

HIDDEN UNIQUE POSSIBILITIES OF MATHEMATICAL PHYSICS EQUATIONS (THE FORMALISM OF SKEW-SYMMETRIC FORMS)

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We show that the equations of mathematical physics that describe real processes and include the equations of the conservation laws for energy, momentum, moment of momentum, and mass have a peculiar feature. The conservation law equations turn out to be inconsistent. This produces the integrability of the mathematical physics equations and the specific features of their solutions, namely the presence of double solutions: on a nonintegrable coordinate space and on integrable structures. As a consequence, we can describe evolutionary processes, such as the emergence of discrete structures and observable formations (waves, vortices, turbulent pulsations). Since double solutions are defined on different spatial objects, they can be obtained only in two coordinate systems and using simultaneously two methods of equation solving: numerical and analytical. Appropriate results have been derived in the framework of skew-symmetric forms.

Keywords: inconsistency of the conservation laws equations, integrability of mathematical physics equations, double solutions, modeling features, skew-symmetric forms.

Introduction

In this article, we consider partial-differential equations of mathematical physics describing continuous media (such as thermodynamic, hydrodynamic, or space media), charged particle systems, and so on. These mathematical physics equations describe not only evolutionary processes and changes of physical variables (such as, for instance, energy, pressure, density) but also have possibilities for describing discrete processes, such as those involving the formation of various structures, quantum jumps, the appearance of observable formations (waves, vortices, turbulent pulsations) and so on.

Such possibilities, however, are hidden. They become apparent only in the process of equation solving. This is associated with the integrability of the mathematical physics equation, which is realized discretely.

Mathematical physics equations describing real processes are non-integrable on the original tangential space. In this case, the solution of the equations is not a function: the derivatives of this solution do not form differentials.

Integrability of the equations can be achieved only discretely in the process of solution in the presence of some degrees of freedom. The resulting solution is a discrete function.

We shall see that these features of the integrability of mathematical physics equations open unique possibilities for mathematical physics equations.

In Chapter 1, we review the current approaches to the solution of differential equations of mathematical physics that depend on their integrability and associated issues.

We describe the characteristic properties of integrable equations, specifically, the presence of differentials and directly integrable relations in differentials. These properties of integrable equations and the process of realization of integrability can be described only by means of skew-symmetric differential forms.

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Chapter 2 presents some properties and features of exterior and evolutionary skew-symmetric differential forms required for the investigation of the integrability of differential equations.

In Chapter 3 we investigate the integrability of differential equations of mathematical physics that depends on the consistency of the derivatives of the relevant functions and the consistency of the equations in systems of mathematical physics equations. (In practice, this issue is ignored when solving differential equations.)

Our study has shown that the integrability of differential equations is realized discretely. The process reveals the hidden properties of mathematical physics equations, such as invariance, double solutions, discrete transitions, and so on.

We show in Chapter 4 that these specific properties of mathematical physics equations are responsible for the unique possibilities of mathematical physics equations in the description of discrete processes and phenomena.

1. Approaches to the Solution of Differential Equations of Mathematical Physics

We distinguish between analytical and numerical methods for the solution of differential mathematical physics equations.

Analytical methods may be applied if the differential equations are reducible to integral form or if the integrability conditions are satisfied.

Since the issue of integrability of differential equations involves insurmountable difficulties, various numerical methods have been developed for the solution of the mathematical physics equations.

What is the correspondence between the solutions obtained by analytical and numerical methods, and how accurate are the descriptions of the real phenomena provided by mathematical physics equations?

We propose the formalism of skew-symmetric differential forms as a framework that not only answers these questions but also reveals the physical meaning of analytical and numerical solutions of mathematical physics equations.

Issues with the solution of differential mathematical physics equations are connected with the issue of integrability of differential equations.

Characteristic properties of integrable equations are obviously the existence of differentials and the possibility of reducing an equation to an identical relation in differentials. This reduction would allow explicit integration of the differential equation.

Skew-symmetric differential forms have such properties. Unlike other existing mathematical formalisms, they can take the form of differentials and differential expressions.

Differentials, as we know, are closed exterior forms.

In this article (as in other publications of the present author), we reveal yet another unique property of skew-symmetric differential forms. It turns out that some skew-symmetric differential forms are obtained from differential equations and generate closed exterior forms, i.e., differentials. These skew-symmetric differential forms are evolutionary forms and enable us to investigate integrability and properties of solutions of differential mathematical physics equations.

The investigation of mathematical physics equations using skew-symmetric differential forms has shown that mathematical physics equations have properties that do not follow explicitly from the differential equations; rather, these are properties realized in the process of solution. Such hidden properties connected with the integrability conditions of the equations reveal unique possibilities of mathematical physics equations.

Below we describe the properties of exterior and evolutionary skew-symmetric forms and the features of the mathematical apparatus that reveals these hidden properties and possibilities of mathematical physics equations.

More detailed information on skew-symmetric exterior and evolutionary forms can be found in [1].

