accepted by the real options literature, the NPV (Net Present Value) method tends to undervalue projects that can be postponed. In fact, the NPV does not capture the value of the investor's right to delay its implementation, which may have a major impact on the value of the investment opportunity.

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So, a firm with an opportunity to invest is holding an "option", analogous to financial options. According to Copeland and Antikarov (2003), a real option is " the right, but not the obligation, to take an action (e.g. deferring, expanding, contracting, or abandoning) at a predetermined cost called the exercise price, for a predetermined period of time - the life of option".

Several models, such as is assumed to be in Majd and Pindyck (1987), Trigeorgis (1991), Ingersoll and Ross (1992), Lee (1997), are based on this definition, in which the exercise price is fixed. Since a company cannot make an accurate estimate of the future costs, the exercise price is uncertain, particulary for R&D investments. So, the investment opportunity corresponds to an exchange option and not to a simple call option: it's the exchange of an uncertain investment cost, for an uncertain gross project value. The most relevant models that value investment opportunities with two stochastic variables are given in Margrabe (1978), McDonald and Siegel (1985), Carr (1988, 1995), Armada et al. (2007). We can synthesize the main characteristics of these models.

Margrabe's (1978) model values an European exchange option that can be exercised only at maturity. The Margrabe model can value an American exchange option only where the underlying asset does not distribute dividends, since an American option should never be exercised prior to maturity, in the absence of dividends. In a real options context, "dividends" are the opportunity costs inherent in the decision to defer an investment project (Majd and Pindyck 1987). Furthermore, in a real options context, deferment implies the loss of the project's cash flows.

McDonald and Siegel's (1985) model values an European exchange option considering that the assets distribute dividends.

Carr (1988) develops a model to value an American exchange option using the Geske and Johnson (1984) approximation for valuing American puts, and he presents also a model to value an European compound exchange option. Finally, Armada et al. (2007) correct the Carr's extrapolation formula to value an American exchange option. So, we can use real options methodology, and particulary Exchange options to value an opportunity investment but we have to consider that most projects are open to more than one company in the same line of business. The strategic equilibrium in exercise policies is ignored in the financial options literature, because their exercise does not influence the characteristics of the underlying security or the options themselves. Differently, real investment opportunities are not held by one firm in isolation and so, the optimal exercise strategies can be derived with consideration of strategic interactions across option holders.

The combined options and games approach is particularly appropriate for R&D valuation. Weeds (2002) derives optimal investment strategies for two firms that compete for a patent that may help explain strategic delay in patent races, shedding light on the role of first vs. second movers. Lambrecht (2000) and Weeds (2000) consider innovation with uncertainty over completion and time delays, which can explain phenomena like faster exit and delayed commercialization. Mason and Weeds (2002) consider more general strategic interactions with externalities that may explain why investment might sometimes be speeded up under uncertainty.

The aim of this paper is to investigate a real option game model where two firms invest in R&D. The first firm that invests, defined as the Leader, acquires a first mover advantage that we assume as a higher share of market. But, several R&D investments, present positive externalities and so, the option exercise by the Leader, generates an "Information-Revelation" that benefits the Follower.

This paper follows the Dias and Teixera's (2004) model concerning an oligopoly game with information revelation, but we differentiate because we use exchange options to value the stochastic processes for R&D development costs and value. Moreover many authors, such as Smit and Trigeorgis (2004), follow the two-stage binomial option tree to value the R&D opportunity and therefore the uncertain is model considering only two expected trajectory (up and down) of the gross project value for each stage. This method allows to analyse better the game in the second stage in case of high or low expectation since we have the exact values of gross project but our model captures the complex uncertainty deriving by R&D investments.

Finally, the R&D competitive investment setting has the essential feature of the prisoners' dilemma game. The equilibrium in such situations it that the outcome is individually rational (in the sense that no firm has an incentive to do anything different from what it is doing in the equilibrium) but it is collectively irrational (in the sense that coordinated behaviour could make all players better off). However, as is well-known, if we consider the game infinitely repeat, we allow firms to wholly or partly repair the inefficiency in the equilibrium of one shot game and to model better the complicated R&D process. But, to simplify the computation of exchange strategic options, we abstract from scenarios in which players repeatedly interact with each other and in this paper we consider a one-shot game.

The paper is organized as follows. The Sect. 2 reviews some of the relevant option pricing literature and the Sect. 3 derives the final payoffs of two firms. In the Sect. 4, we present a real model implementation with computation of Nash equilibriums and critical market values while the Sect. 5 shows the influence of first mover's advantage and information revelation on the equilibrium behavior of both players. Finally, the Sect. 6 concludes.

2 Exchange Options Methodology

In this section we present the final results of the principal models to value European exchange options.

2.1 Simple European Exchange Option (SEEO)

McDonald and Siegel's model (1985) gives the value of a SEEO to exchange asset D for asset V at time T. The asset given up is termed the delivery asset while the asset received is the optioned asset. Denoting with $s(V, D, T-t)$ the value of SEEO at time t, the final payoff at the option's maturity date T is $s(V, D, 0) = \max[0, V_T - D_T]$.

So, assuming that V and D follow a geometric Brownian motion process given by:

$$\frac{dV}{V} = (\mu_v - \delta_v)dt + \sigma_v dZ_v \tag{1}$$

$$\frac{dD}{D} = (\mu_d - \delta_d)dt + \sigma_d dZ_d \tag{2}$$

$$cov\left(\frac{dV}{V}, \frac{dD}{D}\right) = \rho_{vd}\sigma_v\sigma_d\, dt \tag{3}$$

where:

- V and D are the Gross Project Value and the Investment Cost, respectively;
- μ_v and μ_d are the equilibrium expected rate of return on asset V, and the expected growth rate of the investment cost;
- δ_v and δ_d are the " dividend-yields", of V and D, respectively;
- Z_v and Z_d are the brownian standard motions of asset V and D;
- σ_v and σ_d are the volatility of V and D, respectively;
- ρ_{vd} is the correlation between changes in V and D.

McDonald and Siegel (1985) show that the value of a SEEO on dividend-paying assets, when the valuation date $t = 0$, is given by:

$$s(V, D, T) = Ve^{-\delta_v T} N(d_1(P, T)) - De^{-\delta_d T} N(d_2(P, T)) \tag{4}$$

where:

- $P = \frac{V}{D}$; $\quad \sigma = \sqrt{\sigma_v^2 - 2\rho_{v,d}\sigma_v\sigma_d + \sigma_d^2}$; $\quad \delta = \delta_v - \delta_d$;
- $d_1(P, T) = \frac{\log P + \left(\frac{\sigma^2}{2} - \delta\right)T}{\sigma\sqrt{T}}$; $\quad d_2 = d_1 - \sigma\sqrt{T}$;
- $N(d)$ is the cumulative standard normal distribution.

