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I. MATHEMATICAL MODELING

SOLVING THE NAVIER-STOKES EQUATION FOR A VISCOUS INCOMPRESSIBLE FLUID IN AN n-DIMENSIONAL BOUNDED REGION AND IN THE ENTIRE SPACE $\mathbb{R}^n$

A. V. Baev

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We consider initial-value problems for a system of motion equations of a viscous incompressible fluid in Lagrangian variables with $n = 2$ or $n = 3$. We show that the fluid motion is independent of pressure. In the absence of external forces, the pressure is constant and the fluid is in free motion. This motion is purely turbulent and is described by quasi-linear equations of parabolic type. We prove existence and uniqueness of the classical solution of the initial-value problem in a bounded region and in the entire space. Necessary conditions of solvability are given. Steady-state equations of fluid motion are derived. Applied problems involving fluid flow in a pipe, onset of turbulence, and existence of Taylor vortices in a solid torus are solved.

Keywords: Lagrangian variables, Navier–Stokes equations and equations of parabolic type, prior bounds, turbulence, Taylor vortices.

1. Introduction

Many authors have studied existence and uniqueness issues for the problem of viscous incompressible flow. Most of these studies use the Navier–Stokes equation in Eulerian coordinates [1] in $\mathbb{R}^n,\ n \geq 2$, and consider the existence and uniqueness of generalized solutions (in various senses) in spaces of type $L^p,\ H^s$ and also in spatial scales [2–6]. A more detailed bibliography can be found in review [7]. By now, the generalized solutions of the Navier–Stokes equations have been shown to be well-posed in the small and in the large for initial conditions of different smoothness and different behavior at infinity. The well-posedness of steady-state solutions is also of considerable interest due to the physical applications of time-independent problems. Stronger results have been obtained in this direction, especially for generalized solutions, due to the application of further technical capabilities, [8–13].

The existence and uniqueness issues for the classical solution of the Navier–Stokes equation in $\mathbb{R}^n$ and T^n, and also in a bounded region with a smooth boundary, remain unresolved, however. Furthermore, the following question is open at the present: is there a finite time limit for the existence of a classical solution with an arbitrary initial condition. This situation is the reason for the inclusion of the Navier–Stokes equation in the list of most important unsolved mathematical problems [14, 15]. The aim of the present study is to try and answer these questions by imposing supplementary conditions associated with the physical interpretation of the Navier–Stokes equations and the formulation of the boundary conditions.

The use of Lagrangian coordinates as having the best physical interpretation leads to a proof of existence and uniqueness and also to the solution of some applied problems. Earlier investigations of the Navier–Stokes

Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, Moscow, Russia; email: drbaev@mail.ru.

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equation did not utilize one additional condition relating to the statement of the problem. This condition implies constancy of pressure in the absence of external forces, which allows to exclude pressure from the equation. In the presence of external potential fields, pressure simply annuls them. A strict proof of these assertions is provided by Lemma 1 and Corollary 1. From the physical point of view, these properties of pressure are attributable to the infinite velocity of sound in an incompressible fluid, and in other words to the instantaneous establishment of the pressure. We should stress that all our propositions apply to closed bounded regions and the entire space. Our results do not apply to problems involving the egress of jets and flow past nonsmooth bodies.

The elimination of pressure from the Navier–Stokes equation reduces the flow problem to the solution of a quasilinear equation of parabolic type for the velocity of a fluid particle. The nonlinearity is determined by the dependence of the equation coefficient on the particle coordinate. This dependence enables us to prove the existence of the solution first in the small and then in the large. The incompressibility condition plays an essential role in the proof. In the end, it turns out that the solution of the Navier–Stokes equation has typical properties of the solutions of parabolic equations such as infinite smoothness and the maximum principle.

In this study, we establish a fundamental property of fluid motion. We prove that it is purely turbulent flow, which leads to the appearance of eigenstates. Such stationary states are well-known in experimental physics, such as, for instance, Taylor waves and turbulent motion. We show that these states are a manifestation of mathematical non-uniqueness and not the result of instability, as previously assumed. In this study, we also provide a correct solution for fluid flow in a closed pipe and an exact solution of the turbulent flow problem.

The mathematical aspects of the problem have been considered using the results of the theory of parabolic equations from [9, 16]. The Navier–Stokes equations have not been solved largely because their solution has been treated as a purely mathematical problem. In fact, this is a deeply physical model. Unlike the heat equation, whose solution produces infinitely fast temperature propagation, in the Navier–Stokes equation the flow velocity is finite with infinite velocity of sound. The latter factor is attributable to the instantaneous establishment of constant pressure in the absence of external fields.

1.1. Notation and Basic Equations.

The Model (the Geometry)

$\mathbb{R}^n$ — the Euclidean space with the norm $|\alpha| = \sqrt{(\alpha,\alpha)},\ 2 \le n \le 3$,

$D \subset \mathbb{R}^n$ — an open bounded region with the closure $\overline{D}$ in $\mathbb{R}^n,\ \overline{D}_T = \overline{D} \times [0,T]$,

$S = \partial D$ — the boundary of D of class $C^{2+\alpha}$, ν — the unit inner normal to S,

$\mathbb{R}^n/\mathbb{Z}^n = T^n$ — an n-dimensional torus, $\overline{T}^3 = D^2 \times S^1$ — a solid torus, $\partial\overline{T}^3 = T^2$,

$(a) = (a^i)$ — contravariant Lagrangian coordinates of fluid particles,

$(x) = (x^i)$ — Cartesian Eulerian coordinates in the space $\mathbb{R}^n$,

$x = x(a,t)$ — a smooth trajectory of motion of particle a in $D,\ t$ is time,

$J = J(x) = \mathscr{D}(x)/\mathscr{D}(a)$ — the matrix of the transformation $x\colon a \to x,\ \det J \equiv |J|$,

$g = (g^{ik}(x))$ — the fundamental metric tensor in contravariant components, det $\det g \equiv |g|,\ I$ is the identity tensor,

$\mathrm{grad},\ \nabla \equiv \mathrm{grad},\ \mathrm{div},\ \mathrm{rot},\ \Delta = \mathrm{div}\,\mathrm{grad} = \mathrm{grad}\,\mathrm{div} - \mathrm{rot}\,\mathrm{rot}$ — invariant differential operators in scalar, vector, and tensor fields.

The Model (the Medium)

ε — kinematic viscosity,

$\mathrm{v} = \mathrm{v}(a, t)$ — the vector field of fluid particle velocities,

$p = p(a, t)$ — the hydrodynamic pressure in the fluid,

$u = u(a, t)$ — the potential of the external force field $f = -\nabla u$,

$j = j(a, t)$ — the external field of solenoidal forces j, $\operatorname{div} j = 0$,

$C^\alpha(D)$ — the class of Hoelder-continuous functions $\mathrm{v}(a)$, $a \in D$, with index $\alpha \in (0, 1)$, Hoelder constant $H_D^\alpha(\mathrm{v})$, and $\|\mathrm{v}\|_{C^\alpha(D)} = \|\mathrm{v}\|_{C(D)} + H_D^\alpha(\mathrm{v})$,

$C^{k+\alpha}(D)$ — the class of functions $\mathrm{v}(a) \in C^k(D)$ such that $\partial_a^k \mathrm{v} \in C^\alpha(D)$, $\|\mathrm{v}\|_{C^{k+\alpha}(D)} = \|\mathrm{v}\|_{C^\alpha(D)} + \|\partial_a \mathrm{v}\|_{C^\alpha(D)} + \ldots + \|\partial_a^k \mathrm{v}\|_{C^\alpha(D)}$,

$C^{2k+\alpha, k+\alpha}(D_T)$ — the class of functions $\mathrm{v}(a, t) \in C^{2k,k}(D_T)$ such that $\partial_a^{2k}\mathrm{v}, \partial_t^k \mathrm{v} \in C^\alpha(D_T)$, $\|\mathrm{v}\|_{C^{2k+\alpha,k+\alpha}(D_T)} = \|\mathrm{v}\|_{C^\alpha(D_T)} + \|\partial_a \mathrm{v}\|_{C^\alpha(D_T)} + \ldots + \|\partial_a^{2k}\mathrm{v}\|_{C^\alpha(D_T)} + \|\partial_t \mathrm{v}\|_{C^\alpha(D_T)} + \ldots + \|\partial_t^k \mathrm{v}\|_{C^\alpha(D_T)}$.

