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I. MATHEMATICAL MODELING

TWO-DIMENSIONAL INVERSE PROBLEM OF MAGNETOTELLURIC SOUNDING IN A NONHOMOGENEOUS MEDIUM

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Unique solvability theorems are proved for the inverse two-dimensional sounding problem in a nonhomogeneous conducting half-space with different primary-field polarizations.

Keywords: inverse problems, differential equations, solution uniqueness.

Introduction

Magnetotelluric sounding (MTS) studies the structure of the Earth by using its natural electromagnetic field. It is assumed that the field source is at a large distance from the Earth's surface, where the electromagnetic field is measured. Under these assumptions, we may treat the primary field as constant in the observation region and assume that its variations are associated with the nonhomogeneous distribution of the subsurface conductivity.

The strength of the source creating the Earth's natural electromagnetic field is unknown, and we observe the relationship between the electric and magnetic fields:

$$\begin{cases} E_x = Z_{xx} H_x + Z_{xy} H_y, \\ E_y = Z_{yx} H_x + Z_{yy} H_y. \end{cases} \tag{1}$$

The linear coefficients in (1) constitute the impedance tensor, which is independent of the strength of the distant field source and is determined only by the field frequency and the distribution of the electrical conductivity below the Earth's surface $(z > 0)$.

The forward MTS problem determines the impedance tensor $\hat{Z}$ at $z = 0$ when the conductivity distribution $\sigma(M)$ is known for $z > 0$. The field source is a plane wave normally incident on the Earth's surface from the half-space $z < 0$.

The inverse MTS problem determines $\sigma(M)$, $z > 0$ given the impedance tensor on the Earth's surface $z = 0$ as a function of the observation point and the frequency $\hat{Z}(x, y, \omega)$.

In the one-dimensional case, when the conductivity $\sigma(z)$, $z > 0$, is a function of depth only (a layered medium), the inverse problem has a unique solution for piecewise-analytical functions $\sigma(z)$ with $\sigma(z) = \sigma_H = \text{const}$ given for $z > H$ [1]. The uniqueness of the inverse-problem solution has been proved for a layered medium containing a thin layer with longitudinally varying conductivity [2]. We apply the method of [2] to investigate

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the unique solvability of the inverse problem in the two-dimensional case, when the conductivity is $\sigma(y,z)$ and the impedance tensor has the form

$$\hat{Z} = \begin{pmatrix} 0 & Z_E(y,\omega) \\ Z_H(y,\omega) & 0 \end{pmatrix}. \tag{2}$$

Statement of the Problem

In the general three-dimensional case, the forward MTS problem involves solving the Maxwell's equations

$$\operatorname{rot}\bar{H} = \sigma\bar{E}, \quad \operatorname{rot}\bar{E} = i\omega\mu\bar{H}, \tag{3}$$

where $\mu = \text{const}$, $\sigma = \sigma_0 \approx 0$ for $z < 0$, whereas for $z > 0$

$$\sigma = \begin{cases} \sigma(M), & z \in [0, H], \\ \sigma_H = \text{const}, & z > H, \end{cases} \tag{4}$$

the nonhomogeneity is local, i.e., $\sigma(M) = \sigma_c(z)$ for $|x| > L_x$ and for $|y| > L_y$. Inside the nonhomogeneity with $M \in V_H$, $V_H : \left(|x| < L_x, |y| < L_y, z \in [0, H]\right)$, the conductivity is a piecewise-continuous function. The field source is a plane wave normally incident on the plane $z = 0$, with two polarizations:

$$1. \quad \bar{E} = (E_x^0, 0, 0), \quad \bar{H} = (0, H_y^0, 0),$$

$$2. \quad \bar{E} = (0, E_y^0, 0), \quad \bar{H} = (H_x^0, 0, 0).$$

Given the fields for the two plane-wave polarizations, we determine the components of the impedance tensor from Eqs. (1).