Moreover, considering the case without dividends, it results that:

$$\frac{\partial s}{\partial V} = N(d_1(P, T)) \tag{5}$$

2.2 Compound European Exchange Option (CEEO)

Exchange option may be simple or compound. If the underlying asset is another option, then the option is called compound. Carr (1988) develops a model to value the CEEO $c(s, \varphi D, t_1)$ whose final payoff at maturity date t_1 is:

$$c(s, \varphi D, 0) = \max[0, s - \varphi D]$$

The CEEO value, considering the valuation date $t = 0$, is given by:

$$\begin{aligned} c(s(V, D, T), \varphi D, t_1) = {} & Ve^{-\delta_v T} N_2\left(d_1\left(\frac{P}{P^*}, t_1\right), d_1(P, T); \rho\right) \\ & - De^{-\delta_d T} N_2\left(d_2\left(\frac{P}{P^*}, t_1\right), d_2(P, T); \rho\right) \\ & - \varphi De^{-\delta_d t_1} N_1\left(d_2\left(\frac{P}{P^*}, t_1\right)\right) \end{aligned} \tag{6}$$

where:

- φ is the exchange ratio of CEEO;
- t_1 is the expiration date of the CEEO;
- T is the expiration date of the SEEO, where $T > t_1$;
- $\tau = T - t_1$ is the time to maturity of the SEEO and $\rho = \sqrt{\frac{t_1}{T}}$;
- $d_1\left(\frac{P}{P^*}, t_1\right) = \frac{\log\left(\frac{P}{P^*}\right) + \left(-\delta + \frac{\sigma^2}{2}\right)t_1}{\sigma\sqrt{t_1}}$; $d_2\left(\frac{P}{P^*}, t_1\right) = d_1\left(\frac{P}{P^*}, t_1\right) - \sigma\sqrt{t_1}$;
- P^* is the critical price ratio that solves the following equation:

$$P^* e^{-\delta_v \tau} N(d_1(P^*, \tau)) - e^{-\delta_d \tau} N(d_2(P^*, \tau)) = \varphi \tag{7}$$

- $N_2(a, b, \rho)$ is the standard bivariate normal distribution function evaluated at a and b with correlation coefficient ρ.

Moreover, as is computing in Carr (1988), if we consider the case without dividends we have that:

$$\frac{\partial c}{\partial V} = N_2\left(d_1\left(\frac{P}{P^*}, t_1\right), d_1(P, T); \rho\right) \tag{8}$$

As $N_2(a, b; \rho) = N(b) - N_2(-a, b; -\rho)$, comparing the Eqs. 5 and 8 it results:

$$\frac{\partial s}{\partial V} > \frac{\partial c}{\partial V} > 0 \tag{9}$$

3 The Basic Model Game

This model aims to capture the uncertain dynamics in the value of competitive R&D strategies. An R&D investment opportunity depends on the resolution of several source of uncertainity that may influence the investment decision of each competing firm. Several R&D investments yield positive externalities and so, the optimal decision depends on the degree of information about the resolution of the other firm's technical R&D uncertainity (R&D success or failure). The information revelation is an important factor influencing the timing of R&D investment.

Our model describes a two-stage competitive R&D investment game assuming that each of two competitors (A and B) faces a decision if to realize the R&D investment at time t_0 or to postpone the decision at time t_1 waiting for better information.

We state also that the Leader is the pioneer firm (A or B) that invests in R&D at time t_0 earlier than other one, namely the Follower, that postpones the R&D investment decision at time t_1. So, we denote by R the R&D investment for the development of a new product, V the overall market value deriving by R&D innovations and D is the total investment cost to realize new goods. We consider that the production investment of each firm is proportional to its market share and it can be realized only at time T, that is the time needed for to develop the new product. Hence, we suppose that the option to enter in the market is like an European exchange option. According to exchange options, we assume that V and D follow the geometric Brownian motion defined in the Eqs. 1 and 2, respectively, and $R = \varphi D$ is a ratio φ of asset D, so R assumes the identical stochastic process of D.

We state that the Leader can take an advantage of being first in the market and, in particular way, we suppose that the Leader achieves the market share opportunity $\alpha \in (\frac{1}{2}, 1]$ of V higher than Follower's one, that is $1-\alpha \in [0, \frac{1}{2})$. In return the Follower benefits of the information revelation from the previous Leader's R&D investment. In our model we assume that firms A and B have an R&D success probability q and p, respectively. We can represent this situation with two Bernoulli distributions S and X:

$$S : \begin{cases} 1 & q \\ 0 & 1-q \end{cases} \qquad X : \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

After the information revelation [1], if the Leader's R&D is successful, the Follower's probability p changes in positive information revelation p^+, while p changes in negative information revelation p^-, in case of Leader's failure. To compute p^+ and p^- we present, in the Table 1, the bivariate Bernoulli distribution and the marginal distributions. So, it results that:

$$\rho(X, S) = \frac{p_{11} - pq}{\sqrt{p(1-p)q(1-q)}} \tag{10}$$

$$p^+ = Prob[X = 1/S = 1] = \frac{p_{11}}{q} = p + \sqrt{\frac{1-q}{q}} \cdot \sqrt{p(1-p)} \cdot \rho(X, S) \tag{11}$$

$$p^- = Prob[X = 1/S = 0] = \frac{p_{10}}{1-q} = p - \sqrt{\frac{q}{1-q}} \cdot \sqrt{p(1-p)} \cdot \rho(X, S) \tag{12}$$

We can interpret the correlation as the learning measure, namely the intensity of information revelation from S to X. Finally, if both players invest simultaneously in R&D or they wait to invest, we can consider that they share the market equally and there is not information revelation, so $\rho(X, S) = 0$ and consequently it results that $p = p^+ = p^-$. Before starting to examine the game, we assume that initial time $t_0 = 0$.

3.1 The Follower's Payoff

We analyze the game where the firm A (Leader) invests in R&D at time t_0 and the firm B (Follower) decides to postpone its R&D investment decision at time t_1. If

[1] For more details, please see Dias (2004): "Real Options, Learning Measures and Bernouilli Revelation Processes", Working paper presented at 9th Annual International Conference on Real Options, Paris, June 2005.

Table 1 Bivariate Bernoulli distribution and marginal distribution

		Variable S	Variable S	Marginal distr. of X
		$S = 1$	$S = 0$	
Variable X	$X = 1$	p_{11}	p_{10}	p
Variable X	$X = 0$	p_{01}	p_{00}	$1 - p$
Marginal distr. of S		q	$1 - q$	

the Leader's R&D investment is successfull, the Follower's R&D success probability changes in p^+. So, after the investment R, the Follower holds, in case of success with a probability p^+, the development option $s((1-\alpha)V, (1-\alpha)D, \tau)$ to invest $(1-\alpha)D$ at time T and claims a share $1-\alpha$ of the overall market V. Of course, a necessary condition to invest R at time t_1 is that the exploratory option is in the " money", namely, in case of success, the development option $s((1-\alpha)V, (1-\alpha)D, \tau)$ is higher than R. So, the Follower's payoff at time t_0 is a CEEO with maturity t_1, exercise price equal to R and the underlying asset is the development option $s((1-\alpha)V, (1-\alpha)D, \tau)$, as shown in the Fig. 1a.

