Computational Economics (2006) 27: 453–481
DOI: 10.1007/s10614-005-9000-8

A Classification System for Economic Stochastic Control Models

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Accepted 11 August 2005

Abstract. The lack of a clear classification structure and the use of a variety of names for the same solution method for stochastic control models in economics, create communications inefficiencies in the field. A proposal is made for a classification system based on a number of attributes of these models including stochastic elements, solution classes, estimation method, forward-looking variables and policies-to-parameters effects. Tables are provided which categorize some well-known example models into this structure. Our work focuses on models with quadratic criterion functions and linear systems equations and without game theory elements. Thus it is a mere start of a larger effort which is much needed since there has been a proliferation of stochastic control models in economics in recent years.

Key words: stochastic control, economic stochastic control models, classification system, naming system, adaptive control, min–max control, robust control

1. Introduction

The use of stochastic control methods in economics has increased rapidly in recent years and the complexity of the models has grown at the same time. As this has occurred it has become apparent that communication between modelers would be greatly facilitated by the availability of a classification system for these models. At times in the past we have had the experience of hearing a lecture on a new method and having considerable difficulty relating the new approach to existing methods. However, if we had had in place a classification system, communication about the new method could have been achieved much more quickly.

Also, the need for a classification system stems, in part, from a confusing set of names that we economists have inherited from control engineers and used in ways that have added to the confusion. In addition, game theory contains a set of names that sometimes overlap with those used in control theory. As a consequence of the confusion, the two authors of this paper have debated on numerous occasions what name should be applied to one or another solution method.

Thus this paper provides a classification scheme for stochastic control models in economics. It might be possible, as in Rausser and Pekelman (1978), to return to our roots and adopt a classification scheme closely akin to that used in the engineering literature. However, the development of the methodology in economics has taken a separate course since the famous meeting on economics and control theory in May of 1972 at Princeton University hosted by Edwin Kuh, Gregory Chow, M. Ishaq Nadiri and Michael Athans, cf. Kendrick (2005). Also, it is useful while setting out the classification scheme to cite well-known models in economics as examples of each of the different methods. These models can provide landmarks on the terrain already crossed and the hills yet to be climbed in the application of stochastic control methods to economics.

The classification system used here is restricted to models with quadratic criterion functions and linear systems equations.[1] Many economic models are either naturally quadratic-linear models or can be reasonably approximated in the neighborhood of the solution by quadratic-linear systems. Also, the models can be iteratively solved with the approximations redone at each step. None-the-less, at a later date we would hope to extend the system to include general nonlinear models. Also, we have not yet extended the classification system in any significant way to game theory models – rather the reader is referred to Basar and Olsder (1999). Thus our work is a mere start of a much larger effort that is needed to cover the proliferation of stochastic control models that have been developed in economics in recent years.[2]

Prior to beginning the discussion of the classification systems it is useful to write the mathematics of the basic quadratic-linear optimal control models in order to establish notation that will be helpful later. This is followed by a section on the major attributes of the classification system such as the various types of stochastic elements, the solution procedure and the estimation procedure. These attributes are then used in a section on the core of the classification system and in the following section this system is applied to a small set of example models. The focus then shifts briefly to the naming system with a discussion of the defaults for that system while the number of options for each attribute is still small.

In the following section the core classification systems is extended to more options within each attribute in order to expand the coverage to a substantial portion of the quadratic linear stochastic control models that have been used in the past. Then in the next-to-last section of the paper this extended classification system is applied to a larger set of example models. The last section of the paper contains the conclusions.

2. Mathematics of the Basic Quadratic-Linear Optimal Control Models

The criterion function for most economic control theory models with finite horizons may be written as

$$J = E\left\{L_N(x_N) + \sum_{k=0}^{N-1} L_k(x_k, u_k)\right\} \tag{2.1}$$

where J = criterion value; E = expectations operator; L_N = criterion function for the terminal period N; x_N = state vector for the terminal period N – an n vector; L_k = criterion function for period k; x_k = state vector for period k – an n vector; u_k = control vector for period k – an m vector, and the system equations may be written as

$$x_{k+1} = f_k(x_k, u_k) + \xi_k \tag{2.2}$$

where f_k = vector of n system equation functions for period k; ξ_k = vector of additive noise terms – an n vector.

Most of the time a quadratic criterion function is normed. It contains desired paths for both state and control variables but does not contain cross terms between states and controls. Thus the criterion function may be written as

$$L_N(x_N) = \frac{1}{2}(x_N - \tilde{x}_N)' W_N (x_N - \tilde{x}_N) \tag{2.3}$$

and

$$L_k(x_k, u_k) = \frac{1}{2}[(x_k - \tilde{x}_k)' W_k (x_k - \tilde{x}_k) + (u_k - \tilde{u}_k)' \Lambda_k (u_k - \tilde{u}_k)] \tag{2.4}$$

where $\tilde{x}_N$ = desired state vector for terminal period N – an n vector; W_N = symmetric state variable penalty matrix for terminal period, N; $\tilde{x}_k$ = desired state vector for period k – an n vector; W_k = symmetric state variable penalty matrix for period k; $\tilde{u}_k$ = desired control vector for period k – an m vector; Λ_k = symmetric control variable penalty matrix for period k.

In other cases the criterion function is written as the more general quadratic form as shown below. It contains quadratic and linear terms in the state and control vectors as well as cross terms between the states and controls. In this case the criterion function elements are

$$L_N(x_N) = \frac{1}{2}x_N' W_N x_N + w_N' x_n \tag{2.5}$$

$$L_k(x_k, u_k) = \frac{1}{2}x_k' W_k x_k + w_k' x_k + x_k' F_k u_k + \frac{1}{2}u_k' \Lambda_k u_k + \lambda_k' u_k \tag{2.6}$$

where w_N = linear state coefficient vector for period N; w_k = linear state coefficient vector for period k; λ_k = linear control coefficient vector for period k; F_k = coefficient matrix for the $x_k - u_k$ cross term for period k.