2. Some Properties and Features of Exterior and Evolutionary Skew-Symmetric Differential Forms Used in the Investigation of Differential Equations

The properties of exterior differential forms (skew-symmetric differential forms on integrable manifolds and structures) are the foundation of various mathematical formalisms. However, due to historical factors, they are primarily applied in differential geometry and topology, although the properties of skew-symmetric forms have relevance, e.g., for the Frobenius theorem on the integrability conditions of Pfaffian systems.

An exterior differential form of degree p (a p -form) can be written as [1, 2]

$$\theta^p = \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}, \quad 0 \leq p \leq n. \quad (1)$$

Here $a_{i_1 \dots i_p}$ are functions of the independent variables x^1, \dots, x^n , n is the spatial dimension, and $dx^i, dx^i \wedge dx^j, dx^i \wedge dx^j \wedge dx^k, \dots$ is the local basis satisfying the skew-symmetry conditions

$$\begin{aligned} dx^i \wedge dx^i &= 0 \\ dx^i \wedge dx^j &= -dx^j \wedge dx^i \quad i \neq j \end{aligned} \quad (2)$$

The differential of the exterior form θ^p is expressed by the formula

$$d\theta^p = \sum_{i_1 \dots i_p} da_{i_1 \dots i_p} \wedge dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_p} \quad (3)$$

Since exterior forms are defined on integrable manifolds or structures, the differential of exterior forms does not contain the differential of the basis (because it vanishes). Such skew-symmetric forms may be closed.

2.1. Closed Exterior Differential Forms. Closed differential forms with invariant properties are the most useful in mathematical formalisms.

A form is said to be closed if its differential vanishes:

$$d\theta^p = 0 \quad (4)$$

By condition (4), a closed form is a conserved quantity. (This means that it corresponds to a conservation law, i.e., to some conserved physical variable.)

If an exterior form is an inexact closed form, i.e., it is defined only on some structure (which by its metric properties is a pseudostructure), the closure condition is written in the form

$$d_\pi \theta^p = 0 \quad (5)$$

The pseudostructure π satisfies the condition

$$d_\pi^* \theta^p = 0 \quad (6)$$

where $^* \theta^p$ is the dual form. In other words, the dual form describes a pseudostructure on which a closed inexact exterior form is defined. In this case, the (incomplete) interior differential vanishes.

A closed inexact exterior form and the corresponding dual form constitute a differential-geometric structure that describes a conserved object, specifically a conserved quantity on a pseudostructure. (This may correspond to some conservation law, i.e., a conserved object).

The integrability of partial-differential equations is associated with the following properties of closed exterior differential forms and the specific features of their mathematical apparatus.

(i) By the closure of the exterior form

$$d\theta^p = 0 \quad (7)$$

the closed form is a *differential*.

The differential form is closed because

$$dd\phi = 0 \quad (8)$$

where ϕ is an arbitrary exterior form.

A closed exterior inexact form is an interior (on a pseudostructure) differential.

(ii) Since a closed form is a differential, this implies that a closed form is *invariant* under all differential-conserving transformations.

A closed exterior inexact form and the corresponding dual form constituting a differential-geometric structure describe an invariant object.

(iii) Differential-conserving transformations of exterior skew-symmetric forms are *nondegenerate transformations*. Examples of such transformations are unitary transformation (the 0-form), tangent and canonical transformations (the 1-form), gradient and gauge transformations (the 2-form), etc.

(iv) Identical relations are an important element of the mathematical formalism of closed exterior forms. Identical relations express the fact that each closed exterior form is a differential of some exterior form. In general, an identical relation can be written as

$$d\phi = \theta^p \quad (9)$$

In this relation, the form in the right-hand side is a *closed form*.

Identical relations occur in virtually all branches of physics, mechanics, and thermodynamics. For instance, the Poincare invariant $ds = -H dt + p_j dq_j$, the second principle of thermodynamics $dS = (dE + p dV)/T$ etc., are such identical relations.

Closed exterior forms are, first, differentials (i.e., integrable) and, second, closed exterior forms are building blocks of integrable identical relations.

This suggests that if we understand the connection of differential equations with closed exterior forms, the issue of the integrability of differential equations is resolved.

The properties of *evolutionary skew-symmetric forms* reveal the connection of differential equations with closed exterior forms.

2.2. Properties and Features of Evolutionary Skew-Symmetric Differential Forms. Evolutionary skew-symmetric forms [1], unlike exterior forms, are defined on nonintegrable manifolds (such as the differential equation tangent manifolds, Lagrangean manifolds, etc.).

An evolutionary form may be written similarly to an exterior differential form [1]. However, unlike the differential of an exterior form, the differential of an evolutionary form contains an additional term. This is so because the basis of the evolutionary form changes since it is defined on a nonintegrable manifold.

The differential of an evolutionary form is written as

$$d\theta^p = \sum_{i_1 \dots i_p} da_{i_1 \dots i_p} \wedge dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_p} + \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} d(dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_p}) \quad (10)$$

where the second term is connected with the differential of the basis being nonzero: $d(dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}) \neq 0$. (For an exterior form defined on an integrable manifold, $d(dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}) = 0$).