The CEEO payoff at expiration date t_1 with positive information revelation is:

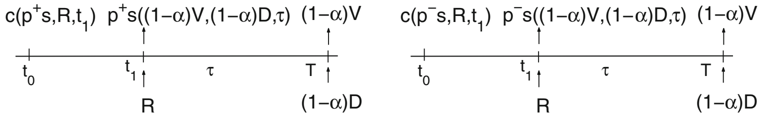
$$c(p^+ s((1-\alpha)V, (1-\alpha)D, \tau), R, 0) = \max[p^+ s((1-\alpha)V, (1-\alpha)D, \tau) - R, 0]$$

According to Carr's (1988) model, we consider that the investment $R = \varphi D$ is a ratio φ of asset D. Hence, denoting with $c(p^+)$ the CEEO at time t_0, i.e.:

$$c(p^+) \equiv c(p^+ s((1-\alpha)V, (1-\alpha)D, \tau), \varphi D, t_1)$$

we can write, using the Eq. 6, the value of CEEO with positive information:

$$\begin{aligned} c(p^+) = {} & p^+(1-\alpha)Ve^{-\delta_v T} N_2\left(d_1\left(\frac{P}{P^*_{up}}, t_1\right), d_1(P, T); \rho\right) \\ & -p^+(1-\alpha)De^{-\delta_d T} N_2\left(d_2\left(\frac{P}{P^*_{up}}, t_1\right), d_2(P, T); \rho\right) \\ & -\varphi De^{-\delta_d t_1} N_1\left(d_2\left(\frac{P}{P^*_{up}}, t_1\right)\right) \end{aligned} \tag{13}$$



(a) Follower's Payoff in case of Leader's success

(b) Follower's Payoff in case of Leader's failure

Fig. 1 Follower's payoffs

where $P = \frac{(1-\alpha)V}{(1-\alpha)D} = \frac{V}{D}$ and P^*_{up} is the critical value that makes the underlying asset of CEEO $c(p^+)$ equal to exercise value. Hence P^*_{up} solves the following equation:

$$p^+ s((1-\alpha)V, (1-\alpha)D, \tau) = \varphi D$$

and assuming the asset $(1-\alpha)D$ as numeraire we can rewrite the above equation as:

$$p^+ \left(P^*_{up}\, e^{-\delta_v \tau} N(d_1(P^*_{up}, \tau)) - e^{-\delta_d \tau} N(d_2(P^*_{up}, \tau)) \right) = \frac{\varphi}{1-\alpha} \tag{14}$$

Alternatively, in case of Leader's R&D failure, the Follower's R&D success probability changes in p^-. So, as we shown previously, the Follower's payoff at time t_0 is a CEEO with maturity t_1, exercise prise equal to R and the underlying asset is the development option $s((1-\alpha)V, (1-\alpha)D, \tau)$ as shown in the Fig. 1b. Hence, the CEEO payoff at expiration date t_1 with negative information revelation is:

$$c(p^- s((1-\alpha)V, (1-\alpha)D, \tau), R, 0) = \max[p^- s((1-\alpha)V, (1-\alpha)D, \tau) - R, 0]$$

So, denoting with $c(p^-)$ the CEEO at time t_0, i.e.:

$$c(p^-) \equiv c(p^- s((1-\alpha)V, (1-\alpha)D, \tau), \varphi D, t_1)$$

we can write, using the Eq. 6 , the value of CEEO with negative information:

$$\begin{aligned} c(p^-) = {} & p^-(1-\alpha)V e^{-\delta_v T} N_2 \left(d_1 \left(\frac{P}{P^*_{dw}}, t_1 \right), d_1(P, T); \rho \right) \\ & - p^-(1-\alpha)D e^{-\delta_d T} N_2 \left(d_2 \left(\frac{P}{P^*_{dw}}, t_1 \right), d_2(P, T); \rho \right) \\ & - \varphi D e^{-\delta_d t_1} N_1 \left(d_2 \left(\frac{P}{P^*_{dw}}, t_1 \right) \right) \end{aligned} \tag{15}$$

where P^*_{dw} is the critical value the solves the following equation:

$$p^- \left(P^*_{dw} e^{-\delta_v \tau} N(d_1(P^*_{dw}, \tau)) - e^{-\delta_d \tau} N(d_2(P^*_{dw}, \tau)) \right) = \frac{\varphi}{1-\alpha} \tag{16}$$

The Follower obtains the CEEO $c(p^+)$ in case of Leader's success with a probability q or the CEEO $c(p^-)$ in case of Leader's failure with a probability $(1-q)$. Hence, the Follower's payoff at time t_0 is the expectation value:

$$F_B(V, D) = q\, c(p^+) + (1-q)\, c(p^-) \tag{17}$$

Similarly, if we consider that firm B (Leader) invests in R&D at time t_0 and firm A (Follower) decides to wait to invest we have that:

$$F_A(V, D) = p\, c(q^+) + (1-p)\, c(q^-) \tag{18}$$

We can observe that the critical market values P^*_{up} and P^*_{dw} depending on φ that is the ratio of R&D. So, in case of Leader's success, the Follower will realize its R&D investment at time t_1, namely it will exercise its exploratory option at time t_1 if $\frac{(1-\alpha)V_{t_1}}{(1-\alpha)D_{t_1}} = \frac{V_{t_1}}{D_{t_1}} \geq P^*_{up}$, otherwise the Follower renounces to the R&D project. In case of Leader's failure, the Follower will invest in R&D at time t_1 if $\frac{(1-\alpha)V_{t_1}}{(1-\alpha)D_{t_1}} = \frac{V_{t_1}}{D_{t_1}} \geq P^*_{dw}$.

3.2 The A and B Payoffs when Both Firms Invest Simultaneously in R&D

In this situation, both players decide to realize the R&D investment simultaneously at time t_0. Hence, we can setting that there is not information revelation and consequently it results that $\rho(X, S) = 0$. Since the investment R is equal for both firms, we can setting that A and B can capture the same fraction $\alpha = \frac{1}{2}$ of the total market value. So, after the investment R in t_0, the firm A (firm B) holds the development option $s\left(\frac{1}{2}V, \frac{1}{2}D, T\right)$ to invest $\frac{1}{2}D$ at time T as illustrated in the Figs. 2a and 2b.

Recalling that A and B success probability is q and p, respectively, according to Eq. 4, we can write the final A and B payoffs in case of simultaneous R&D investment:

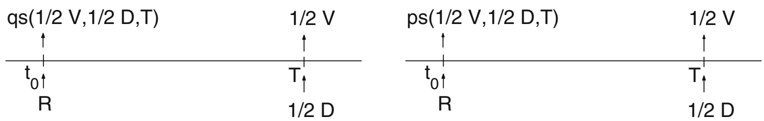
$$\begin{aligned} S_A(V, D) &= -R + q \cdot s\left(\frac{1}{2}V, \frac{1}{2}D, T\right) \\ &= -R + q\left(\frac{1}{2}Ve^{-\delta_v T}N\left(d_1\left(P, T\right)\right) - \frac{1}{2}De^{-\delta_v T}N\left(d_1\left(P, T\right)\right)\right) \end{aligned} \tag{19}$$

while:

$$\begin{aligned} S_B(V, D) &= -R + p \cdot s\left(\frac{1}{2}V, \frac{1}{2}D, T\right) \\ &= -R + p\left(\frac{1}{2}De^{-\delta_v T}N\left(d_1\left(P, T\right)\right) - \frac{1}{2}De^{-\delta_v T}N\left(d_1\left(P, T\right)\right)\right) \end{aligned} \tag{20}$$

3.3 The Leader's Payoff

Now we focus on the game in which firm A (Leader) invests in R&D at time t_0, assuming that firm B (Follower) decides to wait to invest. In this situation, the Leader spends the investment R at time t_0 and obtains, in case of R&D success, the development option $s(\alpha V, \alpha D, T)$ to invest αD at time T and claims a share $\alpha > \frac{1}{2}$ of the total market V, as illustrated in the Fig. 3.