Quadratic-linear computer codes are usually based on the more general form in Equation (2.5) so quadratic tracking functions in Equation (2.3) are transformed in the computer codes to the quadratic form before the model is solved (see Kendrick, 1981, 2002, pp. 7–8).

The systems Equations[3] (2.2) in the quadratic linear case are specialized to

$$x_{k+1} = A_k x_k + B_k u_k + C_k z_k + \xi_k \tag{2.7}$$

where $k \in [0, N-1]$ is the time index; z_k= the exogenous vector for period k with ℓ elements; A_k = state vector coefficient matrix for period k; B_k = control vector coefficient matrix for period k; C_k= exogenous vector coefficient matrix for period k; ξ_k = vector of additive noise terms for period k.

In many cases the z_k vector consists of a single variable that is one in all time periods and the elements of the only column of the C_k matrix are the intercept terms in the system equations. In the additive noise case to be discussed below the only uncertainty is in the ξ_k term and theA_k, B_k and C_k matrices contain constant parameters.

In contrast, in the case of uncertain parameters a subset of the elements of these parameter matrices have true constant values but these true values are unknown to the policy maker. Rather the policy maker knows only the first two moments (mean and variances) of the parameter estimates. This situation is represented in the notation by creating a vector θ_k that contains the uncertain parameters. For example if the model had three state variables and two control variables the A_k matrix would be 3×3, theB_k matrix would 3×2 and the C_k matrix would be 3×1. Then if only the coefficients a_{11}, a_{23}, b_{22} and c_{31} were to be treated as uncertain the θ_k vector of uncertain parameters would be

$$\theta_k = \begin{bmatrix} a_{11,k} \\ a_{23,k} \\ b_{22,k} \\ c_{31,k} \end{bmatrix} \tag{2.8}$$

with this notation in hand we can begin a description of the classification system with a discussion of the main attributes of stochastic control models.

3. Attributes of the System

Some of the confusion of the past stems from the failure to appreciate fully the different attributes of stochastic control models. In this regard we find it useful to classify solution methods using the following attributes:

1. stochastic elements
2. solution classes
3. estimation procedure
4. forward-looking variables
5. policies-to-parameters effects

Each of these attributes is considered in turn in the following.

3.1. STOCHASTIC ELEMENTS

The simplest stochastic control models have a single uncertain vector, namely the additive noise terms, ξ_k, in the systems Equations (2.7). More complicated models have uncertain parameters, measurement errors, uncertain initial state vectors and time-varying parameters.

3.2. SOLUTION CLASSES

The term solution "classes" is used here because it is useful to create first a set each of whose elements is a solution class, i.e. a broad category of solution methods. Then, since the method used to solve a model depends on the other attributes of the model as well as on the solution class, a set is created for each solution class whose elements are solution methods. For example "optimal feedback rule" and "expected optimal feedback rule" are two solution classes. In turn, the optimal feedback rule class consists of methods that differ when they are applied to models without and with additive noise terms. Thus it is useful to think of a set of solution classes with each solution class consisting of a set of solution methods.

The use of a "set of sets" here is confusing at first; however it can prove to be most useful in helping individuals to communicate with one another about the methods used to solve stochastic control models.

Five solution classes are used as options for the solution classes attribute. The first of these is handcrafted feedback rules (Taylor rules) that are developed without the benefit of optimization procedures. The second is optimal feedback rules in which dynamic programming is used for optimization that yields Riccati equations.[4] The third is the min–max solution class in which the criterion function is minimized over some arguments and maximized over others. The fourth, expected optimal feedback, is use when there is parameter uncertainty. In this case expectations operators are used in the optimal feedback rules. Finally, methods in the dual control class consider tradeoffs between reaching target paths and improving parameter estimates.

3.3. ESTIMATION PROCEDURE

The simplest models treat the parameters as constant and known so there is no estimation and no parameter updating. More complex models use the Kalman filter, least-square or other estimation procedures.

3.4. FORWARD-LOOKING VARIABLES

Forward-looking variables are used in the systems equations of some models, i.e. the state variables in period $k + 1$ are a function not only of states and controls in period k but also of expected values of state variables in periods $k+2$, $k+3$, etc. This is a specification that is used to incorporate rational expectation into models. Since the presence of forward-looking variables has substantial effect on the solution method it is an important attribute of the classification system.

3.5. POLICIES-TO-PARAMETERS EFFECTS

In a critique of the use of optimal control in macroeconomics Lucas (1976) expressed concern that the announcement of policy variables would results in changes in behavioral parameters and thus invalidation of the previously computed optimal feedback rule.[5]

Though this critique can be addressed with game theory, a simpler approach is to use a model with time-varying parameters and to employ a Kalman filter to update parameter estimates each time period. Thus the optimal controller is only one period behind the change in behavior, so if the changes are relatively small the optimal feedback policy will be close. In order to evaluate "close" one needs a policy based on prior knowledge of the changes in behavior in response to changes in policy. This policy is an example of different information sets and is called "insight" by Amman and Kendrick (2003).

Rather than dealing with all of these attributes at once it is useful to begin with a simple structure that uses only the core of the classification system.

4. The Core of the Classification System

The core system focuses on the first two attributes, namely the stochastic elements and the solution classes. Moreover, the core system is limited to three options for each attribute.

The three options for the stochastic elements are (1) none (deterministic), (2) additive noise terms in the systems equations and (3) uncertain parameters in the systems equations.[6] The three options for the solution classes are (1) handcrafted feedback rules, (2) optimal feedback rules obtained from the Riccati equations or (3) optimal feedback rules obtained when the expected value operator is applied in the Riccati equations. The core classification scheme for this situation is as shown in Figure 1.