Here and in what follows, the summation symbol \sum and the exterior product symbol \wedge are omitted. Summation over repeating indexes is implied.

An evolutionary form, unlike an exterior form, is not a closed form because its differential is not zero, since the second term of the differential of the evolutionary form connected with the differential of the basis is not zero.

The second term in the expression for the differential of the evolutionary skew-symmetric form connected with the differential of the basis is expressed in terms of a metric-form commutator.

For instance, consider the first-degree form $\omega = a_\alpha dx^\alpha$.

The differential of this form may be written as $d\omega = K_{\alpha\beta} dx^\alpha dx^\beta$ where $K_{\alpha\beta}$ is the commutator of the evolutionary form.

If the nonintegrable manifold is described using the connectednesses $\Gamma_{\beta\alpha}^\sigma$, the commutator $K_{\alpha\beta}$ may be written as

$$K_{\alpha\beta} = \left(\frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta} \right) + (\Gamma_{\beta\alpha}^\sigma - \Gamma_{\alpha\beta}^\sigma) a_\sigma \quad (11)$$

The commutator $K_{\alpha\beta}$ does not vanish because the coefficients a_μ are nonpotential and the connectednesses of the corresponding manifold (which is a deformable integrable manifold) are nonsymmetric [3]. The commutator of the form ω and its differential are thus nonzero.

Since the evolutionary form is not closed, it cannot be a differential like a closed exterior form.

The evolutionary form has a nontraditional mathematical apparatus, which includes nonidentical self-changing relations and degenerate transformations.

2.3. Nonidentical Self-Changing Relations. Identical relations have been shown to be the foundation of the mathematical apparatus of exterior differential forms. *Nonidentical relations*, on the other hand, are the foundation of the mathematical apparatus of evolutionary differential forms. Nonidentical relations appear in many processes.

A nonidentical relation may be written also as an identical relation

$$d\phi = \theta^p \quad (12)$$

However, the right-hand side of this relation is an evolutionary skew-symmetric form, which is not a differential.

An evolutionary relation is nonidentical because the left-hand side is a differential, and the right-hand side is a skew-symmetric form, which is not a differential.

An evolutionary nonidentical relation is self-changing because this is an evolutionary relation containing two terms, one of which is unmeasurable and is not comparable with the second term.

2.4. Degenerate Transformations. The degenerate transformation is one of the main elements of the mathematical formalism of the theory of evolutionary forms. Unlike the differential-conserving nondegenerate transformation of closed exterior forms, this degenerate transformation does not conserve the differential. (The Legendre transformation is an example of a degenerate transformation. However, it is applied only implicitly.)

Degenerate transformations reveal a unique property of the evolutionary form. An evolutionary form may generate closed exterior forms that are differentials, have invariant properties, and correspond to integrability of differential equations.

It will be shown below that the formalism of evolutionary forms with its nontraditional elements, such as nonidentical self-changing relations and degenerate transformations, reveals the hidden properties of mathematical physics equations, resolves the issue of integrability of differential equations, and reveals their unique possibilities.

3. On Integrability of Partial Differential Equations

We have noted previously that the presence of differentials and identical relations in differentials are a characteristic property of integrable equations, allowing explicit integration of the differential equation. We have shown that closed exterior forms have these properties. The issue of integrability of differential equations can be resolved if we detect a connection of differential equations with closed exterior forms and clarify how this connection is realized. These desiderata can be accomplished using evolutionary skew-symmetric forms, which, as noted, may generate closed exterior forms.

The qualitative investigation of differential equations has shown that the integrability of differential mathematical physics equations depends on the consistency of the derivatives of the relevant functions and the consistency of the equations when they are part of a system of mathematical physics equations.

3.1. Investigation of the Correspondence Between Derivatives in Differential Equations. The correspondence between the derivatives of the sought functions in differential equations can be traced for the case of a partial-differential equation of first order:

$$F(x^i, u, p_i) = 0, \quad p_i = \partial u / \partial x^i \quad (13)$$

The derivatives are clearly consistent if they form a differential. This requires the relation

$$du = \theta \quad (14)$$

where $\theta = p_i dx^i$ (summation over repeating indices is implied).

In the general case, Eq. (13) does not (explicitly) imply that the derivatives $p_i = \partial u / \partial x^i$, satisfying the equation form a differential. The derivatives p_i may constitute a differential only if the closure conditions are satisfied for the form $\theta = p_i dx^i$ and the corresponding dual form (in this case, the functional F is the dual form for θ):

$$dF(x^i, u, p_i) = 0, \quad d(p_i dx^i) = 0. \quad (15)$$

Expanding the differentials, we obtain a system of homogeneous equations in dx^i and dp_i (in the $2n$ dimensional cotangent space):

$$\left(\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} p_i \right) dx^i + \frac{\partial F}{\partial p_i} dp_i = 0, \quad dp_i dx^i - dx^i dp_i = 0. \quad (16)$$

The condition of solvability for this system (i.e., that the determinant formed from the coefficients of dx^i and dp_i vanishes) is

$$\frac{dx^i}{\partial F / \partial p_i} = \frac{-dp_i}{\partial F / \partial x^i + p_i \partial F / \partial u} \quad (17)$$

This condition defines an integrable structure: a pseudostructure on which the form $\theta = p_i dx^i$ is closed, i.e., becomes a differential.