(a) Firm A's payoff **(b)** Firm B's payoff

Fig. 2 A and B payoffs in case of simultaneous investment

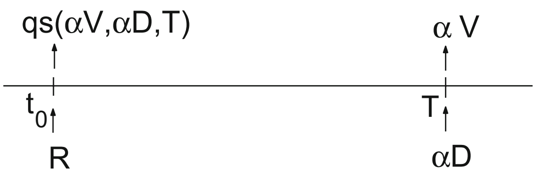


Fig. 3 Leader's payoff

Thus, the Leader's payoff is the following:

$$\begin{aligned} L_A(V,D) &= -R + q \cdot s\,(\alpha\, V, \alpha D, T) \\ &= -R + q\left(\alpha\, V e^{-\delta_v T} N\,(d_1\,(P,T)) - \alpha D e^{-\delta_d T} N\,(d_2\,(P,T))\right) \end{aligned} \quad (21)$$

Symmetrically, if we assume that the firm B (Leader) invests in R&D and the firm A (Follower) waits to invest, the final value of firm B is the following:

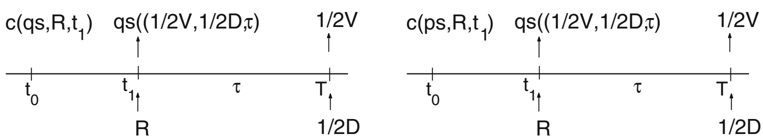
$$\begin{aligned} L_B(V,D) &= -R + p \cdot s\,(\alpha\, V, \alpha D, T) \\ &= -R + p\left(\alpha\, V e^{-\delta_v T} N\,(d_1\,(P,T)) - \alpha D e^{-\delta_d T} N\,(d_2\,(P,T))\right) \end{aligned} \quad (22)$$

3.4 The A and B Payoffs when Both Firms Wait to Invest

We suppose now that both players decide to delay their R&D investment decision at time t_1 and, specifically, we can assume that there is not information revelation and consequently $\rho(X,S) = 0$. Moreover we can setting that both players share the market equally, so $\alpha = \frac{1}{2}$. Then, after the investment R in t_1, each player holds the development option $s\left(\frac{1}{2}V, \frac{1}{2}D, \tau\right)$ to invest $\frac{1}{2}D$ at time T and claims a market share $\frac{1}{2}V$. So, at time t_0, the A and B payoffs are CEEOs with maturity t_1, exercise price equal to R and the underlying asset is the development option s, as summarized in the Figs. 4a and 4b.

Thus, according to Carr's (1988) model and recalling that $R = \varphi D$, the A and B payoffs at time t_0 are given by:

$$W_A(V,D) = c\left(q \cdot s\left(\frac{1}{2}V, \frac{1}{2}D, \tau\right), \varphi D, t_1\right) \quad (23)$$



(a) Firm A's payoff **(b)** Firm B's payoff

Fig. 4 A and B payoffs in case of waiting

$$W_B(V, D) = c\left(p \cdot s\left(\frac{1}{2}V, \frac{1}{2}D, \tau\right), \varphi D, t_1\right) \tag{24}$$

Using the Eq. 6, we can write:

$$\begin{aligned} c\left(q \cdot s\left(\frac{1}{2}V, \frac{1}{2}D, \tau\right), \varphi D, t_1\right) = {} & q\,\frac{1}{2}Ve^{-\delta_v T} N_2\left(d_1\left(\frac{P}{P^*_{wA}}, t_1\right), d_1\left(P, T\right); \rho\right) \\ & -q\,\frac{1}{2}De^{-\delta_d T} N_2\left(d_2\left(\frac{P}{P^*_{wA}}, t_1\right), d_2\left(P, T\right); \rho\right) \\ & -\varphi De^{-\delta_d t_1} N_1\left(d_2\left(\frac{P}{P^*_{wA}}, t_1\right)\right) \end{aligned} \tag{25}$$

where P^*_{wA} is the critical value that solves the following equation:

$$q \cdot s\left(\frac{1}{2}V, \frac{1}{2}D, \tau\right) = \varphi D$$

and assuming the asset $\frac{1}{2}D$ as numeraire we can rewrite the above equation as:

$$q \cdot \left(P^*_{wA}e^{-\delta_v \tau} N(d_1(P^*_{wA}, \tau)) - e^{-\delta_d \tau} N(d_2(P^*_{wA}, \tau))\right) = \frac{\varphi}{\frac{1}{2}} \tag{26}$$

and:

$$\begin{aligned} c\left(p \cdot s\left(\frac{1}{2}V, \frac{1}{2}D, \tau\right), \varphi D, t_1\right) = {} & p\,\frac{1}{2}Ve^{-\delta_v T} N_2\left(d_1\left(\frac{P}{P^*_{wB}}, t_1\right), d_1\left(P, T\right); \rho\right) \\ & -p\,\frac{1}{2}De^{-\delta_d T} N_2\left(d_2\left(\frac{P}{P^*_{wB}}, t_1\right), d_2\left(P, T\right); \rho\right) \\ & -\varphi De^{-\delta_d t_1} N_1\left(d_2\left(\frac{P}{P^*_{wB}}, t_1\right)\right) \end{aligned} \tag{27}$$

where, as seen before, P^*_{wB} is the critical value that solves the following equation:

$$p \cdot \left(P^*_{wB}e^{-\delta_v \tau} N(d_1(P^*_{wB}, \tau)) - e^{-\delta_d \tau} N(d_2(P^*_{wB}, \tau))\right) = \frac{\varphi}{\frac{1}{2}} \tag{28}$$

If neither firm wishes to invest in R&D at time t_0, we will have four scenarios at time t_1. If they decide to don't invest in R&D either at time t_1, their payoffs will be zero:

$$W^{t_1}_A(V_{t_1}, D_{t_1}) = 0; \quad W^{t_1}_B(V_{t_1}, D_{t_1}) = 0 \tag{29}$$

If both players decide to realize their R&D project at time t_1, they share fairly the market ($\alpha = \frac{1}{2}$). Hence, using the Eqs. 19 and 20, we obtain:

$$S_A^{t_1}(V_{t_1}, D_{t_1}) = -\varphi D_{t_1} + qs\left(\frac{1}{2} V_{t_1}, \frac{1}{2} D_{t_1}, \tau\right) \tag{30}$$

$$S_B^{t_1}(V_{t_1}, D_{t_1}) = -\varphi D_{t_1} + ps\left(\frac{1}{2} V_{t_1}, \frac{1}{2} D_{t_1}, \tau\right) \tag{31}$$