The Basic Equations (Apply in $\overline{D}$)

the continuity equation (the incompressibility condition)

$$\operatorname{div} \mathrm{v} = 0 \quad (\,|J| = |g| = 1), \tag{1}$$

the Navier–Stokes equation in Eulerian coordinates (x)

$$v_t + (v, \nabla)v = \varepsilon \Delta v - \nabla p - \nabla u + j, \quad v = v(x, t),$$

the Navier–Stokes equation in Lagrangian coordinates (a)

$$\mathrm{v}_t = \varepsilon \Delta \mathrm{v} - \nabla p - \nabla u + j, \quad \mathrm{v} = \mathrm{v}(a, t), \tag{2}$$

the representation of the vector field v in terms of the potentials φ, ψ (Helmholtz theorem)

$$\mathrm{v} = -\nabla\varphi + \operatorname{rot}\psi, \quad \operatorname{div}\psi = 0. \tag{3}$$

1.2. Statement of the Problem in a Bounded Region. The initial-value problem involves finding sufficiently smooth functions v and p that satisfy system (1), (2) in $\overline{D}$ for $t > 0$ and the following boundary and initial conditions:

$$\mathrm{v}|_S = 0, \quad t \geq 0, \tag{4}$$

$$\mathrm{v}|_{t=0} = 0, \quad a \in \overline{D}. \tag{5}$$

The functions u and j are assumed known and sufficiently smooth. Also let

$$j|_S = 0, \quad t \geq 0, \tag{6}$$

$$u = 0, \quad j = 0, \quad t = 0, \tag{7}$$

$$u = 0, \quad j = 0, \quad t \geq t_* > 0. \tag{8}$$

This form of the supplementary conditions allows the problem to be considered both with and without an external force field. In other words, the sources create some initial state during the time from 0 to $t_* < t_0$ and then turn off. In this way, we can elucidate what initial conditions v_0 and p_0 for t_0 are physically realizable, setting

$$\mathrm{v}_0 = \mathrm{v}(a, t_0), \quad p_0 = p(a, t_0), \quad a \in \overline{D}. \tag{9}$$

We establish that the physically realizable conditions are

$$\operatorname{div} \mathrm{v}_0 = 0, \quad p_0 = \mathrm{const}, \tag{10}$$

$$(\mathrm{v}_0, \nabla)\mathrm{v}_0 = 0, \tag{11}$$

and also $\mathrm{v}_0 \in C^\infty(\overline{D})$, i.e., no other initial conditions can be created by an external action. However, we also show that, from the mathematical point of view, the smoothness condition of the function v_0 for the existence of a classical solution can be relaxed to $\mathrm{v}_0 \in C^{2+\alpha}(\overline{D})$ in the same sense as in the initial-value problem for the heat equation [9, 16]. This can be proved by reducing the system (1), (2) to a quasilinear parabolic equation while using the incompressibility condition in the form $|J| = |g| = 1 \ \forall t > 0$.

At the same time, the conditions $p_0 = \mathrm{const}$, $(\mathrm{v}_0, \nabla)\mathrm{v}_0 = 0$ are necessary for the solvability of the initial-value problem (1), (2), (4), (6), (8), (9). From the physical point of view, the equality $p_0 = \mathrm{const}$ reflects the obvious fact of infinite velocity of sound in an incompressible medium. Equality (11) is the matching condition for the initial values (10).

2. Solution of the Problems for Pressure and Velocity

2.1. The Problem for Pressure. Let us first consider the supplementary boundary condition $\partial^2 \mathrm{v}_\nu / \partial \nu^2|_S = 0$ which is of key importance in the formulation of boundary-value problems for pressure. It is well known [1] that viscous forces arise due to friction only in a moving fluid. The fluid particles directly adjoining the wall are at rest (but not as a result of sticking, although the term sticking conditions is often used) and thus do not experience viscous forces in the normal direction.

This implies that the sum of potential forces (including pressure) in the direction of the normal is also zero. Since the total potential of these forces (allowing for the incompressibility condition) satisfies the Laplace equation, this total is constant as the solution of the corresponding Neumann problem. This assertion is proved rigorously in Lemma 1 and Corollary 1.

Since the operators

$$\operatorname{div} = 1/\sqrt{|g|} \sum_{i=1}^{n} \partial/\partial a^i \sqrt{|g|} \qquad \text{and} \qquad \partial/\partial t$$

are commutative for $|g| = 1,$ we have from (1), (2)

$$\Delta(p + u) = 0. \tag{12}$$

Let $\mathrm{v}_\nu = (\mathrm{v}, \nu)|_S$ be the normal component of the velocity v on S. Obviously, $\mathrm{v}_\nu|_S = 0$. The following lemma is of key importance for everything that follows.

Lemma 1. *For every $t \geq 0$ and $n \geq 2$, the solution of the boundary-value problem (1), (2), (4), (5) satisfies the boundary conditions*

$$\frac{\partial^2 \mathrm{v}}{\partial \nu^2}\Big|_S = 0, \qquad \frac{\partial(p+u)}{\partial \nu}\Big|_S = 0.$$

Proof. Represent the vector v in some neighborhood of a fixed point on S in the form $\mathrm{v} = \mathrm{v}_\nu \nu + \mathrm{v}_\tau \tau$, where v_τ is the velocity component along the direction τ tangential to S. Since $\partial^2 \mathrm{v}/\partial \tau^2|_S = 0 \ \forall \tau \perp \nu$, we obtain from (2)

$$\frac{\partial \mathrm{v}_\nu}{\partial t}\Big|_S = \varepsilon \frac{\partial^2 \mathrm{v}_\nu}{\partial \nu^2}\Big|_S - \frac{\partial(p+u)}{\partial \nu}\Big|_S. \tag{13}$$

But $\partial^2 \mathrm{v}_\nu/\partial \nu^2|_S$ determines the viscous forces acting on a stationary particle near the wall [1] and thus, given the physics of the process and equality (13), we obtain

$$\frac{\partial^2 \mathrm{v}_\nu}{\partial \nu^2}\Big|_S = 0 \iff \frac{\partial(p+u)}{\partial \nu}\Big|_S = 0. \tag{14}$$

We stress that condition (14) cannot be derived mathematically, and it is accepted from physical considerations as a supplementary boundary condition.

Remark 1. Condition (14) is fully consistent with Eq. (1). Indeed, let $\tau_i \perp \nu,\ \tau_i \perp \tau_j,\ i \neq j$. Then

$$\frac{\partial}{\partial \nu}\left(\frac{\partial \mathrm{v}_\nu}{\partial \nu} + \sum_{1 \leq i \leq n-1} \frac{\partial \mathrm{v}_{\tau_i}}{\partial \tau_i}\right)\Bigg|_S = \frac{\partial^2 \mathrm{v}_\nu}{\partial \nu^2}\Big|_S + \sum_{1 \leq i \leq n-1} \frac{\partial}{\partial \tau_i}\frac{\partial \mathrm{v}_{\tau_i}}{\partial \nu}\Big|_S = 0.$$

By Newton's law of viscous friction, $\partial \mathrm{v}_{\tau_i}/\partial \nu|_S$ is the stress of viscous forces in the direction τ_i on S, whence it follows that the last sum is zero.