The stochastic elements are in the top part of the diagram and the solution classes are shown in the bottom part. For example, if there are no stochastic element and the solution class is "no feedback" the method will be like the one used by Pindyck (1972, 1973a,b). This method is also frequently called "open loop" because the feedback loop is not closed and the solution to the model is computed for all time periods at once without the necessity to wait to solve the model in later periods after the stochastic elements from earlier periods have been manifest. The use of

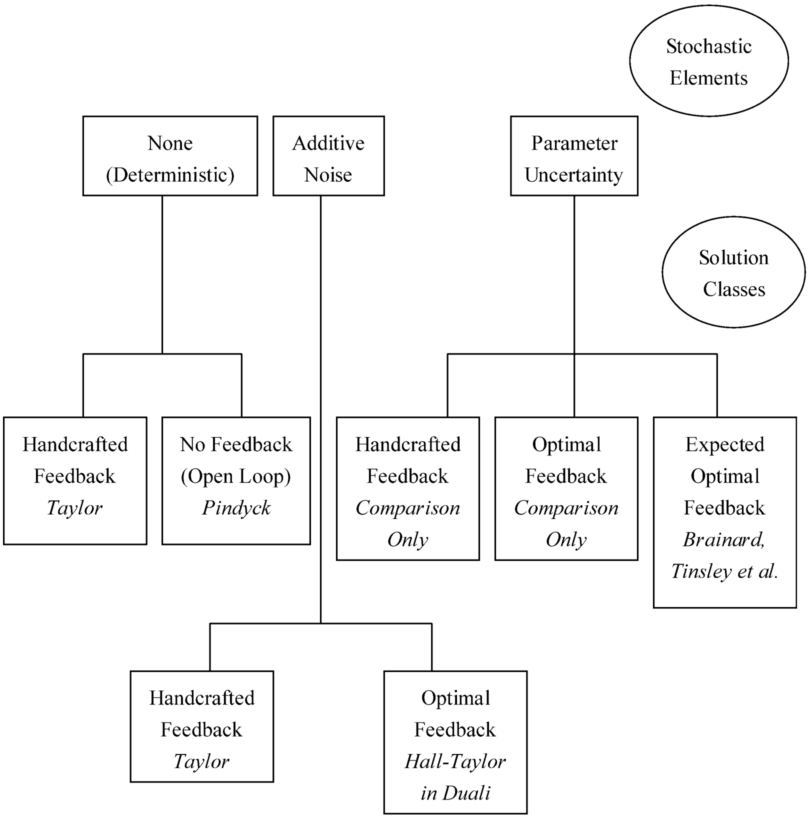

Figure 1. The core of the classification system in tree format.

the term "open loop" here is akin to its use in game theory where it means that a player devises his strategy without taking into account feedback from other players. In the control case "open loop" means that there is no account taken of feedback from nature in the form of uncertain events.

While the quadratic-linear Riccati method used by Pindyck also provides a feedback rule, it is not necessary to use that rule in a feedback manner to solve the model since there are no stochastic elements.

The third column in Figure 1 is the case when the stochastic elements include parameter uncertainty and the solution class is "expected optimal feedback". In this case the method is like the method developed by Farison, Graham and Shelton (1967) and Aoki (1967) that is sometimes called "open loop feedback". It is similar to the methods used by Brainard (1967) and Tinsley, Craine and Havenner (1974) and Chow (1975) on macroeconomic models.

Examples are not given for some of the methods shown in Figure 1 because these methods are only used for purposes of comparison. For example, when there is parameter uncertainty in a model, it can be useful to solve the model with (1)

Table I. The core classification system in table format.

	Solution Classes		
Stochastic elements	Handcrafted feedback	Optimal feedback	Expected optimal feedback
None	Handcrafted feedback (HF)	Open loop (OL)	
Additive noise	Handcrafted feedback (HF)	Optimal feedback (OF)	
Parameter uncertainty	Handcrafted feedback (HF)	Optimal feedback (OF)	Expected optimal feedback with EOF

methods that ignore and (2) methods that take account of that source of uncertainty. See for example Amman and Kendrick (1999b) for a comparison of optimal feedback and expected optimal feedback solution classes applied to a macroeconomic model with parameter uncertainty.

The tree format used in Figure 1 becomes too unwieldy when more than two attributes are used; therefore we will mostly use a table format like that shown in Table I to depict the classification system.

In this table the row and column headings are "options" for the attributes. The entries in the table are "methods". Thus the method that uses the optimal feedback solution class to solve a model that has no stochastic elements is called Open Loop (OL) and the method that uses this same solution class to solve a model that includes additive noise terms is called Optimal Feedback (OF).[7]

In this and subsequent tables the stochastic element options will be treated as nested. Thus models with parameter uncertainty will also have additive noise terms. The handcrafted feedback rules are those that are crafted without the benefit of optimization methods, but which nonetheless may be carefully developed based on experience with simulating alternate rules. They might alternatively be called "Phillips rules" after A. W. Phillips who was one of the pioneers of the use of classical control feedback rules for use in economics models. Currently the macroeconomic applications of handcrafted feedback rules are sometimes called "Taylor rules" after John B. Taylor; however the Taylor label is also applied at times to a variety of types of feedback rules and not just to handcrafted rules.

Linear optimal feedback rules are obtained for models with quadratic criterion functions and linear systems equations when one uses either of the latter two solution class options. These two options differ only in the use of the expectations operator over the uncertain parameters. In the first of these two solution classes the optimal feedback rule is written (see Kendrick, 2002, Ch. 2) as

$$u_k = G_k x_k + g_k \tag{4.1}$$

where G_k = the feedback gain matrix; g_k = the feedback gain vector, with

$$G_k = -[B_k' K_{k+1} B_k + \Lambda_k']^{-1} [F_k' + B_k' K_{k+1} A_k] \tag{4.2}$$

and the g_k vector computed in a similar manner.

In contrast, the expected optimal feedback rule (see Kendrick (2002), Ch. 6) is written as

$$u_k = G_k^{\dagger} x_k + g_k^{\dagger} \tag{4.3}$$

with

$$G_k^{\dagger} = -[E\{B_k' K_{k+1} B_k\} + \Lambda_k']^{-1} [F_k' + E\{B_k' K_{k+1} A_k\}] \tag{4.4}$$

and with the g_k vector computed in a similar manner. In this expression the E is the expectations operator that is taken over the uncertainty in the parameter estimates in the matrices A_k, B_k and C_k. Also the expectations operator plays a similar role in the computation of the Riccati matrices, K_k. This expectation operator comes from the objective function for the model.