If condition (17), which we may call the integrability condition, is satisfied, then the derivatives form a differential $\delta u = p_i dx^i = du$ (on the pseudostructure). In this case, the solution becomes a function.

Such solutions, i.e., functions on pseudostructures, are called generalized solutions. The derivatives of generalized solutions constitute a closed exterior form on a pseudostructure, which is an interior (on a pseudostructure) differential.

Note that a pseudostructure which is an integrable structure exists on a cotangent space. In other words, we pass from a tangent to a cotangent space. This transition is facilitated by a degenerate transformation (i.e., a transformation that does not conserve the differential), which requires that the determinant vanishes. (Below we will show that the integrability of mathematical physics equations is realized under a degenerate transformation, which requires that certain functional expressions, such as the determinant, the Jacobian, residues, and so on, vanish.)

We thus conclude that on the original tangent space the differential equation is not integrable. In this case, the derivatives of the differential equation do not constitute a differential. Relation (14) is not identical. The right-hand side is a skew-symmetric form, which is not a differential. The corresponding relation cannot be exactly integrated.

On the other hand, on the integrable structures of the cotangent manifold, the derivatives of the generalized solution constitute a differential, which indicates the integrability of the differential equation.

Examples of tangent and cotangent spaces are Lagrangean and Hamiltonian manifolds. A known example of a cotangent space is the phase space.

3.2. On the Integrability of Mathematical Physics Equations. Mathematical physics equations in fact are systems of equations consisting of several differential equations. Examples of mathematical physics equations are equations describing material media, such as thermodynamic, hydrodynamic, and space media, etc. They consist of the equations of conservation laws for energy, momentum, moment of momentum, and mass.

Integrability of a system of equations, such as a system of mathematical physics equations, depends on the consistency of the equations that comprise the given system.

The integrability of mathematical physics equations has been investigated for the particular case of a system of conservation-law equations.

3.3. The Consistency of Conservation-Law Equations. The Evolutionary Relation. In the investigation of the consistency of equations entering a system of equations, we use two systems of coordinates: the inertial system and the attached system fixed to the manifold of particle trajectories. (Examples are Eulerian and Lagrangean coordinate systems.)

In addition, the conservation-law equations are transformed into equations for functionals characterizing the state of the medium. Such state functionals include the wave function, the entropy, the action functional, the Poynting vector, the Einstein tensor, and so on (they are all field-theory functionals) [4].

The investigation of the consistency of conservation-law equations leads to an evolutionary relation in skew-symmetric forms for the state functional. This relation reveals the specific features of the integrability of mathematical physics equations.

Consider the consistency of the energy and momentum equations. In the attached system of coordinates, the energy and momentum equations can be written as

$$\frac{\partial \psi}{\partial \xi^1} = A_1 \quad (18)$$

$$\frac{\partial\psi}{\partial\xi^\nu} = A_\nu \quad (19)$$

Here ψ is the state functional, ξ^1 and ξ^ν are the coordinates along the trajectory and normal to the trajectory, A_1 is a quantity dependent on the specific features of the material medium and on external energy interactions with the medium, and A_ν is a quantity dependent on the specific features of the material medium and external forces.

Here we observe a certain peculiarity: two equations obtain for the same quantity (the functional ψ). How do we investigate such a system?

Equations (18) and (19) can be folded into the relation

$$d\psi = A_\mu d\xi^\mu \quad (20)$$

Relation (20) may be rewritten as

$$d\psi = \omega \quad (21)$$

where $\omega = A_\mu d\xi^\mu$ is a skew-symmetric differential form of the first degree.

Since the conservation-law equations are evolutionary, this relation and the skew-symmetric form ω are also evolutionary.

If we further use the conservation laws for the moment of momentum and the mass, the evolutionary relation takes the form

$$d\psi = \omega^p \quad (22)$$

where the degree p of the form takes the values $p = 0, 1, 2, 3$.

3.4. The Properties and Features of the Evolutionary Skew-Symmetric Forms and Evolutionary Relations.

The evolutionary skew-symmetric form, unlike the exterior form, is not closed.

The evolutionary forms and the evolutionary relation have been obtained in the attached coordinate system fixed to the manifold of particle trajectories in the relevant medium. The manifold generated by particle trajectories is deformable, i.e., nonintegrable. As shown in sec. 2.2, the evolutionary form defined on a nonintegrable manifold may not be closed because its commutator (which contains a nonzero metric-form commutator) and thus also the differential do not vanish.

Since the evolutionary form ω is not a differential, the evolutionary relation is *nonidentical* (the left-hand side is a differential and the right-hand side is a skew-symmetric form, which is not a differential).