Thus, all first and second derivatives of the function v_ν on S are zero. Furthermore, every partial derivative of the function v_ν on S is zero and thus v_ν has on S a zero of infinite order on S.

Condition (14) leads to important corollaries.

Corollary 1. *Let $n \geq 2$. Then $\forall t \geq 0$, we have the equality*

$$p + u = \mathrm{const}. \tag{15}$$

Proof. From (13), (14), for $t \geq 0$ we obtain the Neumann problem for the Poisson equation

$$\Delta(p+u) = 0, \quad \frac{\partial(p+u)}{\partial \nu}\Big|_S = 0 \implies p + u = \Phi(t). \tag{16}$$

Φ is an arbitrary function of time, and in what follows we take $\Phi = \mathrm{const}$. Pressure is thus established in (15) in accordance with the potential of external forces. With $u = 0$, we may take without loss of generality $p = 0$.

A physical confirmation of constant pressure at $u = 0$ is provided by the fact that the velocity of sound in an incompressible fluid is infinite, which implies instantaneous establishment of a baric equilibrium. This conclusion would seem to contradict known solutions of stationary problems, in particular the problem of pipe flow. A valid solution of the stationary pipe flow problem is given in Sec. 3.4. This solution again corroborates the known fact remarked on in [1]: the stationary flow velocity is independent of pressure.

Corollary 2. *Assume that a solution of problem (1), (2), (4)–(8) exists. Then* $\forall t \geq 0$ *we have the equality* $(\mathrm{v}, \nabla)\mathrm{v} = 0$.

Proof. The equality $\Delta(p+u) = 0$ implies the equality $\operatorname{div}(v, \nabla)\, v = 0$ in Eulerian coordinates (x). Since $\mathrm{v}(a,t) = v(x,t)$ at the point $x = x(a,t)$, we have

$$(v, \nabla)v = \lim_{\delta \to 0} \frac{v(x + \delta v, t) - v(x,t)}{\delta} = \lim_{\delta \to 0} \frac{\mathrm{v}(a + \delta \mathrm{v}, t) - \mathrm{v}(a,t)}{\delta} = (\mathrm{v}, \nabla)\mathrm{v}.$$

The directional derivative and divergence are invariant with respect to the choice of the coordinate system, and so $\operatorname{div}(\mathrm{v}, \nabla)\,\mathrm{v} = 0$.

We will show that $\operatorname{rot}(v, \nabla)v = 0$. By known formulas of vector analysis, we have

$$\operatorname{rot}(v, \nabla)v = -[\nabla, [v, \operatorname{rot} v]] = (\nabla, v)\ \operatorname{rot} v - (\nabla, \operatorname{rot} v)\, v = 0.$$

Since the divergence and the curl of a vector are zero, we have $(v, \nabla)\, v = (\mathrm{v}, \nabla)\, \mathrm{v} = 0$.

In what follows, we assume that conditions (10), (11) hold.

2.2. The Problem for Velocity. First assume that a solution of problem (1), (2), (4)–(8) exists. Let $\mathrm{v} = c + \omega$ and look for v in representation (3), where

$$c = -\nabla\varphi, \quad \omega = \operatorname{rot}\psi, \quad \operatorname{div}\psi = 0, \quad c|_S = \omega|_S = 0.$$

From (1), (3), (12) we obtain a problem for the potential φ:

$$\Delta\,\varphi = 0, \quad \nabla\varphi|_S = 0, \quad t \geq 0, \tag{16}$$

Hence in $\overline{D}$

$$\varphi = \Phi(t) \quad \Longrightarrow \quad c = 0, \quad t \geq 0. \tag{17}$$

For the vortex part of the velocity

$$\omega_t = \varepsilon\Delta\,\omega + j, \quad \operatorname{div}\omega = 0, \quad \omega|_S = 0, \quad t \geq 0, \tag{18}$$

where the vector potential ψ is the solution of the problem

$$\Delta\,\psi = -\operatorname{rot} v, \quad \psi|_S = 0.$$

From (16)–(18) it follows that the fluid motion in a bounded volume is a pure vortex.

If the metric tensor $g = g(x)$ is known at the point $x = x(a,t)$, then the problem for Eq. (2) may be treated as linear with a unique classical solution, because

$$\Delta = \Delta(x) \equiv \frac{\partial}{\partial a^i}\, g^{ik}(x(a,t))\frac{\partial}{\partial a^k},$$

which incorporates the incompressibility condition in the form $|g| = 1$. In fact, the tensor g is not known and the problem for $t \geq 0$ reduces to an initial-boundary value problem for a quasilinear system of parabolic type

$$\begin{gathered} \mathrm{v}_t = \varepsilon\Delta(x)\mathrm{v} - \nabla p - \nabla u + j, \quad \mathrm{v}|_S = 0, \quad \mathrm{v}(a,0) = 0, \\ x_t = \mathrm{v}, \quad x|_S = a, \quad x(a,0) = a. \end{gathered} \tag{19}$$

If we consider $t \geq t_0$, the problem takes the form

$$\begin{gathered} \mathrm{v}_t = \varepsilon\Delta(x)\mathrm{v} - \nabla p, \quad \mathrm{v}|_S = 0, \quad \mathrm{v}(a,t_0) = \mathrm{v}_0(a), \\ x_t = \mathrm{v}, \quad x|_S = a, \quad x(a,t_0) = a, \end{gathered} \tag{20}$$

where the boundary and initial conditions are consistent.

It is easy to see that the existence of a solution of problems (19), (20) is fully determined by the variation of the tensor g along the trajectory x. The solvability problem thus completely shifts to the domain of existence of solutions of quasilinear equations of parabolic type, albeit with the essential constraint $|g| = 1$.

It may seem that by passing to the description of fluid motion in Lagrangian variables and thus getting rid of the nonlinear term $(v, \nabla)v$ in the substantive derivative dv/dt, we essentially linearized the problem. This is evidently not so.

While in the original problem the nonlinearity, or more precisely, the quasi-nonlinearity was quadratic, in the Lagrangian coordinates the nonlinearity has shifted into the metric tensor $g(x)$, where x in turn functionally depends on g through (19), (20). The nonlinearity of the original problem is thus preserved, but in the process we have utilized the incompressibility condition $|g| = 1$. As we shall see from Theorems 1, 2 this is a highly relevant factor for the proof of existence.

2.3. Existence of a Solution. Let $\overline{D}_t = \overline{D} \times [t_0, t]$ and assume that $\mathrm{v}(t) = \mathrm{v}$ is an element of the functional space $L_2(\overline{D})$ or $C^{2+\alpha}(\overline{D})$. Condition (11) holds for v_0. We have the following existence and uniqueness theorem for the initial-value problem (20).

Theorem 1. *Problem (20)* $\forall \mathrm{v}_0 \in C^{2+\alpha}(\overline{D})$ *has a unique classical solution* $\mathrm{v}, x \in C^\infty(\overline{D} \times (t_0, \infty))$, $p = 0$. *Also,* $\forall t \geq t_0$ *we have the bound*

$$\|\mathrm{v}\|_{C(\overline{D})} \leq \|\mathrm{v}_0\|_{C(\overline{D})}$$

and the identity

$$\|\mathrm{v}(t)\|^2_{L_2(\overline{D})} = \|\mathrm{v}_0\|^2_{L_2(\overline{D})} - 2\varepsilon\|\nabla\mathrm{v}\|^2_{L_2(\overline{D}_t)}.$$

Proof. We divide the proof into two stages as follows: the proof of the theorem in the small for t and then the proof globally.