It is significant that the only difference between the optimal feedback rule in Equation (4.2) and the optimal feedback rule with expectations is the expectations operator used in the calculation of the feedback gain matrix, $G_k^{\dagger}$, and vector, $g_k^{\dagger}$.[8]

Consider next examples of each of the types of stochastic control listed in Table I.[9]

5. Examples from the Core Classification

In the discussion below the examples are categorized primarily by the attribute of stochastic elements and then secondarily by the solution classes.

5.1. NO STOCHASTIC ELEMENTS

One of the first applications of handcrafted feedback rules to economic models was Phillips (1954) who used PID (proportional, integral, differential) controllers from engineering on macroeconomic models. Twenty years later Healey and Summers (1974) used a procedure in which a linear feedback rule was written down and a systematic search was performed to find the best values for the feedback gain matrix and vector. Later Karakitsos and Rustem (1984, 1985) applied handcrafted feedback rules to large macroeconomic models. Recently handcrafted rules have achieved substantial influence from the prominence they have been given by the works of John B. Taylor including Taylor (1993, 1999). The rules are crafted by beginning with a feedback gain matrix, G_k, and vector, g_k, and using this rule to simulate the model. Then the elements are altered and the simulation is repeated until useful stabilization policies are obtained.

In contrast, optimal feedback rules are computed using dynamic programming to obtain Riccati equations that in turn provide the key elements to compute the feedback gain matrix and vector as in Equation (4.1) above. One of the first applications of this approach to macroeconomic models was Holt (1962). This was followed ten years later by the work of Pindyck (1972, 1973a,b) on a ten-equation macroeconomic model of the U.S. economy. Models of this type are labeled in Table I as "Open Loop (OL)". This means that there is no explicit use of feedback and thus the feedback loop is "open". Thus the solution for a multiperiod model can be computed for all periods at one time. While a feedback rule is computed that could be used period by period, the controls and states are computed for all periods at time zero and do not change with the evolution of the system over time. This is so because these are deterministic solutions with no uncertain elements, not even an additive noise term. For an innovative use of this modeling approach to the relative macroeconomic performance of U.S. presidents see Fair (1978).

In the past this method has sometimes been called QLP (Quadratic Linear Problems). This name is not descriptive but has come to be associated with methods used to solve models with quadratic criterion functions and linear systems equations with no uncertainty. The term Open Loop is used in both engineering and economics circles as a label for solution methods used to solve models with no uncertainty so it seems nicely descriptive of this method.

5.2. ADDITIVE NOISE TERMS

Additive noise terms are the most common addition to deterministic models when stochastic control is first considered, viz Simon (1956) and Theil (1957). To focus on these terms, consider a subset of elements of Table I that contains only the simplest two stochastic element options and the simplest two solution classes as in Table II below.

The distinction to be made here is between systems equations without and with the additive noise terms, ξ_k, in Equation (2.7) i.e.

$$x_{k+1} = A_k x_k + B_k u_k + C_k z_k + \xi_k \quad k = 0, \ldots, N-1 \tag{5.1}$$

Table II. Illustrative examples with a focus on additive noise elements.

Stochastic elements	Solution classes	
	Handcrafted feedback rule	Optimal feedback rule
None	HF Taylor (1993, 1999)	OL Holt (1962), Pindyck (1973a), Fair (1978)
Additive noise	HF Taylor (1993, 1999)	OF Hall and Taylor (1993) in Duali

When these noise terms are absent the solution can be computed for all time periods from period zero. In contrast, when there is an additive noise term the control in each period cannot be computed from the feedback rule until the state is known and the state will not be known until the additive noise is manifest in the previous period. Thus ξ_k is instrumental in determining x_{k+1} through the systems Equations (5.1) before x_{k+1} can be used to compute u_{k+1} with the feedback rule

$$u_{k+1} = G_{k+1}x_{k+1} + g_{k+1} \tag{5.2}$$

This applies whether the feedback rule is handcrafted or optimal.

The Optimal Feedback (OF) method in the lower right corner of Table II has frequently been called Certainty Equivalence (CE) in the past; however we have used Optimal Feedback here to distinguish this method from the Expected Optimal Feedback (EOF) method to be discussed later. Also the phrase "certainty equivalence" gives the connotation that the uncertainty can be ignored. While this is true in computing the feedback rule in each time period, it is not true in computing the state for the next time period when the additive noise term must be used.

For examples of variants of models with additive noise terms see Amman and Kendrick (1999c). This *User's Guide* for the Duali software provides additive noise versions of many well-known models such as those of Pindyck (1973a), Abel (1975), MacRae (1972) and Hall and Taylor (1993) [in Mercado and Kendrick (1999)].[10]

Next we return to the core classification system in Table I and progress to the third stochastic element option, namely parameter uncertainty.

5.3. PARAMETER UNCERTAINTY

Parameter uncertainty occurs when a subset of the parameters in the A_k, B_k and C_k matrices in the systems equations are treated as uncertain. Two cases need to be distinguished here. The one occurs when the true parameters are constant and the estimates are time varying and treated as uncertain. The other occurs when the true parameters are themselves time-varying. We will consider in this part of the paper only the case where the true parameters are constant and postpone until later discussion of the time-varying true parameters case.

As was discussed in the mathematics near the beginning of this paper a subset of the parameters from the A_k, B_k matrices and C_k matrices are treated as uncertain and a stacked up in a vector like

$$\theta_k = \begin{bmatrix} a_{11,k} \\ a_{23,k} \\ b_{22,k} \\ c_{31,k} \end{bmatrix} \tag{5.3}$$

Table III. Illustrative examples for models with parameter uncertainty.

	Solution classes		
Stochastic elements	Handcrafted feedback	Optimal feedback	Expected optimal feedback
Parameter uncertainty	HF for comparison only	OF for comparison only	EOF Brainard (1967), Tinsley, Craine and Havenner (1974), Amman and Kendrick (1999b)

in a case where the (1, 1) and (2, 3) elements from the A_k matrix, the (2, 2) element from the B_k matrix and the (3, 1) element from the C_k matrix are treated as uncertain. The estimates of these parameters include both the means $\hat{\theta}_k$ and the covariance estimates $\Sigma_{k|k}^{\theta\theta}$.