Consider the two-dimensional case, when $L_x = \infty$ and $\dfrac{\partial\sigma(M)}{\partial x} = 0$. Then the electromagnetic field decomposes into two field polarizations depending on the plane-wave polarization.

1. E-polarization: $\bar{E} = (E_x(y,z), 0, 0)$, $\bar{H} = (0, H_y, H_z)$.
Then

$$H_y = \frac{1}{i\omega\mu}\frac{\partial E_x}{\partial z}, \quad H_z = -\frac{1}{i\omega\mu}\frac{\partial E_x}{\partial y} \tag{5}$$

and the electric field is the solution of the problem

$$\Delta E_x + i\omega\mu\sigma(y,z)E_x = 0, \tag{6}$$

E_x and $\dfrac{\partial E_x}{\partial n}$ are continuous at the discontinuity boundaries of $\sigma(y, z)$ (n is the normal to the boundary). As $|y| \to \infty$, $E_x(y, z) \to E_x^0(z)$, where $E_x^0(z)$ is the field of the plane wave in the layer with $\sigma_c(z)$, which is the limit of $\sigma(y, z) \to \sigma_c(z)$ as $|y| \to \infty$.

The impedance, in this case, is given by

$$Z_E = \frac{E_x(y, z=0)}{H_y(y, z=0)}. \tag{7}$$

The inverse problem determines $\sigma(y, z)$ given the impedance $Z_E(y, \omega)$.

2. H-polarization: $\bar{E} = (0, E_y, E_z)$, $\bar{H} = (H_x, 0, 0)$.

In this case, the magnetic field is given by

$$H_x(y, z) = \frac{1}{i\omega\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right). \tag{8}$$

The electric field is the solution of the following system of equations for $z > 0$, $y \in (-\infty, \infty)$

$$\begin{cases} \dfrac{\partial}{\partial z}\left(\dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z}\right) = i\omega\mu\sigma E_y, \\[2ex] \dfrac{\partial}{\partial y}\left(\dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z}\right) = -i\omega\mu\sigma E_z, \end{cases} \tag{9}$$

at $z = 0$ the boundary conditions are

$$E_z(y, z=0) = 0; \qquad H_y(y, z=0) = \frac{1}{i\omega\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = 1 \tag{10}$$

and at the discontinuity boundaries of $\sigma(y, z)$ both E_τ and σE_n are continuous (E_n is the normal component of the electric field, E_τ is the tangential component). As $|y| \to \infty$, we have $E_z \to 0$ and $E_y \to E_y^0$, where E_y^0 is the plane-wave field in the layer with $\sigma_c(z) = \lim\limits_{|y| \to \infty} \sigma(y, z)$.

The Inverse Sounding Problem in a Nonhomogeneous Layered Medium with E-Polarized Field

The inverse problem, as we have noted above, involves determining $\sigma(y, z)$ given the impedance $Z_E(y, \omega)$ at $z = 0$. Once the impedance is known, we can find the field on the Earth's surface. To this end, consider the problem for the field at $z < 0$:

$$\Delta E_x(y, z) + i\omega\mu\sigma_0 E_x = 0, \qquad z < 0, \qquad y \in (-\infty, \infty), \tag{11}$$

$\frac{\partial E_x}{\partial z} = i\omega\mu Z_E(y,\omega)E_x$ at $z = 0$. As $|y| \to \infty$, we have $E_x(y,z) \to E_x^0(z)$, where $E_x^0(z)$ is the field of the plane wave reflected from the plane with the impedance $Z_E^0(\omega) = \lim_{|y|\to\infty} Z_E(y,\omega)$. This reflected field is given by

$$E_x^0(z) = e^{ik_0 z} - \frac{Z_E^0 - \gamma}{Z_E^0 + \gamma} e^{-ik_0 z}, \tag{12}$$

where $k_0 = \sqrt{i\omega\mu\sigma_0}$, $\operatorname{Re} k_0 > 0$, $\gamma = \frac{k_0}{\omega\mu}$.