We will show that $\forall \mathrm{v}_0 \in C^{2+\alpha}(\overline{D})$ there is a number $\delta > 0$ such that the classical solution of problem (20) exists in the region $\overline{D}_\delta = \overline{D} \times [t_0, t_\delta],\ t_\delta = t_0 + \delta$. Now let $\|\mathrm{v}_0\|_{C(\overline{D})} \le M$. Represent problem (20) in the form

$$\mathrm{v}_t = \varepsilon\Delta(a)\mathrm{v} + \varepsilon(\Delta(x) - \Delta(a))\mathrm{v} \quad \mathrm{v}|_S = 0, \quad \mathrm{v}(a, t_0) = \mathrm{v}_0(a), \tag{21.1}$$

$$x_t = \mathrm{v}, \quad x|_S = a, \quad x(a, t_0) = a. \tag{21.2}$$

Let us evaluate $\Delta(x) - \Delta(a)$. Recall that

$$g(x) = J^T(x)J(x),$$

where J^T is the transpose of the Jacoby matrix J. For $\xi = x - a$ — a we obtain

$$g(a + \xi) - g(a) = J^T(a + \xi)J(a + \xi) - J^T(a)J(a) = P(\xi, a).$$

The form $P(\xi, a)$ is determined exclusively by the initial values of problem (20).

Problem (21.1) thus takes the form

$$\mathrm{v}_t = \varepsilon\Delta(x)\mathrm{v} = \varepsilon \sum_{i=k} \frac{\partial}{\partial a^i}\frac{\partial \mathrm{v}}{\partial a^k} + \varepsilon \sum_{i,k} \left[\frac{\partial}{\partial a^i}\, P(x - a, a)\right] \frac{\partial \mathrm{v}}{\partial a^k},$$

$$\mathrm{v}|_S = 0, \quad \mathrm{v}(a, t_0) = \mathrm{v}_0(a).$$

Given the function $x(a, t)$, this problem has a unique classical solution and the maximum principle holds.

For problem (21) consider in $\overline{D}_\delta$ the iterative process over $m = 1, 2, \ldots$:

$$x_t^m = \mathrm{v}^{m-1}, \quad x^m|_S = a, \quad x^m(a, t_0) = a, \quad \mathrm{v}^0(a) = \mathrm{v}_0(a), \tag{22.1}$$

$$\mathrm{v}_t^m = \varepsilon\Delta(x^m)\mathrm{v}^m, \quad \mathrm{v}^m|_S = 0, \quad \mathrm{v}^m(a, t_0) = \mathrm{v}_0(a). \tag{22.2}$$

It is easy to see that $\forall m \in \mathbb{N}$ & $\forall \delta > 0$ problem (22.2) has the unique solution $\mathrm{v}^m \in C^{2+\alpha,1+\alpha}(\overline{D}_\delta)$ and on the set $X = \left\{x^m \mid \|x^m - a\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)} \le M_0\right\}$ we have the uniform bound

$$\|\mathrm{v}^m\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)} \le M_1(M_0)\|\mathrm{v}_0\|_{C(\overline{D})} \le M_2.$$

From (22.1) we obtain the inequality

$$\|x^m - a\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)} \le M_2\delta.$$

There is obviously δ_0 such that for $\delta < \delta_0$ we have

$$\|x^m - a\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)} \le M_0.$$

Note that δ_0 depends only on M_0 and M_2.

Every solution v^m continuously depends on the coefficients of Eq. (22.2) and $\forall x^m \in X$ we have [9, 16] the following bound uniformly in m:

$$\|\mathrm{v}^{m+1} - \mathrm{v}^m\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)} \leq M_3 \|x^{m+1} - x^m\|_{C^\alpha(\overline{D}_\delta)},$$

where M_3 depends only on M_0.

From the first equation in system (22) for $t - t_0 \leq \delta$ we obtain

$$\|x^{m+1} - x^m\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)} \leq \delta \, \|\mathrm{v}^m - \mathrm{v}^{m-1}\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)}.$$

We thus get

$$\|\mathrm{v}^{m+1} - \mathrm{v}^m\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)} \leq q \, \|\mathrm{v}^m - \mathrm{v}^{m-1}\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)},$$

where the inequality $q < 1$ can be satisfied for sufficiently small $\delta < \delta_0$. Hence it follows that v^m is a fundamental sequence in $C^{2+\alpha,1+\alpha}(\overline{D}_\delta)$, and by completeness of the space $C^{2+\alpha,1+\alpha}(\overline{D}_\delta)$ [16],

$$\mathrm{v}^m \overset{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)}{\longrightarrow} \mathrm{v} \in C^{2+\alpha,1+\alpha}(\overline{D}_\delta), \quad m \to \infty.$$

Now, the sequence $x^m \overset{C^{2+\alpha,1+\alpha}(\overline{D}_\delta)}{\longrightarrow} x \in C^{2+\alpha,1+\alpha}(\overline{D}_\delta)$ as $m \to \infty$ and we return to problem (20) for a parabolic equation with a smooth principal part. Substituting the functions v^m, x^m in (20) and passing to the limit as $m \to \infty$ we prove that $\mathrm{v},\ x$ are the solution of problem (20). The infinite smoothness of the solution is determined by the form of the equation, i.e., the presence of variable coefficients that depend only on the solution itself.

It follows from the above that problem (20) has the solution $\mathrm{v} \in C^\infty(\overline{D}_\delta)$ and $\|\mathrm{v}\|_{C(\overline{D}_\delta)} \leq \|\mathrm{v}_0\|_{C(\overline{D})}$ [9, 16]. The number δ is determined by the constants $M,\ M_0$ and the geometry of D. This completes the proof of the theorem in the small.

Let $t_1 = t_0 + \delta$ and take $b = x_1(a),\ x_1(a) \equiv x(a, t_1)$ as the new Lagrangian variables. Since the map $x_1(a)\colon a \to b$ is one-to-one and $x_1 \in C^\infty(\overline{D})$, the problem (20) in the new coordinates b for the velocity $\mathrm{w}(b,t)$ and the flow trajectory $y(b,t)$ takes the form

$$\begin{aligned} &\mathrm{w}_t = \varepsilon \Delta(y)\, \mathrm{w}, \quad \mathrm{w}|_S = 0, \quad \mathrm{w}(b, t_1) = \mathrm{w}_1(b) = \mathrm{v}(x_1^{-1}(b), t_1), \\ &y_t = \mathrm{w}, \quad y|_S = b, \quad y(b, t_1) = x(x_1^{-1}(b), t_1) = b, \end{aligned} \tag{23}$$

where the function x_1^{-1} is the inverse of x_1 with the same smoothness as x_1 so that $g = J = I$ for $t = t_1$, and $\|\mathrm{w}_1\|_{C(\overline{D})} \leq M$ by the maximum principle.

Problems (23) and (20) are obviously identical apart from notation. Thus, both (23) and (20) are standard problems, i.e., independent of $t_0, t_1, \ldots,$ and a classical solution exists on every closed interval $[t_k, t_k + \delta]$, $t_k = t_0 + k\delta,\ k = 1, 2, \ldots$. Smooth matching at $t = t_k$ is ensured by condition (23) allowing for the smoothness of the map $x_k(a,t)$. For every $t > t_0$ the solution of problem (20) can be constructed from finitely many solutions of problem (23).

Let us now prove uniqueness. Without loss of generality, assume that two solution $\mathrm{v}_1 \neq \mathrm{v}_2$ exist on $[t_0, t_1]$. Then from the previous bounds we obtain

$$\|\mathrm{v}_1 - \mathrm{v}_2\|_{C^{2+\alpha,1+\alpha}(\overline{D}_t)} \leq M_3 \| \int_{t_0}^{t} (\mathrm{v}_1(\tau) - \mathrm{v}_2(\tau))\, d\tau \|_{C^{2+\alpha,1+\alpha}(\overline{D}_t)},$$

where $\overline{D}_t = \overline{D} \times [t_0, t]$. From the integral inequality

$$\|\mathrm{v}_1 - \mathrm{v}_2\|_{C^{2+\alpha,1+\alpha}(\overline{D}_t)} \le M_3 \int\limits_{t_0}^{t} \|\mathrm{v}_1 - \mathrm{v}_2\|_{C^{2+\alpha,1+\alpha}(\overline{D}_\tau)}\, d\tau$$

and Gronwall's lemma it follows that $\mathrm{v}_1 = \mathrm{v}_2$.