We can similarly show that relation (22) is also nonidentical. In this case, the evolutionary forms of degrees 1-3 are not closed because the commutator of their evolutionary forms contains the nonzero metric-form commutators of degree 1, 2, and 3 that respectively define torsion, rotation, and bending.

(Note that the resulting evolutionary relation is nonidentical irrespective of the accuracy with which the conservation-law equations are written. This makes physical sense.)

A nonidentical evolutionary equation is self-changing (self-varying) because it is evolutionary and contains terms that are inconsistent with one another.

3.5. Nonintegrability of Mathematical Physics Equations on the Original Tangent Space.

A nonidentical evolutionary relation is not directly integrable because the right-hand side is a nonclosed form and thus not a differential. Mathematical physics equations cannot be folded into an identical relation for direct integration.

The mathematical physics equations are thus nonintegrable on the original tangent space. The solutions of mathematical physics equations in this case depends not only on variables. They also depend on the commutator of the evolutionary form. The solutions are not analytical solution, i.e., functions. The derivatives of these solutions do not form a differential. (We will show below that these solutions are physically meaningful.)

3.6. Realization of the Integrability of Mathematical Physics Equations. The integrability of mathematical physics equations is realizable only if the nonidentical relation can be transformed into an (integrable) identical relation.

To obtain an identical relation from an evolutionary nonidentical relation, we have to derive a closed exterior form from the evolutionary form entering the evolutionary relation.

Since the differential of an evolutionary form does not vanish while the differential of a closed exterior form is zero, the transition from the evolutionary form to a closed exterior form is possible only with a degenerate transformation, i.e., a transformation that does not conserve the differential.

Degenerate transformations may arise under additional conditions determined by the realization of some degrees of freedom. These additional conditions correspond to the situation when functional expressions such as determinants, Jacobians, Poisson brackets, residues, and so on vanish. Equality of these functional expressions to zero requires the closure of the dual form describing the integral structure being realized.

The additional conditions may be realized through self-variation of a nonidentical evolutionary relation. This may occur only discretely.

The degenerate-transformation conditions specify integrable structures (pseudostructures): the characteristics (the determinant vanishes), the singular points (the Jacobian vanishes), etc. A degenerate transformation is realized as a transition from the original tangent space to an integrable structure. Mathematically this is realized as a transition from a tangent nonintegrable manifold to sections of a cotangent integrable manifold.

The Legendre transformation is an example of a degenerate transformation. However, the Legendre transformation is usually executed implicitly and the discrete nonequivalent transition from one space to another is ignored. (An example of such a transition is the transition from a Lagrangean to a Hamiltonian manifold.)

3.7. The Realization of a Closed Inexact Exterior Form. Deriving an Identical Relation from a Nonidentical Relation. When the condition of degenerate transformation is satisfied, we have the transition

$$d\omega^p \neq 0 \rightarrow (\text{degenerate transformation}) \rightarrow d_\pi \omega^p = 0, d_\pi^* \omega^p = 0$$

The condition $d_\pi^* \omega^p = 0$ defines a closed dual form describing some integrable structure π (pseudostructure by its metric properties). The condition $d_\pi \omega^p = 0$ defines a closed inexact (on an integrable structure) exterior form.

Here we should stress the following point. The realization of a closed exterior form which is a differential and therefore an invariant implies that the mathematical physics equations have invariant properties.

On an integrable structure we obtain from the evolutionary relation (22)

$$d\psi_\pi = \omega_\pi^p. \quad (23)$$

This relation is identical because the form ω_π^p is an (interior) differential.

An identical relation may be integrated, which implies that the mathematical physics equation is integrable on an integrable structure. In other words, the solution of a mathematical physics equation is a function. But this analytical solution is a discrete function because it is defined only on the structure. This is a so-called generalized solution.

We thus find that the mathematical physics equations have double solutions: one on the original coordinate space and one on an integrable structure. The solution on the original coordinate space is not a function, while the solution on an integrable structure is a discrete function.

We have established a discretely realized connection between these solutions. When the degenerate-transformation conditions are satisfied, integrable structures are realized and the mathematical physics equations transit from the original tangent space with a solution (on the coordinate space) which is not a function to integrable structures of the cotangent manifold with a generalized solution which is a discrete function.

3.8. The Physical Meaning of Double Solutions. We have shown that mathematical physics equations have double solutions: one on the coordinate space and the other on integrable structures. This feature has a physical meaning. Inexact solutions describe a nonequilibrium state of the medium. Exact solutions (these are discrete functions) describe a local-equilibrium state of the medium.

This follows from the evolutionary relation $d\psi = \omega^P$ and is connected with the fact that the evolutionary relation contains the functional ψ characterizing the state of the medium.

The presence of the differential of a state functional indicates the presence of a state function, which corresponds to the equilibrium state of the medium. But there is a nuance, here. Since the evolutionary relation is nonidentical, it cannot be used to obtain the differential of the state functional $d\psi$. This implies the absence of a state function and indicates that the system state is nonequilibrium. Inexact solutions describe such a nonequilibrium state of the medium. The internal forces that produce the nonequilibrium obviously should be described by the commutator of a nonclosed evolutionary form ω^P .