Consider the secondary (anomalous) field

$$E_x^s(y,z) = E_x(y,z) - E_x^0, \qquad z < 0.$$

Then for E_x^s we obtain the problem

$$\Delta E_x^s + k_0^2 E_x^s = 0, \qquad z < 0, \qquad y \in (-\infty, \infty) \tag{13}$$

with the boundary condition at $z = 0$

$$\frac{\partial E_x^s}{\partial z} = i\omega\mu Z_E E_x^s + i\omega\mu\left(Z_E(y) - Z_E^0\right)E_x^0, \tag{14}$$

$E_x^s \to 0$ as $\sqrt{y^2 + z^2} \to \infty$.

Problem (13)–(14) is reduced to an integral equation by applying the Green's function

$$G(y - y_0, z, z_0) = \frac{i}{4} H_0^{(1)}(k_0 R) + \frac{i}{4} H_0^{(1)}(k_0 R^*),$$

where $R = \sqrt{(y - y_0)^2 + (z - z_0)^2}$, $R^* = \sqrt{(y - y_0)^2 + (z + z_0)^2}$, and $H_0^{(1)}(x)$ is the zeroth-order Hankel's function of the first kind.

Since

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = 0, \qquad G|_{z=0} = g(y - y_0, z_0) = \frac{i}{2} H_0^{(1)}\left(k_0\sqrt{(y - y_0)^2 + z^2}\right),$$

we obtain from Green's formula

$$E_x^{(s)}(y,z) = i\omega\mu \int_{-\infty}^{\infty} Z_E(y_0) E_x^s(y, z_0 = 0) g(y - y_0, z)\, dy_0$$

$$+ i\omega\mu \int_{-\infty}^{\infty} \left(Z_E(y_0) - Z_E^0\right) E_x^0(z_0 = 0) g(y - y_0, z)\, dy_0. \tag{15}$$

For $z = 0$ we obtain from (15) an integral equation for the secondary field $E_x^s(y, z = 0)$

$$E_x^s(y) - i\omega\mu \int_{-\infty}^{\infty} Z_E(y_0) E_x^s(y_0) g(y - y_0, z = 0)\, dy_0$$

$$= i\omega\mu \int_{-\infty}^{\infty} \left(Z_E(y_0) - Z_E^0 \right) E_x^0(z_0 = 0) g(y - y_0, z_0 = 0)\, dy_0 \,. \tag{16}$$

This is a Fredholm integral equation whose kernel $g(y - y_0, z = 0) = \frac{i}{2} H_0^{(1)}(k_0 | y - y_0 |)$ has a weak (logarithmic) singularity. This equation has a unique solution. Therefore, in the inverse problem, we can replace the impedance $Z_E(y, \omega)$ on the surface $z = 0$ with the secondary (anomalous) electric field $E_x(y, \omega)$.

Consider the inverse sounding problem for a homogeneous conducting half-space $z > 0$ with finitely many N conducting bodies in the shape of thin nonhomogeneous layers. Each layer is at depth z_n with thickness h_n, $n \in [1, N]$ and conductivity $\sigma_n(y)$, $y \in [-L_n, L_n]$ that varies only longitudinally. This implies that for $z > 0$ the conductivity is specified as

$$\sigma(M) = \begin{cases} \sigma_n(y) & \text{for} \quad M \in Q_n, \quad n \in [1, N], \\ \sigma^* & \text{for} \quad M \notin Q_n, \quad n \in [1, N], \end{cases} \tag{17}$$

where $U_n : \{| y| < L_n, z \in [z_n, z_n + h_n]\}$ are the nonhomogeneous layers with longitudinally varying conductivity.