The bound in L_2 follows from the obvious chain of equalities

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\overline{D}} |\mathrm{v}|^2\, da = \varepsilon \int\limits_{\overline{D}} (\mathrm{v}, \Delta \mathrm{v})\, da = -\varepsilon \int\limits_{\overline{D}} |\nabla \mathrm{v}|^2\, da.$$

The last expression determines the energy loss due to viscosity and vanishes when the fluid comes to rest. Q.E.D.

Now consider the initial-value problem on the torus $\overline{T}^3 \subset \mathbb{R}^3$ without self-intersection points of the surface $\partial \overline{T}^3 = T^2$ (a so-called solid torus). In this case $\partial \overline{T}^3 \in C^\infty$, and in $\overline{T}^3$ we thus have the following corollary.

Corollary 3. *Problem (20)* $\forall \mathrm{v}_0 \in C^{2+\alpha}(\overline{T}^3)$ *has the unique classical solution* $\mathrm{v}, x \in C^\infty(\overline{T}^3 \times (t_0, \infty))$, $p = 0$. *Also* $\forall t \ge t_0$ *we have the bounds* $\|\mathrm{v}\|_{C(\overline{T}^3)} \le \|\mathrm{v}_0\|_{C(\overline{T}^3)}$ *and* $\|\mathrm{v}\|_{L_2(\overline{T}^3)} \le \|\mathrm{v}_0\|_{L_2(\overline{T}^3)}$.

Let us consider the problem with a force term in the right-hand side (19). To prove existence, we apply the same approach as in Theorem 1. The result is therefore given without proof and it relies on prior bounds for quasilinear equations of parabolic type [9, 16]. Let $f = -\nabla u$, $\overline{D}_T = \overline{D} \times [0, T]$.

Theorem 2. *Problem (19)* $\forall f, j \in C^\alpha(\overline{D}_T)$ *and* $\forall T > 0$ *has a unique classical solution*

$$\mathrm{v}, x \in C^{2+\alpha,1+\alpha}(\overline{D}_T), \quad p + u = \mathrm{const}.$$

2.4. Initial-Value Problem in $\mathbb{R}^n$. The problem in the Euclidean space $\mathbb{R}^n$ contains boundary conditions as $|a| \to \infty$ which we constraint to be zero for the velocity v based on physical considerations. Let condition (11) hold for v_0.

Pose the initial-value problem

$$\begin{aligned} &\mathrm{v}_t = \varepsilon \Delta\, \mathrm{v} - \nabla p, \quad \mathrm{v}|_\infty = 0, \quad \mathrm{v}(a, t_0) = \mathrm{v}_0(a), \\ &x_t = \mathrm{v}, \quad x|_\infty = a, \quad x(a, t_0) = a, \quad t_0 < t \le T, \end{aligned} \tag{24}$$

where the functions v_0, v for $m, l \ge 1$, $\alpha \in (0,1)$ and every $r > 0$ decrease as $|a| \to \infty$ according to the following conditions:

$$\|\mathrm{v}_0\|_{C^{2l+\alpha}(\overline{\Omega})} \le K_{lm\alpha}/(1+r)^m \quad \forall \overline{\Omega} \subset \{|a| > r\}, \tag{25.1}$$

$$\|\mathrm{v}\|_{C^{2l+\alpha,l+\alpha}(\overline{\Omega})} \le K_{lm\alpha}/(1+r)^m \quad \forall \overline{\Omega} \subset \{|a| > r\}\,\&\,\forall t \in (t_0, T]. \tag{25.2}$$

The space of these functions v_0, v is denoted by $\mathscr{E}^{2l,\alpha}_m$, $\mathscr{E}^{2l,\alpha}_{T,m}$, respectively. Conditions (25) are obviously an analogue of the boundary condition $\mathrm{v}|_S = 0$ for a bounded region in $\mathbb{R}^n$.

If conditions (25) hold for all $m, l \ge 1$, $\alpha \in (0,1)$, we denote these spaces as $\mathscr{E}$ and $\mathscr{E}_T$, respectively. As we know, these spaces can be endowed with the topology of a countably-normed space [17]. The behavior of the solutions at infinity described above has been established also in [18].

Consider in $\mathbb{R}^n$ the sequence of n-balls $D^k = \{\xi \mid |\xi| < k\}$, $k = 1, 2, \ldots$, and let $S^k = \partial D^k$, $\overline{D}^k = D^k \cup S^k$. Take $\overline{D}^k_T = \overline{D}^k \times [t_0, T]$. Then the sequence of norms

$$\|\mathrm{v}\|_k = \sup_{\overline{D}^k_T} \|(1+|a|)^m \mathrm{v}\|_{C^{2l+\alpha, l+\alpha}(\overline{D}^k_T)}$$

introduces in $\mathscr{E}^{2l,\alpha}_{T,m}$ a countably-normed space structure. Similar introduction of norms is observed for $\mathscr{E}_T$.

In $\mathscr{E}^{2l,\alpha}_{T,m}$ we can define the concept of converging sequence, and this space may be thus treated as complete. We say that the sequence $\{\mathrm{v}^s\}$ converges in $\mathscr{E}^{2l,\alpha}_{T,m}$ if $\{\mathrm{v}^s\}$ converges uniformly in every bounded subregion of $\mathbb{R}^n$ $\forall t \in [t_0, T]$. Completeness of a countably-normed space implies convergence of every sequence fundamental in each of the norms $\|\cdot\|_k$ [17].

Now let $\|\mathrm{v}_0\|_{C(\mathbb{R}^n)} \le M$, $\overline{\Omega}_T = \overline{\Omega} \times (t_0, T]$, where $\overline{\Omega}$ is a closed bounded region in v with a boundary of class $C^{2+\alpha}$.

Theorem 3. *Problem (24), (25) $\forall \mathrm{v}_0 \in \mathscr{E}^{2l,\alpha}_m$ and $\forall T > t_0$ has the unique classical solution $\mathrm{v}, x \in \mathscr{E}^{2l,\alpha}_{T,m}$, $p = 0$. Also $\forall t \ge t_0$ we have the bound*

$$\|\mathrm{v}\|_{C(\mathbb{R}^n)} \le \|\mathrm{v}_0\|_{C(\mathbb{R}^n)},$$

and for $m > n$ the identity

$$\|\mathrm{v}(t)\|^2_{L_2(\mathbb{R}^n)} = \|\mathrm{v}_0\|^2_{L_2(\mathbb{R}^n)} - 2\varepsilon \int_{t_0}^{t} \|\nabla \mathrm{v}(\tau)\|^2_{L_2(\mathbb{R}^n)}\, d\tau$$

holds.

Proof. In the relevant class of solutions, $\|\mathrm{v}\|_{C^{2l+\alpha,l+\alpha}(|a|\ge r)} \to 0$ as $|r| \to \infty$, and from (2) it follows that $\nabla p \to 0$ as $|a| \to \infty$. Then for the Neumann problem we have

$$\Delta p = 0, \quad \nabla p|_\infty = 0 \quad \Longrightarrow \quad p = \mathrm{const}.$$

The term ∇p is eliminated from Eq. (24). Without loss of generality we take $p = 0$.