Instead, if A decides to invest alone in R&D at time t_1 while B renounces to its project we have that:

$$L_A^{t_1}(V_{t_1}, D_{t_1}) = -\varphi D_{t_1} + qs(\alpha V_{t_1}, \alpha D_{t_1}, \tau); \quad F_B^{t_1}(V_{t_1}, D_{t_1}) = 0 \tag{32}$$

In the symmetrical case, it results:

$$F_A^{t_1}(V_{t_1}, D_{t_1}) = 0; \quad L_B^{t_1}(V_{t_1}, D_{t_1}) = -\varphi D_{t_1} + ps(\alpha V_{t_1}, \alpha D_{t_1}, \tau) \tag{33}$$

As we can observe, if both players decide to wait to invest at time t_0, the final payoffs at time t_1 depending on the market value V_{t_1} and the development cost D_{t_1} evaluated at time t_1 that evolve according the stochastic process given in the Eqs. 1 and 2. But we know only the initial value of V and D at time t_0 and this statement we make difficult to carry out the analysis of the game using a backward induction. So, denoting with P_i^s and P_i^l the critical market ratios between V_{t_1} and D_{t_1} that make $S_i^{t_1} = 0$ and $L_i^{t_1} = 0$, respectively, it results that $P_i^l < P_i^s$ since $L_i^{t_1} > S_i^{t_1}$, for $i = A, B$. Hence, we can conclude that:

- If $\frac{V_{t_1}}{D_{t_1}} > \max[P_A^s, P_B^s]$ then $L_i^{t_1} > S_i^{t_1} > 0$ for $i = A, B$.
- If $\frac{V_{t_1}}{D_{t_1}} < \min[P_A^l, P_B^l]$ then $0 > L_i^{t_1} > S_i^{t_1}$ for $i = A, B$.

3.5 Analysis of Final Payoffs at Time t_0

The two-by-two matrix represented in the Fig. 5 summarizes the final payoffs. The first value in each cell indicates the strategic investment opportunity for A at time t_0, while the second represents the firm B's value. We can distinguish four basic cases: (*i*) when both firms decide to postpone the R&D investment at time t_1; (*ii*) and (*iii*) when one firm invests first (as a Leader) and the other decides to invest later (as a Follower); (*iv*) when both firms decide to invest simultaneously in R&D at time t_0.

Now, we analyse the relations among the strategic payoffs according to several expected market values V at time t_0 considering fixed the investment cost D at time t_0. So the final payoffs depending only by the asset V. We remember that V and D evolve always stochastically according to Eqs. 1 and 2, respectively. First of all, using the Eq. 9, comparing the Leader's payoff with the Waiting one we can observe that:

		FIRM B	
		Wait	*Invest*
FIRM A	*Wait*	(W_A, W_B)	(F_A, L_B)
	Invest	(L_A, F_B)	(S_A, S_B)

Fig. 5 Final payoffs at time t_0

- $L_i(0) = -R; \quad W_i(0) = 0;$
- $\frac{\partial L_A}{\partial V} = q\alpha N(d_1(P,T)); \quad \frac{\partial L_B}{\partial V} = p\alpha N(d_1(P,T));$
- $\frac{\partial W_A}{\partial V} = q\frac{1}{2}N_2\left(d_1\left(\frac{P}{P^*_{wA}}, t_1\right), d_1\left(P,T\right); \rho\right);$
- $\frac{\partial W_B}{\partial V} = p\frac{1}{2}N_2\left(d_1\left(\frac{P}{P^*_{wB}}, t_1\right), d_1\left(P,T\right); \rho\right);$
- $\frac{\partial L_i}{\partial V} > \frac{\partial W_i}{\partial V} > 0;$

for $i = A, B$. Then the following proposition holds:

Proposition 1 *If $\alpha > \frac{1}{2}$ then there exists, for each firm $i = A, B$, a unique critical market value V_i^W that makes $L_i(V_i^W) = W_i(V_i^W)$. Denoting by $V_W^* = \min(V_A^W, V_B^W)$ and $V_Q^* = \max(V_A^W, V_B^W)$ it results that:*

$$L_i(V) < W_i(V) \ \mathit{for} \ \ V < V_W^*;$$
$$L_i(V) > W_i(V) \ \mathit{for} \ \ V > V_Q^*.$$

If A's success probability is higher than B, for $V \in]V_W^*, V_Q^*[$ it results:

$$L_A(V) > W_A(V); \quad L_B(V) < W_B(V);$$

otherwise, if B's success probability is higher than A, for $V \in]V_W^*, V_Q^*[$ we have:

$$L_A(V) < W_A(V); \quad L_B(V) > W_B(V).$$

In the same way, if we compare the Follower's payoff with the simultaneous value, we can observe that:

- $F_i(0) = 0; \quad S_i(0) = -R;$
- $\frac{\partial S_A}{\partial V} = q\frac{1}{2}N(d_1(P,T)); \quad \frac{\partial S_B}{\partial V} = p\frac{1}{2}N(d_1(P,T));$
- $\frac{\partial F_A}{\partial V} = (1-\alpha)\left[qp^+N_2\left(d_1\left(\frac{P}{P^*_{up}}\right), d_1\left(P\right)\right) + (1-q)p^-N_2\left(d_1\left(\frac{P}{P^*_{dw}}\right), d_1\left(P\right)\right)\right]$
- $\frac{\partial F_B}{\partial V} = (1-\alpha)\left[pq^+N_2\left(d_1\left(\frac{P}{P^*_{up}}\right), d_1\left(P\right)\right) + (1-p)q^-N_2\left(d_1\left(\frac{P}{P^*_{dw}}\right), d_1\left(P\right)\right)\right]$

- $\frac{\partial S_i}{\partial V} > \frac{\partial F_i}{\partial V} \geq 0$;

for $i = A, B$. Then the following proposition holds:

Proposition 2 *If $\alpha > \frac{1}{2}$ then there exists, for each firm $i = A, B$ a unique critical market value V_i^S that makes $S_i(V_i^S) = F_i(V_i^S)$. Denoting by $V_P^* = \min(V_A^S, V_B^S)$ and $V_S^* = \max(V_A^S, V_B^S)$ it results that:*

$$S_i(V) < F_i(V) \ \textit{for} \ \ V < V_P^*;$$
$$S_i(V) > F_i(V) \ \textit{for} \ \ V > V_S^*.$$

If A's success probability is higher than B, for $V \in]V_P^*, V_S^*[$ it results:

$$S_A(V) > F_A(V); \quad S_B(V) < F_B(V);$$

otherwise, if B's success probability is higher than A, for $V \in]V_P^*, V_S^*[$ we have:

$$S_A(V) < F_A(V); \quad S_B(V) > F_B(V).$$

4 Real Applications

4.1 Assumptions and Inputs

The model of strategic R&D investments can be applied to a variety of settings. More recently, in pharmaceutical industry, several drugs are introduced only after years of laboratory and clinical testing without to get initial cash flows, so this investment opportunity can be priced as an European exchange option in which the development investment D will be realized at time T, and good information about drugs is available in the stages of R&D. Consider also technology hybrid electric cars in automotive industry. For example, GM stopped its hybrid project in 1998 and resumed it recently after observing Toyota's success in its hybrid car project of Prius. Now, to illustrate the concepts and equations presented, we develop a numerical example for the competitive R&D game between firm A and B with the following parameters and we focus on the several equilibriums (in Nash meaning) that firms can determine according to different present value of expected cash flows V deriving by $R\&D$ innovations:

- R&D Investment: $R = 150\,000$ $;
- Development Investment: $D = 400\,000$ $;
- Market and Costs Volatility: $\sigma_v = 0.90$; $\sigma_d = 0.23$;
- Fraction of D required for R: $\varphi = \frac{R}{D} = 0.375$
- Correlation between V and D: $\rho_{vd} = 0.15$;
- Dividend-Yields of V and D: $\delta_v = 0.15$; $\delta_d = 0$;
- Expiration Time of Compound Option: $t_1 = 0.5$ years;
- Expiration Time of Simple Option: $T = 3$ years;
- A and B success probability: $q = 0.60$; $p = 0.55$;
- Information Revelation: $\rho(X, S) = 0.70$;
- First mover's advantage: $\alpha = 0.60$;

The total investment cost D is the exercise price for the development option. We consider that the investment cost is proportional to market share, namely if the firm's market share is α then its investment cost will be αD. We assume that D follows the Brownian motion process defined in (2). The total current value of D is 400 000$.

The R&D investment R can be realized at time t_0 or t_1. If it is made in t_0, then R =150 000$ otherwise the investment R assumes the identical stochastic process of D, except that it occurs at time t_1 and it is proportional to $\varphi = 0.375$ of D.

Appropriately, we assume that the volatility of quoted shares and traded options is an adequate proxy for the volatility of asset V and investment cost D.

According to financial options, δ denotes the opportunity cost in holding the option instead of the stock. So, in real option world, δ_v is the opportunity cost of deferring the project and δ_d is the "dividend yield" on asset D. As at the beginning the cash flows are very low, so we assume that $\delta_v = 0.15$ and $\delta_d = 0$.

The time to maturity T denotes project's deferment option after that each opportunity disappears and we adopt $T = 3$ years. Moreover, we state that Follower needs about six months to know the Leader's outcome and consequently to receive the information revelation. So we assume that $t_1 = 0.5$ years.

Finally, we consider that firm A has an higher and more efficient Know-How than firm B and so, the firm A's success probability is $q = 0.60$ while the firm B's one is $p = 0.55$.

4.2 Computation of Nash Equilibriums

In our numerical example, it results that:

$$V_W^* = 1\,349\,400; \quad V_Q^* = 1\,441\,300; \quad V_P^* = 1\,677\,300; \quad V_S^* = 1\,898\,700.$$

The Tables 2 and 3 summarize the strategic A and B payoffs considering several considerable expected total market values.

When the expected market value is $V < V_W^*$, using the Propositions (1) and (2) it results that:

$$L_A(V) < W_A(V); \quad S_A(V) < F_A(V); \quad L_B(V) < W_B(V); \quad S_B(V) < F_B(V).$$

Using this inequality among the four strategic values, the waiting policy is optimal in Nash meaning for both players at time t_0. So the firms A and B prefer to wait for

Table 2 Firm A's final payoffs assuming $\alpha = 0.60$ and $\rho(X, S) = 0.70$

Market value V	Leader's value L_A	Follower's value F_A	Simultaneous value S_A	Waiting value W_A
1 200 000	41 462	44 471	9 552	54 110
1 400 000	82 470	63 118	43 725	78 484
1 600 000	124 216	83 714	78 513	105 733
1 800 000	166 536	105 843	113 780	135 200
2 000 000	209 314	129 195	149 428	166 377

Table 3 Firm B's final payoffs assuming $\alpha = 0.60$ and $\rho(X, S) = 0.70$

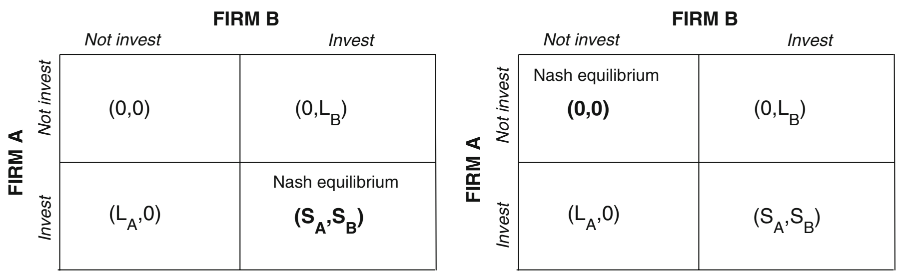
Market value V	Leader's value L_B	Follower's value F_B	Simultaneous value S_B	Waiting value W_B
1 200 000	25 507	40 328	−3 743	45 026
1 400 000	63 097	57 643	27 581	66 187
1 600 000	101 364	76 789	59 470	90 115
1 800 000	140 158	97 341	91 798	116 220
2 000 000	179 371	118 981	124 476	144 027

best market evolution and so they decide to delay their R&D investment decision at time t_1. So, if we consider that $V = 1\,200\,000\$$ we have only one Nash equilibrium (W_A, W_B), as shown in the Fig. 7a, in which each player follows its best response to the other player's strategy. At time t_1, it results that the profitability index $P_A^s = 2.73$ and $P_B^s = 2.93$ while $P_A^l = 2.38$ and $P_B^l = 2.54$. So, if $\frac{V_{t_1}}{D_{t_1}} > 2.93$, we have that $L_i^{t_1} > S_i^{t_1} > 0$ for $i = A, B$ and using this inequality we have one Nash equilibrium in which the R&D investment is optimal at time t_1 for both players, as illustrated in the Fig. 6a. While, if $\frac{V_{t_1}}{D_{t_1}} < 1.65$ we have that $0 > L_i^{t_1} > S_i^{t_1}$ for $i = A, B$ and we obtain one Nash equilibrium in which both players renounce the R&D investment, as shown in the Fig. 6b.

When the expected market value $V \in]V_W^*, V_Q^*[$, using the Propositions (1) and (2) we have that:

$$L_A(V) > W_A(V); \quad S_A(V) < F_A(V); \quad L_B(V) < W_B(V); \quad S_B(V) < F_B(V).$$

Using this inequality, we have one Nash equilibrium (L_A, F_B). In this case the firm A, that has an higher success probability, decides to invest in R&D earlier than player B. In our numerical example it results that $P_{up}^* = 7.76$ and $P_{dw}^* = 15.61$. Hence, at time t_1, if the Leader's R&D effort is successful, the Follower will realize its R&D project if the index profitability $\frac{V_{t_1}}{D_{t_1}} \geq 7.76$, otherwise, in case of Leader's R&D failure, the Follower's R&D investment will be made if $\frac{V_{t_1}}{D_{t_1}} \geq 15.61$. Assuming that $V = 1\,400\,000\$$, the Fig. 7b shows the existence of the Nash equilibrium (L_A, F_B).