As shown in Table III, models of this type are ordinarily solved with the methods from Expected Optimal Feedback (EOF) class making use of both the means $\hat{\theta}_k$ and the covariances $\Sigma_{k|k}^{\theta\theta}$ through the expectations operator that was discussed above.

However, models of this type can also be solved for comparative purposes with Handcrafted Feedback (HF) and Optimal Feedback (OF) methods by ignoring the variances of the uncertain parameters. Thus there are three solution classes that can be used to solve models with these stochastic elements and therefore three methods, namely Handcrafted Feedback (HF), Optimal Feedback (OF) and Expected Optimal Feedback (EOF). This possibility is implemented in the Duali software that was mentioned above. Indeed, it is an important research question as to whether or not HF and OF methods will do almost as well as EOF methods when applied to models with parameter uncertainty. For a discussion of these issues see the paper by Amman and Kendrick (1999b).

There is a rich literature on the application of EOF methods to macroeconomic control models. The engineering literature that was drawn on by economists near the start of this work was the paper by Farison, Graham and Shelton (1967) and the book by Aoki (1967). Some early papers in this genre were those by Brainard (1967), Shupp (1972, 1976c), Henderson and Turnovsky (1972), Chow (1973, 1975),[11] Kendrick and Majors (1974), Tinsley, Craine and Havenner (1974), Turnovsky (1973, 1975, 1977), Craine, Havenner and Tinsley (1976), Kalchbrenner and Tinsley (1976), Craine (1979), Tinsley, von zur Muehlen and Fries (1982) and Soderstrom (2002). In recent years this work has been extended to a rather large model by Lee (1998).

This completes the discussion of the core system with only the two attributes of stochastic elements and solution class. Next we extend the system to include the other three attributes.

6. Extension of the Core System to Other Attributes

The remaining three attributes of stochastic control models are:

estimation procedure
forward-looking variables
policies-to-parameters effects

Each will be discussed in turn.

6.1. ESTIMATION PROCEDURE

The third attribute of the classification system is the estimation procedure for parameter updating. One has estimates of the parameters at period k and receives an additional observation on the state of the system that can be used to update parameter estimates for period $k + 1$. Thus this procedure is embodied in equations that update the means in $\hat{\theta}_k$ and the covariances in $\Sigma_{k|k}^{\theta\theta}$. A common estimation method is the Kalman filter.[12] Other methods are used by Ljung and Soderstrom (1983), Ljung, Pflug and Walk (1992) and Marcet and Sargent (1989).

Of course some stochastic control models do not use estimation, i.e. the parameters are treated as known and constant so no updating is employed. Models of this type are sometimes used as intermediate steps in a project that will progress later to more complicated methods. For example, Chapter 5 of the Duali *User's Guide* (Amman and Kendrick, 1999c)) includes a model that has uncertain parameters but no updating. This is included in this guide in order to let the user first gain an understanding of the use of the expectations operator in the Riccati equations and the feedback rules before progressing to the more complicated case including parameter updating.

Also, estimation and parameter updating may be omitted at times in order to provide a standard for comparison. For example, we used an optimal feedback (OF) method without updating to compare to an expected optimal feedback (EOF) method and a method with "insight" in a comparison of various stochastic control methods in a policy-to-parameter setting in Amman and Kendrick (2003).

The earlier literature focused on learning by the policy makers about the behavior of decentralized agents while a newer line of literature has focused on learning by decentralized agents, viz Marcet and Sargent (1989), Evans and Honkapohja (2001), Kozicki and Tinsley (2001) and Bullard and Mitra (2002).

A discussion of estimation also raises the question of whether the learning implicit in updating is "passive" or "active". However, this can be a source of confusion because the use of these phrases actually refers to solution methods and not to estimation procedures. An active learning solution method is one in which the degree of learning is considered while deciding on the optimal control variables in each time period. In contrast, in passive learning solution methods one does not consider the impact of the policy choices on learning the parameter values. We will return to a more complete discussion of passive and active learning methods later in this paper.

The term "passive" has been used by others, and by us, in the past in a way that is confusing. For example, we have sometimes debated with one another whether or not to call an expected optimal feedback (EOF) solution method an "open loop feedback" (OLF) method or a "passive" method. The engineering literature has used OLF as a labeled for such solutions but we felt that economists might find that term confusing and therefore on occasion elected to label those solutions as "passive". Also, passive learning may be done in a variety of methods such as Handcrafted Feedback (HF) and Optimal Feedback (OF) as well as in Expected Optimal Feedback (EOF) so the term is not narrow enough to uniquely describe a solution method. See Appendix A for a table of alternate names that have been used at various times to describe the same stochastic control methods.

6.2. FORWARD-LOOKING VARIABLES

When the model includes forward-looking variables to represent rational expectations the system equations become

$$x_{k+1} = A_k x_k + B_k u_k + C_k z_k + D_1 x^e_{k+1|k} + D_2 x^e_{k+2|k} + \xi_k \qquad k = 0, \ldots, N-1 \quad (6.1)$$

where $x^e_{k+1|k}$ = the expected value of the state variable at period $k+1$ as projected from; period k; $x^e_{k+2|k}$ = the expected value of the state variable at period $k+2$ as projected from period k; D_1 = forward looking variable parameter matrix for the $k+1 \mid k$ variables; D_2 = forward looking variables parameter matrix for the $k+2 \mid k$ variables.

Thus the state variable in period k is a function of the expected value of future state variables as well as of the lagged state and control variables.

There are many methods to solve models with forward variables including those of Blanchard and Khan (1980), Anderson and Moore (1985), Fisher, Holly and Hughes Hallett (1986), Fair and Taylor (1993), Juillard (1996), Zadrozny and Chen (1999) and Sims (2002). If the model is deterministic the equations for all time periods can simply be stacked up and solved all at once with an open loop (OL) method. If there are additive noise terms or more complicated stochastic specifications a simple recursive method like Fair-Taylor can be used or alternatively the methods of Blanchard-Khan or the QZ method of Sims can be employed. For high-speed solution of large models with forward variables many have found it useful to employ Anderson and Moore's method.