In $\mathbb{R}^n$ introduce the Lagrangian coordinates (a) and pose the initial-value problem for $t \ge t_0$:

$$\mathrm{v}_t = \varepsilon \Delta(x) \mathrm{v}, \quad \mathrm{v}|_{S^k} = \mathrm{v}_0|_{S^k}, \quad \mathrm{v}(a, t_0) = \mathrm{v}_0(a). \tag{26.1}$$

$$x_t = \mathrm{v}, \quad x|_{S^k} = a, \quad x(a, t_0) = a. \tag{26.2}$$

This problem has the unique solution $\mathrm{v}^k, x^k \in C^\infty(\overline{D}^k \times (t_0, T])$ and we accordingly have the uniform bounds

$$\|\mathrm{v}^k\|_{C(\overline{\Omega}_T)} \le M, \quad \|\mathrm{v}^k\|_{C^{2+\alpha,1+\alpha}(\overline{\Omega}_T)} \le M_0\colon \quad \forall \overline{\Omega} \in D^k. \tag{27}$$

The proof of this proposition is a verbatim repetition of the proof of Theorem 1 for a bounded region $D = D^k$.

We will show that the sequence $\{\mathrm{v}^k\}$ converges uniformly in every bounded region $\overline{\Omega}_T$. The bound (27) leads to the inequalities

$$\|\mathrm{v}^k\|_{C^{2,1}(\overline{\Omega}_T)} \le M_0, \quad \|\partial^2_a \mathrm{v}^k\|_{C^{\alpha,1}(\overline{\Omega}_T)} \le M_0 \quad \forall k \in \mathbb{N}.$$

The set $\{\mathrm{v}^k \mid \|\mathrm{v}^k\|_{C^{2,1}(\overline{\Omega}_T)} \le M_0\}$ is thus uniformly bounded and equicontinuous. By the Arzela theorem, this implies the existence of a subsequence $\{\mathrm{v}^{k'}\}$ such that $\{\partial_a^2 \mathrm{v}^{k'}\}$ is uniformly convergent in $\overline{\Omega}_T$. In the same way we find a subsequence from $\overline{\Omega}_T$ that uniformly converges in a together with its derivatives with respect to a to second order inclusive and with respect to t to first order. Denoting this subsequence by $\{\mathrm{v}^{k''}\}$ and its limit by v, we obtain

$$\mathrm{v}^{k''} \rightrightarrows \mathrm{v}, \quad \partial_a \mathrm{v}^{k''} \rightrightarrows \partial_a \mathrm{v}, \quad \partial_a^2 \mathrm{v}^{k''} \rightrightarrows \partial_a^2 \mathrm{v}, \quad \partial_t \mathrm{v}^{k''} \rightrightarrows \partial_t \mathrm{v}, \quad k'' \to \infty.$$

A full proof of the convergence $\|\mathrm{v}^{k''} - \mathrm{v}\|_{C^{2+\alpha,1+\alpha}(\overline{\Omega}_T)} \to 0$ as $k'' \to \infty$ is given in [16]. Thus, v satisfies equation (26.1) in $\overline{\Omega}_T$ and is therefore the sought solution for every T. The corresponding smoothness of this solution follows from Theorem 1.

Let us extend the definition of every function v^k outside D^k as v_0, retaining the notation v^k. Since $\operatorname{div} \mathrm{v}^k = \operatorname{div} \mathrm{v}_0 = 0$, we have $\partial^i \mathrm{v}/\partial \nu^i|_{S^k} = \partial^i \mathrm{v}_0/\partial \nu^i|_{S^k}$ $\forall i \ge 1$. Indeed, along every direction τ_j tangential to S^k we have the equality $\partial \mathrm{v}/\partial \tau_j|_{S^k} = \partial \mathrm{v}_0/\partial \tau_j|_{S^k}$, and thus $\partial \mathrm{v}/\partial \nu|_{S^k} = \partial \mathrm{v}_0/\partial \nu|_{S^k}$ and so on $\forall i$. Here $\{\mathrm{v}^k\} \in \mathscr{E}^{2l,\alpha}_{T,m}$. Hence, by the completeness of $\mathscr{E}^{2l,\alpha}_{T,m}$, we obtain $\mathrm{v} \in \mathscr{E}^{2l,\alpha}_{T,m}$.

Let us prove uniqueness of the solution. Assume that the solutions $\mathrm{v}_1 \ne \mathrm{v}_2$ exist, and let

$$\sup_{\overline{D}^k_T} \|\mathrm{v}_j\|_{C^{2+\alpha,\alpha}(\overline{D}^k_T)} \le M_1, \quad j = 1, 2, \quad \forall k \in \mathbb{N}.$$

Furthermore, since $\mathrm{v}_j \in \mathscr{E}^{2l,\alpha}_{T,m}$, we have $\sup_{\overline{\Omega}_T} \|\mathrm{v}_j\|_{C^{2+\alpha,\alpha}(\overline{\Omega}_T)} \le M_2/k$, $\overline{\Omega} \notin D^k$, where M_2 is independent of k. Then, repeating the argument of Theorem 1 for the region Dk, the prior bounds from [14] lead to the inequality

$$\|\mathrm{v}_1 - \mathrm{v}_2\|_{C^{2+\alpha,1+\alpha}(\overline{D}^k_t)} \le M_3 \int_{t_0}^{t} \|\mathrm{v}_1(\tau) - \mathrm{v}_2(\tau)\|_{C^{2+\alpha,1+\alpha}(\overline{D}^k_\tau)}\, d\tau + \frac{M_4}{k},$$

where $M_3 = M_3(M_1)$, $M_4 = M_4(M_3)$. From this integral inequality, by Gronwall's lemma, we find

$$\|\mathrm{v}_1 - \mathrm{v}_2\|_{C^{2+\alpha,1+\alpha}(\overline{D}^k_t)} \le M_5/k.$$

Since $k \to \infty$ is arbitrary, we obtain $\mathrm{v}_1 = \mathrm{v}_2$. Q.E.D.

The particular problem for $n = 3$ is of special interest in astronomy and astrophysics. From Theorem 3 we obtain

Corollary 4. *Problem (24), (25)* $\forall \mathrm{v}_0 \in \mathscr{E}$ *and* $\forall T > t_0$ *has the unique classical solution* $\mathrm{v}, x \in \mathscr{E}_T$, $p = 0$. *Here,* $\forall t \ge t_0$ *we have the bounds* $\|\mathrm{v}\|_{C(\mathbb{R}^3)} \le \|\mathrm{v}_0\|_{C(\mathbb{R}^3)}$ *and* $\|\mathrm{v}\|^2_{L_2(\mathbb{R}^3)} \le \|\mathrm{v}_0\|^2_{L_2(\mathbb{R}^3)}$.

3. Solution of Applied Problems

3.1. Steady-State Flow. Consider the Navier–Stokes equation in Lagrangian coordinates (a) with steady-state right-hand side. In Sec. 2.1, we have proved that $\nabla(p + u) = 0$ and this term is accordingly eliminated from the equation:

$$\mathrm{v}_t = \varepsilon \Delta \mathrm{v} + j\, \mathrm{e}^{\lambda t}, \tag{28}$$

where $\mathrm{v} = \mathrm{v}(a, t)$ is the velocity of a fluid particle in the Lagrangian coordinates, $x = x(a, t)$ is the trajectory of particle motion, $j = j(x)$ is the vortex force at each point with the Cartesian coordinates x.

The following lemma is relevant for solving problems with steady-state, including stationary, flow, i.e., with $\lambda = 0$.

Lemma 2. *In steady state, Eq. (28) takes the following form in the Cartesian coordinates* (x)

$$\lambda V = \varepsilon \Delta V + j, \tag{29}$$

where $V = V(x)$ *is the velocity at the point* x *and* $v(x,t) = V(x)\mathrm{e}^{\lambda t}$.