(a) $L_i^{t_1} > S_i^{t_1} > 0$ for $i = A, B$ **(b)** $0 > L_i^{t_1} > S_i^{t_1}$ for $i = A, B$

Fig. 6 R&D investment at time t_1

FIRM A \ FIRM B	*Wait*	*Invest*
Wait	Nash Equilibrium **54 110 $** **45 026 $**	**44 471 $** 25 507 $
Invest	41 462 $ **40 328 $**	9 552 $ –3 743 $

(a) I Case: $V = 1\,200\,000\,\$$

FIRM A \ FIRM B	*Wait*	*Invest*
Wait	78 484 $ **66 187 $**	**63 118 $** 63 097 $
Invest	Nash Equilibrium **82 470 $** **57 643 $**	43 725 $ 27 581 $

(b) II Case: $V = 1\,400\,000\,\$$

FIRM A \ FIRM B	*Wait*	*Invest*
Wait	105 733 $ 90 115 $	Nash Equilibrium **83 714 $** **101 364 $**
Invest	Nash Equilibrium **124 216 $** **76 789 $**	78 513 $ 59 470 $

(c) III Case: $V = 1\,600\,000\,\$$

FIRM A \ FIRM B	*Wait*	*Invest*
Wait	166 377 $ 144 027 $	129 195 $ **179 371 $**
Invest	**209 314 $** 118 981 $	Nash Equilibrium **149 428 $** **124 476 $**

(d) IV Case: $V = 2\,000\,000\,\$$

Fig. 7 Final payoffs

Similarly, if $V \in]V_P^*, V_S^*[$, using the inequality among the four strategic values given in the propositions (1) and (2), it results one Nash equilibrium (L_A, F_B).

If the expected market value $V \in]V_Q^*, V_P^*[$, we observe that:

$$L_A(V) > W_A(V); \quad S_A(V) < F_A(V); \quad L_B(V) > W_B(V); \quad S_B(V) < F_B(V);$$

and so we obtain two Nash equilibriums: (L_A, F_B) and (L_B, F_A). In the first equilibrium, A invests immediately at time t_0 while B decides to postpone its R&D decision at time t_1 waiting for better information, vice versa in the second equilibrium. If we consider that $V = 1\,600\,000\$$, we have two Nash equilibriums as it is represented in the Fig. 7c.

At last, if we assume that the expected market value $V > V^*$, it results that:

$$L_A(V) > W_B(V); \quad L_B(V) > W_B(V); \quad S_A(V) > F_A(V); \quad S_B(V) > F_B(V).$$

In this case, both firms decide to invest simultaneously in R&D at time t_0 to take advantage of high market value. In particular, if we assume that $V = 2\,000\,000\$$, the Fig. 7d shows the existence of thi simultaneous Nash equilibrium (S_A, S_B).

Finally, the Fig. 8 illustrates the Nash equilibriums according to the several market values V.

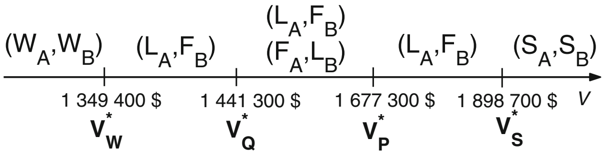


Fig. 8 Nash equilibriums considering $p = 0.55$ and $q = 0.60$

5 The Effects of $\rho(X, S)$ and α on the Equilibriums

In this section we provide a numerical study about the influence of key parameters $\rho(X, S)$ (information revelation) and α (first mover's advantages) on the Nash equilibrium behavior of the firms A and B. Assuming the same parameters summarized in the previous section, the Table 4 shows the effects that the variation of information revelation produces on the game ranges.

First of all, the condition to respect to have $0 \le p^+ \le 1$ and $0 \le p^- \le 1$ (see Eqs. 11 and 12) according to the positive information revelation that benefits the Follower, namely $\rho(X, S) \ge 0$ is that:

$$0 \le \rho(X, S) \le \min\left\{\sqrt{\frac{p(1-q)}{q(1-p)}}, \sqrt{\frac{q(1-p)}{p(1-q)}}\right\} \tag{34}$$

So, for our adopted values, it results that $0 \le \rho(X, S) \le 0.9026$.

We can remark that the Leader's payoff and the Waiting one are independent by $\rho(X, S)$, and so the critical market values V_W^* and V_Q^* are not affected by information revelation. If $\rho(X, S) = 0$ there is no information revelation and the game ranges are given by first mover's market share $\alpha = 0.60$. Moreover, when the information revelation increases, then the game windows $]V_Q^*, V_P^*[$ (in which we have two Nash equilibriums) and $]V_P^*, V_S^*[$ (in which we have one Nash equilibrium) enlarge.
The Table 5 summarizes the variation of first mover's advantage α on the game ranges.

Table 4 Variation of information revelation with $\alpha = 0.60$

$\rho(X, S)$	V_W^*	V_Q^*	V_P^*	V_S^*
0	1 349 400	1 441 300	1 478 200	1 580 600
0.10	1 349 400	1 441 300	1 481 300	1 584 700
0.20	1 349 400	1 441 300	1 490 900	1 597 900
0.30	1 349 400	1 441 300	1 507 500	1 621 600
0.40	1 349 400	1 441 300	1 532 300	1 658 100
0.50	1 349 400	1 441 300	1 566 900	1 711 500
0.60	1 349 400	1 441 300	1 641 000	1 788 200
0.70	1 349 400	1 441 300	1 677 300	1 898 700
0.80	1 349 400	1 441 300	1 763 400	2 060 800
0.90	1 349 400	1 441 300	1 881 700	2 295 000

Table 5 Variation of First mover's advantage with $\rho(X, S) = 0.70$

α	V_W^*	V_Q^*	V_P^*	V_S^*
0.60	1 349 400	1 441 300	1 677 300	1 898 700
0.70	1 049 800	1 120 000	1 333 400	1 452 700
0.80	893 700	952 700	1 198 400	1 288 500
0.90	791 700	843 400	1 148 600	1 230 100
1	717 600	764 000	1 143 300	1 224 100

We can observe that, when the first mover's advantage α increases, then the game ranges $]V_W^*, V_Q^*[$ and $]V_P^*, V_S^*[$ decrease while the range $]V_Q^*, V_S^*[$ increases. Moreover, all the critical market values go down.

Finally, it is interesting to analyse the simplest symmetric situation in which the success probability is equal for both players $(p = q)$. So it results that:

$$L(V) \equiv L_A(V) = L_B(V); \quad F(V) \equiv F_A(V) = F_B(V);$$
$$S(V) \equiv S_A(V) = S_B(V); \quad W(V) \equiv W_A(V) = W_B(V).$$

In this case it's obvious the equality among the critical market values:

$$V_A^W = V_B^W; \quad V_W^* = V_Q^*; \quad V_A^S = V_B^S; \quad V_S^* = V_P^*.$$

If we also assume that there is not information revelation or first mover's advantage, so $\alpha = 0.50$ and $\rho = 0$, we have that

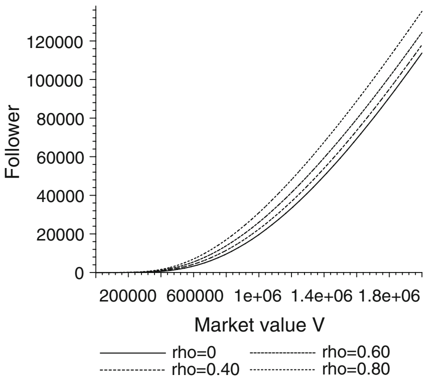
$$L(V) = W(V) \iff F(V) = S(V) \Rightarrow V_W^* = V_S^*$$

and therefore the overall game range $]V_W^*, V_S^*[$ is empty. So, if $V < V^*$ both players decide to wait to invest, otherwise they invest simultaneously in R&D at time t_0.