From the identical relation $d\psi_\pi = \omega_\pi^P$ realized from the evolutionary relation we should be able to obtain the differential $d\psi_\pi$ (internal on an integrable structure). This points to the presence of a state function and the transition of the medium to a local equilibrium state. (The general state of the medium remains nonequilibrium.) The local-equilibrium state of the medium is described by a discrete function.

The transition of the medium from a nonequilibrium state to a local-equilibrium state signifies that the unobservable quantity described by the evolutionary-form commutator and acting as an internal force changes to an observable quantity (a potential quantity for the given medium). This is manifested as the creation of some discrete structures and observable formations, such as waves, vortices, turbulent pulsations.

We should stress again that solutions of mathematical physics equations on integrable structures are connected with solutions on the original coordinate space. This must be taken into consideration when describing real physical processes by mathematical physics equations.

We thus conclude that mathematical physics equations have double solutions: one on the original coordinate space and another on integrable structures.

How can we find these solutions?

Since the mathematical physics equations on the original coordinate space have solutions that are not functions, such solutions can only be obtained by numerical methods.

On the other hand, the solutions on integrable structures are functions and can be determined analytically. These solutions can also be found by numerical methods. But in this case the equations are simulated on an integrable manifold, not on the original space, which means that other, nonequivalent systems of coordinates are used. (We see that double solutions cannot be obtained by a single continuous simulation of the mathematical physics equations.)

Both analytical and numerical solution methods are currently available for mathematical physics equations. The same phenomenon is sometimes described by both numerical methods on the original coordinate space and analytical methods on integrable spaces, such as phase (covariant, cotangent) spaces, which are obtained from the integrability conditions imposed on the equations.

But, in this case, the solutions of the mathematical physics equations describing the same phenomenon but obtained by different methods (numerical methods on the original coordinate space or analytical methods on integral structures) are different. This implies that the solutions are incorrect. The solutions of mathematical physics equations obtained by both numerical and analytical methods are correct but in each case, they describe only one facet of the physical phenomenon. Therefore, to find a complete description of a physical phenomenon, we have to use simultaneously the two methods for the solution of mathematical physics equations and to allow for the interconnection between the two approaches.

In the mechanics of continuous media describing the variation of quantities that characterize material media, we typically use numerical methods on the original coordinate space, whereas in the physics of continuous media describing conserved (under nondegenerate transformations) physical quantities or objects, the mathematical physics equations are usually solved by analytical methods or numerical methods on integrable manifolds. This is attributable to the fact that the mechanics of continuous media focuses on evolutionary processes described by numerical solutions, where the physics of continuous media focuses on conserved objects described by analytical solutions.

Investigation of the integrability of differential equations describing real physical processes and phenomena has shown that the integrability of differential equations is realized discretely in the presence of some degrees of freedom. In the process, unique hidden properties of mathematical physics equations are revealed, such as invariance, double solutions, discrete transition, etc., which enable us to describe evolutionary processes, creation of physical structures, observable formations, and so on.

We should again stress that these results have been obtained using skew-symmetric evolutionary forms, which are defined on nonintegrable manifolds and possess a nontraditional mathematical apparatus operating with non-identical relations and degenerate transformations.

These properties of differential equations reveal unique possibilities of mathematical physics equations for the description of physical processes and phenomena that cannot be described in the frameworks of other mathematical formalisms.

4. Unique Hidden Possibilities of Mathematical Physics Equations

Below we briefly describe other publications of the present author reporting on the properties and possibilities of some mathematical physics equations. We demonstrate the specific features of the described phenomena that are determined by the hidden properties of the mathematical physics equations.

4.1. The Integrability of Euler and Navier–Stokes Equations. The Onset of Vorticity and Turbulence. The Euler and Navier–Stokes equations describing a fluid-dynamic medium, i.e., ideal and viscous gas flows, are examples of mathematical physics equations.

The proof of the existence and smoothness of the solution of Navier–Stokes equations has been declared the problem of the millennium.

However, it follows from this article, and also from the results of [5], that a smooth solution does not exist. The Euler and Navier–Stokes equations, again as mathematical physics equations, have double solutions, specifically, a solution on the original coordinate space and a solution on integrable structures. The transition from the first solution to the second describes the transition of the fluid-dynamic medium from an equilibrium state to a local-equilibrium state, which involves the onset of vorticity (in the ideal gas) or turbulence (in a viscous gas).

This follows from the evolutionary relation obtained from the Euler and Navier–Stokes equations.

The properties of Euler and Navier–Stokes equations are discussed in [5].