Proof. Consider the steady-state solution of Eq. (28) at some time instant $t = t_0$ and define the Lagrangian variables (a) as the position of the fluid particle at time $t = t_0$, i.e., $a = x$. Equation (28) is then solved in these variables. We represent the solution in the form $\mathrm{v}(a,t) = V(a)\,\mathrm{e}^{\lambda t}$. From (28) we have

$$\lambda V = \varepsilon \Delta(x(a,t))V + j(x(a,t)).$$

This equation holds with $x(a,t_0) = a$, where $\Delta(a)$ is the Laplacian $\sum_{1\le i\le n} \partial^2/\partial a_i^2$. Since t_0 is arbitrary, we have $v(x,t) = V(x)\,e^{\lambda t}$ at every point $x \in \overline{D}$. Q.E.D.

Lemma 2 is quite transparent. Indeed, since the steady-state solution is homogeneous in time, any t can be chosen as the initial time. At this instant, the Lagrangian and Eulerian variables are identical and the coordinate system (a) is identical with the Cartesian system (x).

3.2. Taylor Vortices. Consider the steady-state flow in a solid topological torus $\overline{T}^3$ with a circular cross section. This flow is known as Taylor vortices. Their origin is attributed to the instability of the fluid rotational motion and they are regarded as perturbations of regular flow [1]. In reality, this motion is a manifestation of nonuniqueness of the solution.

We apply Lemma 2 to obtain a solution. For $\overline{T}^3$, we introduce the cylindrical coordinates (ρ,θ,z) in $\mathbb{R}^3$ with a periodic condition in z. Then the problem with steady-state perturbations of the vortex velocity component $w = w(\rho,\theta,z)$ represented in Cartesian components takes the following form:

$$\begin{gathered}\lambda w - \varepsilon\Delta w = j(\rho,\theta,z), \quad |w(0,\theta,z)| < \infty, \quad w(r,\theta,z) = 0,\\ w(\rho,\theta,z) = w(\rho,\theta+2\pi,z), \quad w_\theta(\rho,\theta,z) = w_\theta(\rho,\theta+2\pi,z),\\ w(\rho,\theta,0) = w(\rho,\theta,2\pi R), \quad w_z(\rho,\theta,0) = w_z(\rho,\theta,2\pi R),\end{gathered} \tag{30}$$

where λ is a free parameters, $0 < \rho < r$, $0 \le \theta \le 2\pi$, $0 \le z \le 2\pi R$, $\Delta = \Delta_\rho + 1/\rho^2\partial^2/\partial\theta^2 + \partial^2/\partial z^2$, $\Delta_\rho = \partial^2/\partial\rho^2 + 1/\rho\,\partial/\partial\rho$.

We will show that for $j = 0$ the problem (30) has eigenstates, i.e., nontrivial solutions. Substitute one such solution $w = A(\rho)B(z)\{-\sin m\theta, \cos m\theta, 0\}$, $m \in \mathbb{Z}$, in (30). We obtain, after variable separation,

$$\frac{\lambda A - \varepsilon\Delta_\rho A + \varepsilon m^2 A/\rho^2}{A} = \varepsilon\frac{B''}{B} = -\varepsilon\mu^2.$$

By periodicity in z, we have

$$\mu = \mu_k = k/R, \quad k \in \mathbb{Z},$$

whence

$$\lambda\rho^2 A - \varepsilon\rho(\rho A')' + \varepsilon m^2 A = -\varepsilon\mu_k^2\rho^2 A, \qquad |A(0)| < \infty, \quad A(r) = 0, \quad B = \mathrm{e}^{\mathrm{i}kz/R}.$$

We seek the admissible λ in the form $\lambda = -\varepsilon\mu_k^2 - \varepsilon\alpha^2$. Then for $A(\rho)$ we obtain the problem

$$A'' + \frac{A'}{\rho} + \left(\alpha^2 - \frac{m^2}{\rho^2}\right)A = 0, \quad |A(0)| < \infty, \quad A(r) = 0,$$

whose solution exists and is expressed by the Bessel function $\mathrm{J}_m(\alpha_{lm}\rho)$, where $\alpha_{lm} > 0$, $l \in \mathbb{N}$, are the roots of the equation $\mathrm{J}_m(\alpha r) = 0$. As a result, we find the admissible eigenstates of the vortex velocity component

$$w(\rho, \theta, z) = \mathrm{J}_m(\alpha_{lm}\rho)\{-\sin m\theta, \cos m\theta, 0\}\,\mathrm{e}^{\mathrm{i}kz/R}.$$

Since $\alpha_{lm} > 0$, all the modes are asymptotically stable.

We finally find that the steady-state eigenstates of the vortex velocity component ω satisfying the condition $\operatorname{div}\,\omega = 0$ exist only for $m = 1$, and in the cylindrical coordinates have the form

$$\omega_{kl}(\rho, z, t) = \omega^o \mathrm{J}_1(\alpha_l\rho)\,\mathrm{e}^{\mathrm{i}kz/R - \varepsilon(\alpha_l^2 + \mu_k^2)t}, \quad \alpha_l = \alpha_{l1}, \quad k \in \mathbb{Z},\ l \in \mathbb{N},$$

where $\omega^o = \{0, 1, 0\}$ is the vector such that $\operatorname{div}\omega_{kl} = 0$. This vector points along the torus generator, which corresponds to closure of the fluid streamlines.

3.3. Pipe Flow. Consider the pipe as a solid topological torus $\overline{T}^3$ with the lateral surface T^2 and constant cross section Σ. To sustain fluid motion, a force $f = -\nabla u$ should be applied at least in a relatively "small" region. Take $j = 0$.

Consider the problem of steady-state pipe flow. In this case, $\mathrm{v}_t = 0$ in Lagrangian coordinate, and by Lemma 2 the Navier–Stokes equation takes the form

$$\varepsilon\Delta\,\mathrm{v} = \nabla(p + u).$$

In Sec. 2.1, we have shown that $p + u = \text{const} \implies \Delta\mathrm{v} = 0$. Representing v as the sum $\mathrm{v} = c + \omega$, where $c = -\nabla\varphi$, $\omega = \operatorname{rot}\psi$, we obtain from (1), (2)

$$\Delta\varphi = 0, \quad \varphi|_S = 0, \quad \Delta\omega = 0, \quad \omega|_S = 0.$$

Hence,

$$\varphi = 0 \implies c = 0.$$

We thus conclude that fluid motion in a pipe is a pure vortex.

The equation $\Delta\omega = 0$ is solved by a vector function $\omega \neq 0$ such that $\operatorname{div}\omega = 0$. Moreover, in the stationary case, we should additionally have a constant velocity flux in every pipe section. To formulate this condition, we need the following remark.

Remark 2. Since in the Navier–Stokes equation of steady-state flow is a variational Euler equation for the energy functional, we can consider the solution of the original problem in variational form. In particular, this leads to conditional extremum problems with isoperimertric conditions.

Let us determine ω in accordance with Remark 2. We obtain the following conditional-extremum problem

$$\int_{\Sigma} |\nabla\omega|^2 d\sigma \to \min_{\omega}, \quad \omega|_S = 0, \quad \operatorname{div}\omega = 0, \quad \int_{\Sigma} (\omega, \tau)\, d\sigma = q,$$

where τ is the normal to the cross section Σ, q is the fluid discharge.

The Lagrange conditional extremum method gives

$$\Delta\omega - \alpha\tau = 0, \quad \omega|_S = 0, \quad \operatorname{div}\omega = 0, \quad \int_{\Sigma} (\omega, \tau)\, d\sigma = q,$$

where Σ, q is the undefined multiplier determined by an integral condition.

We seek a τ-homogeneous solution of this problem in Cartesian coordinates such that the plane (x, y) overlaps Σ and the axis $z \upuparrows \tau$. Setting $\omega = \{0, 0, \alpha w\}$, we obtain a problem for the Poisson equation in the plane region Σ with the boundary Γ

$$\Delta w = 1, \quad w|_{\Gamma} = 0, \quad \alpha \int_{\Sigma} w\, d\sigma = q.$$

An explicit solution can be obtained only in special regions.