One of the classic examples of models with forward variables in a deterministic framework is Sargent and Wallace (1975). This was followed by a number of papers that dealt with forward variables in the optimal policy context viz Oudiz and Sachs

(1985), Backus and Driffill (1986), Levine and Currie (1987) and Pearlman, Currie and Levine (1986).[13] We have done a series of papers on models with forward variables starting with deterministic models in Amman and Kendrick (1996), then moving to models with additive noise terms (1999a) and finally optimal feedback with expectations (2000). For a recent paper using handcrafted rules in models with forward variables see Levin, Wieland and Williams (2003). See also Giannoni (2002) and Woodford and Svensson (2003) for studies that devise optimal rules that are robust to alternative model specification in the presence of forward variables. For examples of restricted optimal handcrafted decision rules in this context see Tetlow and von zur Muehlen (2001c).

6.3. POLICIES-TO-PARAMETERS EFFECTS

As was discussed above the policies-to-parameters effects have been used by Amman and Kendrick (2003) to represent a comparative but unrealistic situation in which policy makers have the "insight" to know the effect of policy changes on behavioral parameters in the sense of Lucas (1976). This is used as a standard with which to compare more realistic solutions in which the policy makers uses a Kalman filter to update estimates of the behavioral parameters after they change due to policy effects. Our experience with numerical results in this area is mixed so far and we think that much work remains to be done to sort out correctly the effects of insight.

7. Defaults

When there are a number of attributes to consider and each attribute has a number of options it is possible to spell out the specification of a method with a phrase or sentence. This is most useful and can save considerable time in communicating the nature of the stochastic control model under study. For example a method might be described as

> expected optimal feedback in a model with parameter uncertainty and with forward variables

However, in many cases it would be convenient to be able to describe a model with a simple acronym rather than to use a phrase or sentence. This has the drawback that various individuals may use a given acronym to refer to different things. Also, in the early use of a set of acronyms, before they became a part of the lexicon in a field, their use adds to the confusion rather than dissipates it. Thus in reading the next few paragraphs the reader should expect to feel that the proposed acronyms create a fog rather than a clear view. However, it is inevitable that we will want to use a set of acronyms to refer to the various stochastic control methods, so please bear with us.

A set of acronyms can help in communication if there is some agreement about the default specification associated with the more widely used acronyms. Thus the most frequently used acronyms can consist of only a few letters rather than a long list and can still communicate accurately the intent of the user. For example, the default for expected optimal feedback (EOF) could be that there is

(1) parameter uncertainty
(2) updating of parameter estimates
(3) no forward-looking variables

Thus EOF could be used for the specification above and EOF with F could be used for the same case but where there are forward variables. Consider a few such defaults

HF	handcrafted feedback	handcrafted feedback rule, no uncertain elements, no updating
OL	open loop	no uncertain elements, no forward variables
OF	optimal feedback	optimal feedback rule, additive noise, updating of parameter estimates. no forward variables[a]
EOF	expected optimal feedback	optimal feedback rule, uncertain parameters, no forward variables

[a]One might also add to this list OHF for optimal handcrafted rules – a kind of half-way house between HF and OF.

Modifications of these acronyms could be accomplished by adding letters such as

wA	with additive noise
wP	with parameter uncertainty
woU	without updating
wF	with forward variables
wM	with measurement error
wL	with policies-to-parameters (Lucas) effects
wT	with time-varying parameters

Then some example specifications could be described as follows:

OLwF	open loop with forward variables
OFwF	optimal feedback with forward variables
EOFwoU	expected optimal feedback without updating
EOFwM	expected optimal feedback with measurement error
EOFwL	expected optimal feedback with policies-to-parameters (Lucas) effects

8. Extension of the Core Classification System

In the extension, like in the core system, we will focus at the beginning on the first two attributes, namely the stochastic elements and the solution classes. However, instead of limiting the system to three options for each attribute we will include the full array. For example, as sources of uncertainty we will now consider

none
additive
uncertain parameter
measurement noise
time-varying parameters

In models with measurement noise the state variables are not known perfectly but rather are seen through a noisy observer of the form[14]

$$y_k = Hx_k + \zeta_k \tag{8.1}$$

where y_k = observation vector ($rx1$); H = a known matrix (rxn); ζ_k = measurement noise (rx1).

One implication of this is that solution methods may include the use of the mean and covariance of the state, i.e. $\hat{x}_{k|k}$ and $\Sigma_{k|k}^{xx}$ for the estimates of these statistics at time k using data obtained through period k. Thus it is useful to consider that there are really two additional sources of uncertainty in models that have measurement error. The first is the additive noise in the measurement error, ζ_k, in Equation (8.1) and the other is the uncertainty of the state in the initial time period as represented in the initial mean, $\hat{x}_{0|0}$ and covariance $\Sigma_{0|0}^{xx}$.

Models with time-varying parameters are most commonly specified with a first order Markov process in the parameters, i.e.

$$\theta_{k+1} = D\theta_k + \eta_k \tag{8.2}$$

where θ_k = parameter vector; D = a known matrix; η_k = a random vector.

As was discussed above, it is important to make a distinction between models in which the estimates of the parameters are time varying while the true parameters are constant and models in which both the estimates and the true parameters are time varying. Any model in which there is parameter updating will have time-varying parameter estimates but most of these models will have constant true parameters. For example, the solution software, such as Duali, may support the use of Equation (8.2); however in most cases the user sets $D = I$ and $\eta_k = 0$ so that in fact the true parameters are constant.