The evolutionary relation obtained from Euler and Navier–Stokes equations [5] may have the following form:

$$ds = A_\mu d\xi^\mu$$

Here the coefficients A_1 , obtained from the equation of the energy conservation law are respectively $A_1 = 0$ for the ideal gas and

$$A_1 = \frac{1}{\rho} \frac{\partial}{\partial x_i} \left(-\frac{q_i}{T} \right) - \frac{q_i}{\rho T} \frac{\partial T}{\partial x_i} + \frac{\tau_{ki}}{\rho} \frac{\partial u_i}{\partial x_k}$$

for a viscous gas. The coefficients A_ν obtained from the equation of the momentum conservation law are given by [6]

$$A_\nu = \frac{\partial h_0}{\partial \xi^\nu} + (u_1^2 + u_2^2)^{1/2} \zeta - F_\nu + \frac{\partial U_\nu}{\partial t}$$

Here we should stress that the entropy s entering the evolutionary relation depends on the spatial-temporal coordinates ξ^μ , unlike the thermodynamic entropy that depends on the thermodynamic variables. It is this entropy (and not the thermodynamic entropy) that describes the state of a fluid-dynamic medium.

4.2. Connection of Field-Theory Equations with Mathematical Physics Equations. The situation that prevailed in the end of the 19th and the beginning of the 20th century was described by Henry Poincare as the “crisis of physics”. Numerous phenomena and structures had accumulated which could not be described by mathematical physics equations. New equations were accordingly derived in various branches of physics — the field-theory equations (such as Einstein, Maxwell, Schrodinger, Dirac, and Heisenberg equations). These equations were based on the properties of invariance and covariance, which were necessary for the description of observable physical structures and phenomena and which allegedly did not apply to mathematical physics equations.

It turns out that mathematical physics equations possess invariant properties. But these properties are hidden. They are realized discretely, which is connected with the realization of the integrability of mathematical physics equations described by the realization of closed skew-symmetric exterior and dual forms with invariant and covariant properties.

The evolutionary relation obtained from mathematical physics equations during the study of the consistency of conservation-law equations (which is a condition of integrability) possesses the properties of field-theory equations.

In [7], we have shown the correspondence between evolutionary relations and field-theory equations, which indicates the connection of field-theory equations with mathematical physics equations.

We have noted previously that an evolutionary relation is a relation for functionals, such as action functional, entropy, wave function, Lagrangean, Einstein tensor, Poynting vector, and others, which are also field-theory functionals [4].

The field-theory equations also have a form of relations written in terms of skew-symmetric forms or their tensor or differential analogues. Thus,

- the Einstein equation is a relation in skew-symmetric forms (a tensor relation);
- the Maxwell equations have the form of tensor relations;
- the Schrodinger equation has the form of relations expressed in terms of derivatives and their analogues.

The evolutionary relation leads to closed inexact exterior forms, which are solutions of the field-theory equations.

The evolutionary relation of the mathematical physics equations unifies the field-theory equations, reveals their internal connection, and discloses properties common to all field-theory equations. This can provide an approach to a unified and general field theory.

4.3. Duality of Conservation Laws. Mathematical physics equations reveal the duality of conservation laws [8].

Hidden properties of the mathematical physics equations shed light on the specific features of the conservation laws, which have different meanings in thermodynamics and mechanics.

In mechanics and physics of continuous media, the notion of “conservation law” implies conservation of energy, momentum, moment of momentum, and mass. These are conservation laws for material media, and they are described by differential equations.

In physics connected with field theory and in theoretical mechanics, “conservation laws” imply the existence of conserved physical variables or objects. The conservation laws for physical fields fall in this category.

The first principle of thermodynamics as an example of an evolutionary relation is derived from two conservation laws, namely the energy conservation law and the momentum conservation law. The second principle of thermodynamics defining the thermodynamic entropy is an example of an identical relation derived from the first principle of thermodynamics (an evolutionary relation) by the realization of the integrating multiplier $1/T$, where T is temperature.

The realization of closed exterior forms corresponding to conservation laws for physical fields from mathematical physics equations consisting of the conservation laws for material media (conservation laws for energy, momentum, moment of momentum, and mass) reveals the connection of conservation laws for physical fields with conservation laws for material media. This connection is realized discretely in the evolutionary process.

The realization of closed exterior forms corresponding to physical-field conservation laws from mathematical physics equations describing material media indicates that physical fields are generated by material media.

4.4. The Process of Realization of Hamiltonian Systems. The mathematical physics equations reveal the process of realization of integrable Hamiltonian systems.

The hidden properties of mathematical physics equations make it possible to describe the process of realization of Hamiltonian systems of differential equations [9].

In classical mechanics, a Hamiltonian system is realized from the Euler–Lagrange equation as the integrability of the Euler–Lagrange equation [9]. Here we should note that a Hamiltonian system can be obtained from the Euler–Lagrange equation constrained by additional conditions. In this case, the Euler–Lagrange equation is not integrable. However, with the realization of any degrees of freedom (when the constraints become locally holonomic) the integrability condition is satisfied (one of the Hamiltonian system relations) and the Hamiltonian system for the Hamilton function is realized from the Euler–Lagrange equation. The realization of the Hamiltonian system is a discrete process. The transition from the tangent manifold with the Euler–Lagrange equation to sections of the cotangent manifold with the Hamiltonian system is accomplished by the Legendre transformation (a degenerate transformation that does not conserve the differential).