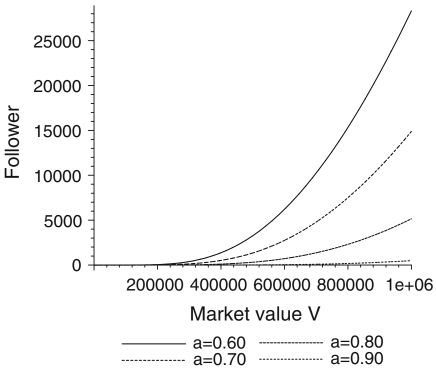
Finally, the Figs. 9 and 10 show the effects that the variation of first mover's advantage and information revelation bring about the Follower's and Leader's strategic option value. In particular way, if we consider a growth of first mover's advantage, we can observe that the Leader's value increases while the Follower's one decreases, as shown comparing the Figs. 10 and 9b. Instead, in case of growth of $\rho(X, S)$, the Follower benefits of higher information revelation, as shown in the Fig. 9a, while the Leader's value does not depend by $\rho(X, S)$.

6 Concluding Remarks

The real options and game theory approach allows to incorporate market opportunities and competitive move in an uncertain environment. Several R&D investments are characterized from positive externalities and uncertainty of development cost D to realize the new product. So, an R&D investment opportunity corresponds to an exchange real option because also the exercise price is stochastic. In this paper, we



(a) Variation of Information Revelation



(b) Variation of First Mover's Advantage

Fig. 9 The effects of α and ρ on Follower's value

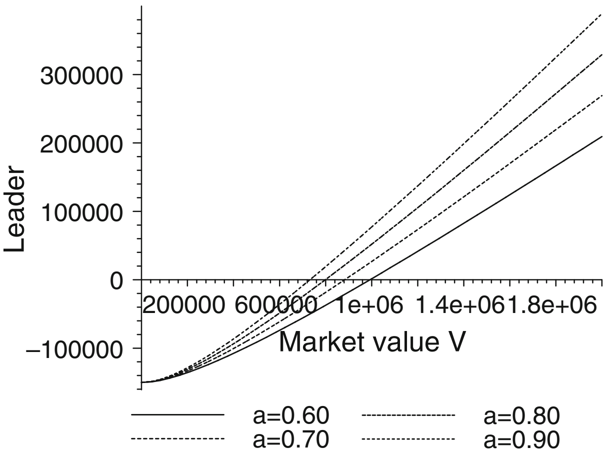


Fig. 10 The effects of α on Leader's value

proposed an option game between two firms that invest in $R\&D$. The first firm that enter in the market has got a first mover's advantage but it reveals an information about its R&D investment that benefits the other one. Using the compound and the simple exchange options, we derive the values of R&D investment opportunities in a competitive environment. The computation of critical market values V_W^* and V_S^* allows to determine the game ranges in which the waiting policy (W_A, W_B) or the simultaneous investment (S_A, S_B) is convenient, in Nash meaning, for both players. In addition, in the ranges $]V_W^*, V_Q^*[$ and $]V_P^*, V_S^*[$, we obtain one Nash equilibrium like (L_A, F_B), in which the firm with an higher success probability invests in R&D earlier then its rival, while in the range $]V_Q^*, V_P^*[$ we have two Nash equilibriums: (L_A, F_B) and (L_B, F_A). The model can be improved using American exchange options to cover the flexibility to invest D before the maturity date T and the theory of stopping time to determine the best time to invest. Finally, we can analyse the cooperation between A and B in R&D via joint research venture and so to repair the inefficiency of one shot game.

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References

Armada, M. R., Kryzanowsky, L., & Pereira, P. J. (2007). A modified finite-lived American exchange option methodology applied to real options valuation. *Global Finance Journal, 17*(3), 419–438.

Carr, P. (1988). The valuation of sequential exchange opportunities. *The Journal of Finance, 43*(5), 1235–1256.

Carr, P. (1995). The valuation of American exchange options with application to real options. In L. Trigeorgis (Ed.), *Real options in capital investment: Models, strategies and applications*. Westport Connecticut, London: Praeger.

Copeland, T., & Antikarov, V. (2003). *Real options: A practitioner's guide*. New York, Texere.

Dias, M. A. G., & Teixeira, J. P. (2004). Continuous-time option games part 2: oligopoly, war of attrition and bargaining under uncertainty. *Working Paper, PUC-Rio, presented at 8th Annual International Conference on Real Options, Montreal, June 2004.*

Dias, M. A. G. (2004). Real options, learing measures, and Bernoulli revelation processes. *Working paper, Puc-Rio, presented at 8th Annual International Conference on Real Options, Paris, June 2005.*

Geske, R., & Johnson, H. E. (1984). The American put valued analytically. *Journal of Finance, 39*(5), 1511–1542.

Ingersoll, J., & Ross, S. (1992). Waiting to invest: Investment and uncertainty. *Journal of Business, 65* (1), 128–156.

Lambrecht, B. M. (2000). Strategic sequential investments and sleeping patents. In M. J. Brennan & L. Trigeorgis (Eds.), *Project flexibility, agency, and product market competition: New developments in the theory and application of real options analysis*. London: Oxford University Press.

Lee, M. H. (1997). Valuing finite-maturity investment-timing options. *Financial Management, 26*(2), 58–66.

Majd, S., & Pindyck, R. S. (1987). Time to build, option value and investment decisions. *Journal of Financial Economics, 18*(1), 7–27.

Margrabe, W. (1978). The value of an exchange option to exchange one asset for another. *The Journal of Finance, 33* (1), 177–186.

Mason, R., & Weeds, H. (2002). Reverse hysteresis': R&D investment and stochastic innovations, mimeo.

McDonald, R. L., & Siegel, D. R. (1985). Investiment and the valuation of firms when there is an option to shut down. *International Economic Review, 28*(2), 331–349.

Smit, H. T. J., & Trigeorgis, L. (2004). *Strategic investment: Real options and games*. Princeton University Press.

Trigeorgis, L. (1991). Anticipated competitive entry and early preemptive investment in deferrable projects. *Journal of Economics and Business, 43*(2), 143–156.

Weeds, H. (2000). Sleeping patents and compulsory licensing: An options analysis. *Warwick economics research paper* (Vol. 579).

Weeds, H. (2002). Strategic delay in a real options model of R&D competition. *Review of Economics Studies, 69*(3), 729–747.