In the core system the solution types included only three options. Here we extend this to five options, namely

handcrafted feedback rule
optimal feedback rule
min–max feedback rule
optimal feedback rule with expectations
dual control

The min–max solution class has been used for some years in economic modeling, viz Becker, Dwolatzky, Karakitsos and Rustem (1986), Deissenberg (1987), Rustem (1992, 1998), Rustem, Wieland and Zakovic (2001) and Rustem and Howe (2002) but has gained recent prominence in the form of robust control, viz Hansen and Sargent (2001). In the previous methods that we have discussed the criterion function is either minimized or maximized. In contrast, in min–max control the criterion is minimized with respect to one set of control variables and maximized with respect to another set. For example, in robust control the criterion function could be minimized with respect to the usual control vector, u_k, and maximized with respect to a second set of additive noise terms,w, which represent misspecification of the model. Thus an engineer designing a bridge may want to select steel girder sizes which minimize the likelihood of collapse of the bridge since nature might challenging the design with the worst possible additive noise terms. For some recent applications of robust control methods to U.S. macroeconomic models see Onatski and Stock (2002) and Tetlow and von zur Muehlen (2001a,b).

Dual control is best understood as an active learning method, in contrast to the passive learning methods discussed so far. In passive learning methods new observations are obtained each period and used to update the parameter estimates; however, no effort is made to choose control variables that will improve the learning. In contrast, in active learning methods control variables are chosen with the dual purpose of moving the system in desired directions and of perturbing the system so as to improve the parameter estimates. Thus the method is called “dual” control. Also, it is referred to as adaptive control; however this term is less descriptive since even in passive learning the system is adapting over time.

The earliest applications of dual control to economic models were by Prescott (1972), Taylor (1974) and MacRae (1975) followed by Norman (1976, 1979, 1981) and Kendrick (1982) and later by Tucci (1989), Mizrach (1991), Wieland (2000a,b), Beck and Wieland (2002), Cosimano (2003) and Cosimano and Gapen (2005).

The extended classification system is shown below in Table IV The abbreviations for the methods used in the table are shown below.

HF	handcrafted feedback rule
HFwA	handcrafted feedback rule with additive noise
HFwP	handcrafted feedback rule with parameter uncertainty
OL	open loop
OF	optimal feedback
OFwP	optimal feedback with uncertain parameters
OFwM	optimal feedback with measurement error

Table IV. The extended classification system.

	Solution class				
Stochastic elements	Handcrafted feedback	Optimal feedback	MinMax (Robust) control	Expected optimal feedback	Dual control
None	Handcrafted feedback (HF)	Open Loop (OL)	Minmax (Robust) MMC (RC)		
Additive noise	Handcrafted feedback (HFwA)	Optimal feedback (OF)	Minmax (Robust) MMC (RC)		
Parameter uncertainty	Handcrafted feedback (HFwA)	Optimal feedback (OF)		Expected optimal feedback EOF	Dual control DC
Measurement error		Optimal feedback (OFwM)	RCwM robust control with meas error	Expected optimal feedback EOFwM	Dual control DCwM
Time-varying parameters			RCwT robust control with time var par	Expected optimal feedback EOFwT	Dual control DCwMT

EOF	expected optimal feedback
EOFwM	expected optimal feedback with measurement error
MMC	min–max control
RC	robust control with two additive noise terms
RCwM	robust control with measurement error
RCwT	robust control with time-varying parameters
EOFwT	expected optimal feedback with time-varying param.
DC	dual control
DCwM	dual control with measurement error
DCwT	dual control with time-varying parameters

Note that in the Handcrafted Feedback column and Parameter Uncertain row of Table IV the method is HFwA, i.e. handcrafted feedback with additive noise. Thus the method in each box need not reflect the stochastic elements. For example, the handcrafted feedback class does not include a class that considers the effects of parameter uncertainty. None-the-less, it may be instructive to use a handcrafted feedback with additive noise method on a model that has parameter uncertainty in order to compare the result to the use of an expected optimal feedback method (in the Expected Optimal Feedback column and Parameter Uncertainty row). Thus the method name indicates the types of uncertainty considered in the method but not necessarily the types of stochastic elements in the model to which the method is applied.

9. Examples from the Extended Classification System

As before, the examples are categorized primarily by the attribute of stochastic elements and then secondarily by the solution class. However the focus here is on the additional options for both attributes that have been added in the extended system.

The names of a few selected authors whose studies are illustrative of the various methodologies are shown in Table V. A listing of many studies using each method is contained later in the paper. Some additional abbreviations for the methods used in the table are shown below.

OFwMF	optimal feedback with measurement error and forward variables
EOFwoU	expected optimal feedback & without updating
EOFwFTL	expected optimal feedback with forward variables, time-varying parameters and policies-to-parameters (Lucas) effects
DC	dual control
DCwMT	dual control with measurement errors and time-varying parameters

We have arranged the columns of Table V from left to right according to the increasing level of complexity of the solution method. Since some methods include allowance for measurement error but not for time-varying parameters and vice-versa, we depart from the nesting rule for stochastic elements in the table lines

Table V. Illustrative studies.

Stochastic elements	Solution classes				
	Handcrafted feedback	Optimal feedback	Min-Max (Robust) control	Expected optimal feedback	Dual control
None	HF Taylor (1993, 1999)	OL Pindyck (1973a)	MMC Rustem (1992)		
Additive noise	HFwA Taylor (1993, 1999)	OF Hall and Taylor 93	RC Hansen and Sargent (01)		
Parameter uncertainty		OF (Amman and Kendrick, 1999b)	RC Giannoni (2002) Onatski and Stock (02) Tetlow and von zur Muehlen (2001a)	EOFwoU Brainard (1967), EOF Tinsley, Craine and Havenner (1974), Amman and Kendrick (1999b)	DC Prescott (1972), Taylor (1974), MacRae (1975), Norman (1979)
Measurement error		OFwMF Coenen and Wieland (2001)	RCwM Onatski and Stock (02), Hansen and Sargent (01)		DCwM Kendrick (1982)
Time-varying parameters			RCwT Tetlow and von zur Muehlen (2001a)	EOFwFTL Amman and Kendrick (2003)	
Meas error & time-varying parameters					DCwT Sarris (73) DCwMT Tucci (89)

below parameter uncertainty. Thus we explicitly include separate categories for measurement error and time-varying parameters as well as a joint category when both sources of uncertainty are included.