In [9], instead of describing the properties of the Hamiltonian system and the Hamilton function (which have been discussed in numerous publications), we show how Hamiltonians are realized from differential equations and describe the realization process. This became possible thanks to the hidden properties of differential equations discovered through skew-symmetric forms.

4.5. Emergence of Physical Structures and the Appearance of Observable Formations. Mathematical physics equations reveal the process of appearance of physical structures and observable formations [10].

As we have shown, degenerate transformation of an evolutionary form realizes a closed inexact exterior form and the corresponding dual form, which constitute a differential-geometric structure, specifically a pseudo-structure with a conserved quantity. This differential-geometric structure describes a physical structure, i.e., a structure on which an exact conservation law is satisfied. Physical fields and corresponding manifolds are formed by such physical structures [10]. Massless particles, structures formed by eikonal surfaces and wave fronts, etc., are examples of physical structures.

The description of the process of emergence of physical structures, i.e., conserved objects, has specific features that do not occur in any of the existing mathematical formalisms. The emergence of physical structures described by a transition from a tangent nonintegrable space to a cotangent integrable space (i.e., from one spatial object to another nonequivalent spatial object). This transition can be described only by skew-symmetric forms.

The emergence of physical structures is accompanied by the appearance of observable formations in the described medium, such as waves, vortices, fluctuations, turbulent pulsations, and so on [10]. In a gas-dynamic medium, the occurrence of observed formations leads to the appearance of vorticity and turbulence.

A physical structure emerges discretely under a degenerate transformation and is realized on an integrable (cotangent) manifold as an integrable structure with a conserved quantity. The integrable structure is a section of a cotangent bundle.

Note that there is a connection between degenerate and nondegenerate transformations. A suitable example is provided by the Lagrangean formalism.

A degenerate transformation is a transition from the *tangent* space (q_j, \dot{q}_j) to the *cotangent* space (q_j, p_j) . A nondegenerate transformation is a transition into a *cotangent space* from some pseudostructure (phase trajectory) (q_j, p_j) to another pseudostructure (Q_j, P_j) . [The canonical transformation formula can be written in the form $p_j dq_j = P_j dQ_j + dW$, where W is the generating function.]

Degenerate transformations describe the emergence of physical structures, while nondegenerate transformations describe transitions from one structure to another.

4.6. Discrete Transitions and Quantum Jumps. Mathematical physics equations may describe discrete quantum transitions [11].

Discrete processes may be described only by mathematical physics equations not constrained by integrability conditions. It is the realization of integrability conditions, taking place discretely in the process of solution of the equations, that detects hidden properties of the mathematical physics equations, which make it possible to describe discrete transitions, emergence of various structures, and appearance of observable formations.

This also reveals the mechanism responsible for quantum jumps. When passing from one structure to another, the conserved quantity described by a closed exterior form and the pseudostructure described by the dual form change discretely. Discrete changes of a conserved quantity and pseudostructure emit a quantum, which is generated during the transition from one structure to another. The evolutionary-form commutator generated at the time of structure emergence determines the characteristics of this quantum.

We can stress again that discrete transitions are described by a degenerate transformation that triggers the transition from the original coordinate space to integrable structures.

5. Conclusion

Hidden unique possibilities of differential equations of mathematical physics were discovered during the investigation of the integrability of mathematical physics equations. In this context, the integrability condition was viewed as consistency of the derivatives of the sought function (i.e., the existence of a differential) and the consistency of the equations forming a system. (These conditions are usually ignored when solving mathematical physics equations.)

The integrability of mathematical physics equations was found to be realized discretely in the process of solution of the equations (it is not a permanent property of mathematical physics equations). This process reveals the hidden properties of mathematical physics equations, which prove to be nontraditional.

First, a transition occurs from the tangent nonintegrable space on which the mathematical physics equations are defined to integrable structures in the cotangent space.

Second, on integrable structures, the solution of mathematical physics equations is a discrete function. This is a so-called generalized solution. Yet, the solution on the original coordinate space remains nonintegrable, i.e., it is not a function. Mathematical physics equations thus have double solutions.

As we have shown, these properties of mathematical physics equations make it possible to describe discrete transitions, quantum jumps, emergence of various structures and observable formations, such as waves, vortices, turbulent pulsations, and so on. Because of these possibilities, the mathematical physics equations reveal the mechanism of numerous processes and phenomena. This feature has been demonstrated in the author's various publications, which are briefly reviewed in the present article.

Such results have been obtained using skew-symmetric differential forms, which include differentials and differential expressions, thus making it possible to tackle integrability issues. In addition to exterior forms possessing invariant properties, we have also used skew-symmetric forms derived by the author, which are defined on nonintegrable manifolds and have evolutionary properties. The theory of evolutionary forms contains elements such as nonidentical relations and degenerate transformation, which are hardly ever applied explicitly in existing mathematical formalisms.

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