Let us solve the classical problem for a pipe with a circular cross section of radius r in the polar coordinates (ρ, θ). We obtain for $w = w(\rho, \theta)$

$$\Delta w = 1, \quad |w(0, \theta)| < \infty, \quad w(r, \theta) = 0,$$

$$w(\rho, \theta) = w(\rho, \theta + 2\pi), \quad w_\theta(\rho, \theta) = w_\theta(\rho, \theta + 2\pi),$$

whence, setting $w = R(\rho)\mathrm{e}^{\mathrm{i}n\theta}$, we get

$$\rho(\rho R')' - n^2 R = \rho^2, \quad |R(0)| < \infty, \quad R(r) = 0.$$

This problem has a solution only for $n = 0$,

$$R = \frac{\rho^2 - r^2}{4}, \quad 2\pi \int_0^r R(\rho)\rho\, d\rho = -\frac{\pi r^4}{8}.$$

In the end, we obtain the known expression

$$w = \frac{2q}{\pi r^4}(r^2 - \rho^2), \quad 0 \le \rho \le r.$$

3.4. Turbulent Motion. Alongside stationary fluid flow, it is interesting to consider so-called turbulent perturbation of the regular flow. This fluid motion is not a manifestation of instability of the stationary flow ($\lambda = 0$) but in fact a realization of the nonuniqueness of the solution of problems (28), (29) in steady-state flow ($\lambda \neq 0$).

Consider the problem of steady-state flow in the topological torus $\overline{T}^3$ with a circular cross section of radius r and generator radius R. We present the solution for problem (30) using its form and notation. For the vortex velocity component $\omega = \{0, 0, w(\rho, \theta, z)\}\, \mathrm{e}^{\lambda t}$ we obtain as a consequence the condition for the flux q through the pipe cross section Σ with the boundary Γ

$$\lambda w - \varepsilon \Delta w = 0, \quad |w(0, \theta, z)| < \infty, \quad w(r, \theta, z) = 0, \quad \int\limits_{\Sigma} (\omega, \tau)\, d\sigma = q,$$

$$w(\rho, \theta, z) = w(\rho, \theta + 2\pi, z), \quad w_\theta(\rho, \theta, z) = w_\theta(\rho, \theta + 2\pi, z),$$

$$w(\rho, \theta, 0) = w(\rho, \theta, 2\pi R), \quad w_z(\rho, \theta, 0) = w_z(\rho, \theta, 2\pi R),$$

where $\Delta = \partial^2/\partial\rho^2 + 1/\rho\, \partial/\partial\rho + 1/\rho^2 \partial^2/\partial\theta^2 + \partial^2/\partial z^2,\ 0 \le \rho \le r,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 2\pi R$. The search for the conditional extremum leads to the following problem in the plane region Σ

$$\lambda w - \varepsilon \Delta w = \alpha, \quad w|_\Gamma = 0, \quad \int\limits_{\Sigma} w\, d\sigma = q, \tag{31}$$

where α is the as yet unknown Lagrange multiplier and q is the sought flux.

We will show that problem (31) has eigenstates, i.e., nontrivial solutions for $\alpha = 0$. Find a solution of the homogeneous equation in the form $w = A(\rho, \theta) B(z)$. After separation of variables we obtain

$$\frac{\lambda A - \varepsilon \Delta A}{A} = \varepsilon \frac{B''}{B} = -\varepsilon \mu^2.$$

Taking $A = \mathrm{e}^{\mathrm{i}\nu\theta} C(r)$, from the condition of periodicity in θ and z we obtain $\nu = n,\ \mu = \mu_k = k/R,\ n, k \in \mathbb{Z}$, whence

$$\lambda \rho^2 C - \varepsilon \rho (\rho C')' + \varepsilon n^2 C = -\varepsilon \mu_k^2 \rho^2 C, \quad B = \mathrm{e}^{\mathrm{i}kz/R}.$$

Bounded solutions of this problem exist only for real λ. We seek admissible λ in the form $\lambda = -\varepsilon\mu_k^2 - \varepsilon\beta^2 \neq 0$. Then for $C(\rho)$ we obtain the equation

$$C'' + \frac{C'}{\rho} + \left(\beta^2 - \frac{n^2}{\rho^2}\right) C = 0,$$

whose solution is expressed by the Bessel function $\mathrm{J}_n(\beta\rho)$. Hence, we obtain the solution of the homogeneous equation (31)

$$w = \mathrm{J}_n(\beta\rho)\, \mathrm{e}^{\mathrm{i}(n\theta + kz/R)}.$$

The condition $\operatorname{div} \omega = 0$ is satisfied only for $k = 0$.

Thus, for the solution w of Eq. (31) we have to find β that satisfies the additional conditions

$$w(r, \theta, z) = 0, \quad 2\pi \int\limits_0^r w \rho\, d\rho = q.$$

Hence we find that β takes countably many values β_{ln} such that $\beta_{ln} r = x_{ln}$, where x_{ln} are the positive roots of the equation $\mathrm{J}_n(x_l) = 0,\ l \in \mathbb{N}$. The eigenstates in cylindrical coordinates have the form

$$\omega_{ln} = \{0, 0, 1\} \mathrm{J}_n(\beta_{ln}\rho)\, \mathrm{e}^{in\theta}.$$

Consider the case $n = 0$, the only one when $q \neq 0$. Simple evaluations of the integral give

$$\int_0^r \mathrm{J}_0(\beta\rho)\rho\, d\rho = \frac{r}{\beta} \mathrm{J}_1(\beta r),$$

whence we obtain equations for β and q

$$\mathrm{J}_0(\beta r) = 0, \quad r\mathrm{J}_1(\beta r) = \beta q.$$

Hence we find that β takes countably many values β_l such that $\beta_l r = x_l$, where x_l are positive roots of the equation

$$\mathrm{J}_0(x_l) = 0, \quad l \in \mathbb{N}, \quad x_1 = 2.40, \quad x_2 = 5.52, \quad \ldots .$$

Finally, the eigenstate $\omega_l(\rho, \theta, z, t)$ of the turbulent flow problem with $n = 0$ and nonzero discharge takes the form

$$\omega_l = \omega^o \mathrm{J}_0(\beta_l r)\, e^{-\varepsilon\beta_l^2 t}, \quad \omega^o = \{0, 0, 1\}, \quad l \in \mathbb{N}.$$

Since $\beta_l > 0$, all the modes are asymptotically stable.

Consider turbulent flow in a pipe with internal radius r [cm]. The longest-lived mode is observed for $l = 1$, which with kinematic viscosity of water $\varepsilon = 0.01\,\mathrm{cm}^2/\mathrm{sec}$ gives the decay factor $0.058/r^2[\mathrm{sec}^{-1}]$. For a pipe with $r = 1$, turbulence-induced beats disappear virtually within 40 sec.

CONCLUSION

Our results show that the flow velocity of viscous incompressible fluid is purely vortex velocity and is unrelated to pressure. The flow velocity is an infinitely smooth function that obeys the maximum principle and tends to zero because of energy dissipation.

The fluid pressure p merely compensates the external potential field u in the form of the equality $p + u = \mathrm{const}$, which initiates the pure vertex motion. In the absence of external fields, this is free motion and the pressure is constant. A physical confirmation of this fact is provided by the infinite velocity of sound in incompressible fluid.

A key to the proof of various assertions in this article is the augmentation of the boundary condition $\mathrm{v}|_S = 0$ with the supplementary condition $\partial^2 \mathrm{v}_\nu / \partial\nu^2|_S = 0$, which implies the absence of viscous forces on the boundary. The last assertion has a simple physical explanation: there is no friction between fluid particles at rest. The condition $\partial^2 \mathrm{v}_\nu / \partial\nu^2|_S = 0$ implies that in an incompressible viscous fluid $p + u = \mathrm{const}$.

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