9.1. MEASUREMENT ERROR

One of the early uses of models with measurement error was the study by Kendrick (1982) using dual control methods on a small macroeconomic model of the U.S. economy. The measurement error was approximated by using revisions data from the primary macroeconomic time series. A measurement equation like Equation (8.1) was used with the specification that $H = I$ so that each state x_k was observed with one and only one observation variable y_k. In contrast, consider recent studies by Coenen and Wieland (2001) and Coenen, Levin and Wieland (2001). The first of these two studies uses a more general form for H. This opens the door for the possibility that the state vector will be augmented with past values and that the more recent states will have noise terms with higher variances since these data have not yet been revised as thoroughly. Also Onatski and Stock (2002) consider noisy observations.

Measurement error can also be used when firms imperfectly observe one another's behavior to result in imperfect common knowledge, viz. Adam (2003). Finally, for a recent study of the Australian economy using observer equations see Herbert (1998).

9.2. TIME-VARYING PARAMETERS

One of the first uses of time-varying parameters in control models was by Sarris (1973) who employed a Kalman filter to estimate the parameters. This was followed later by a number of other studies on estimation of time-varying parameters including the study by Swamy and Tinsley (1980). However one of the first inclusions of this kind of uncertainty in a stochastic control models was by Tucci (1989, 1997). He employed dual control (DC) methods so that one was faced with a tradeoff in which active perturbations today produced coefficients estimates with smaller standard errors. However, these standard errors increased again as the coefficients varied over time. Recently, Amman and Kendrick (2003) have returned to the Tucci framework but have used expected optimal feedback with time-varying parameters and policies-to-parameters (Lucas) effects (EOFwTL) instead of dual control. Also, see Tucci (2004) which deals with time-varying parameters in the forward variables case.

10. Conclusions

It is likely that in the years to come there will be many studies of economic models using stochastic control methods and many of these studies will compare various methods. As this occurs it will greatly facilitate communication between economists if we can move toward a classification system and a naming system that enables us to quickly and easily describe to one another the types of methods in use. In our view the classification system is more important and will be less controversial than the naming system. Thus it would seem useful to edge toward some agreement on a classification system and then follow with discussion of a naming system.

We have proposed both a classification system and a naming system here and are hopeful that discussion within the community over time will enable us to develop a system that enough of us feel comfortable with to improve communications amongst us.

Appendix A: Alternate Names and Acronyms for Stochastic Control Methods

New	Previous
Handcrafted Feedback Rule (HF)	Handcrafted Feedback Rule (HCFR)
Open Loop (OL)	Quadratic Linear Problem (QLP)
Optimal Feedback (OF)	Certainty Equivalence (CE)
Expected Optimal Feedback (EOF)	Open Loop Feedback (OLF), Passive Learning (OLF)
Expected Optimal Feedback with Time-varying Parameters and Policies-to-Parameters (Lucas) Effects (EOFwTL)	Open Loop Feedback with Insight (OLIN)
Dual Control (DC)	Active Learning (DUAL), Adaptive Control (DUAL)

Acknowledgments

We are grateful to Gregory Chow, Ray Fair, Ric Herbert, Ruben Mercado, Berc Rustem, Robert Tetlow, Peter Tinsley, Marco Tucci, Stephen Turnovsky, Peter von zur Muehlen and Volker Wieland for comments on earlier drafts of this paper.

Notes

[1] Most of the papers we refer to here are those that influenced our own intellectual development with dynamic quadratic linear models. For an alternative line of development begin with Tinsley (1971) and work backward in the references to papers on optimal dynamic decisions rules associated with quadratic criteria and cost of adjustment. A more recent paper in this research line is Sargent (1978).

[2] Also, this paper is limited to models in which uncertain parameters are treated as continuous rather than discrete. Thus we have not tried to cover the growing literature on economic models with Markov switching on uncertain parameter estimates.

[3] These equations are called the "state equations" in the control engineering literature.

[4] Also there is a kind of half-way house between handcrafted rules and optimal feedback rules in which there is optimization but using only a subset of the state variables. Such rules might be called

"restricted optimal feedback rules". For discussion of this class of rules see Levine and Curry (1987), Tetlow and von zur Muehlen (2001c) and Onatski and Stock (2002).

[5]See also Kydland and Prescott (1982).

[6]Peter von zur Muehlen has suggested a fourth option for the attribute of stochastic elements, namely model uncertainty, which is Knightian so that no probability distribution can be provided. Peter divides this option into two subclasses of structured and unstructured model uncertainty. In the first the analyst thinks of bounded perturbations around some central values and in the second the analyst thinks of the model as being in some bounded ball. Examples of the first are von zur Muehlen (1982) and Giannoni (2002) and examples of the second are Rustem (1992), Onatski and Stock (2002), Tetlow and von zur Muehlen (2001a) and Hansen and Sargent (2001). For more about this topic see the discussion of robust control later in the paper.

[7]As discussed in an earlier footnote it might be useful to have an intermediate solution class between handcrafted and optimal feedback that is called "restricted optimal feedback" in which the optimization procedure does not make use of the full set of state variables, but rather a restricted set in order to have simple feedback rules. An example in addition to those mentioned above is Soderlind (1999).

[8]See Kendrick (1981, 2002) Appendix B for a discussion of the method of computing the expected values of matrix products.

[9]Also, for a comparison of old and new names for the solution methods see Appendix A.

[10]The Duali software also includes variants of several of these models with parameter uncertainty, viz MacRae (1972) and Mercado and Kendrick (1999).

[11]For a later work by Chow on Lagrangian methods for solving dynamic models see Chow (1997).

[12]For example see Eqs. (10-61) and (10-68) in Kendrick (2002)

[13]For a more recent review of a number of these papers see Soderlind (1999).

[14]Ric Herbert remarks that in much of the control engineering literature this equation is called the "output" equation. He points out that in large models containing many leads or lags this equation may be particularly useful to pick out variables of interest such as consumption in the current period. Also, if the model contains control variables that affect state variables in the same period, it is useful to include the control variables in the output equation.

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