The Green's function for the two half-spaces $z \geq 0$, $z_0 > 0$ has the form:

$$g(y - y_0, z, z_0) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{i\lambda(y - y_0)} \left(e^{-\eta| z - z_0 |} + \frac{\eta - |\lambda|}{\eta + |\lambda|} e^{-\eta(z + z_0)} \right) \frac{d\lambda}{\eta}, \tag{18}$$

where $\eta = \sqrt{\lambda^2 - k^2}$, $\operatorname{Re} \lambda > 0$, $k^2 = i\omega\mu\sigma^*$

Applying the Green's function in accordance with (6), we find the secondary electric field

$$E_x^s(y, z) = i\omega\mu \sum_{n=1}^{N} \int_{-L_n}^{L_n} dy_0 \int_{z_n}^{z_n + h_n} J_n(y_0 z_0) g(y - y_0, z, z_0)\, dz_0 \,, \tag{19}$$

where $J_n(y_0, z_0) = \left(\sigma^* - \sigma_n(y_0) \right) E_x(y_0, z_0)$ is the excess current in nonhomogeneous layer n. If the thin-layer condition holds

$$\omega\mu \max_{y_0} \sigma_n(y_0) h_n^2 << 1$$

we may assume that the electric current in layer n is independent of z and is given by $E_x^{(n)}(y)$ for $z \in [z_n, z_n + h_n]$, $|y| < L_n$. Then (19) may be written as

$$E_x(y,z) = i\omega\mu\sum_{n=1}^{N}\int_{-L_n}^{L_n} J_n(y_0)\cdot\tilde{g}_n(y-y_0,z)\,dy_0\,, \tag{20}$$

where

$$J_n(y_0) = \left(\sigma^* - \sigma_n(y_0)\right)E_x^{(n)}(y_0)\,, \tag{21}$$

$$\tilde{g}_n(y-y_0,z) = \int_{z_n}^{z_n+h_n} g(y-y_0,z,z_0)\,dz_0\,. \tag{22}$$

Representation (20) is used to prove the following lemma.

Lemma 1. *If at least one conductivity distribution* σ_n *in a layer changes by a finite amount, then at least one excess current* J_n *in the layer also changes.*

Proof. Since the boundary-value problem for the secondary field $E_x^s(y,z)$ has a unique solutioin, different fields correspond to different conductivity distributions, i.e., if $\sum_{n=1}^{N}\left\|\sigma^{(1)}(y)-\sigma^{(2)}(y)\right\|^2 > 0$ then $\left\|E_x^{(1)}(y,z)-E_x^{(2)}(y,z)\right\| > 0$. If we further assume that all $J_n^{(1)}(y) = J_n^{(2)}$, then by (20) we obtain $E_x^{(1)}(y,z) \equiv E_x^{(2)}(y,z)$. A contradiction. Hence, at least one $J_n^{(1)}$ is different from $J_n^{(2)}$.

Now consider the condition of the inverse problem when the field is given for $z=0$. By (20), we have

$$i\omega\mu\sum_{n=1}^{N}\int_{-L_n}^{L_n} J_n(y_0)\cdot\tilde{g}_n(y-y_0,z=0)\,dy_0 = E_x^s(y,z=0), \qquad y\in(-\infty,\infty) \tag{23}$$

and $J_n(y_0)=0$ for $|y|>L_n$.

Relationship (23) is the sum of convolution integrals. If we Fourier-transform (23) by y, then we obtain

$$i\omega\mu\sum_{n=1}^{N} S_J^{(n)}(\nu)S_g^{(n)}(\nu) = S_E(\nu), \tag{24}$$

where

$$S_J^{(n)}(\nu) = \int_{-L}^{L} J_n(y_0)e^{-i\nu y_0}\,dy_0\,; \qquad S_g^{(n)}(\nu) = \int_{-\infty}^{\infty}\tilde{g}_n(y,z=0)e^{-i\nu y}\,dy\,,$$

$$S_E^{(n)}(\nu) = \int_{-\infty}^{\infty} E_x^s(y,z=0)e^{-i\nu y}\,dy$$

since $E_x^s(y,z)$ and $\tilde{g}_n(y,z=0)$ decrease as $|y|\to\infty$, the spectral functions $S_g^{(n)}(\nu)$ and $S_E^{(n)}(\nu)$ exist. From (24) we obtain the following proposition.

Theorem 1. *A change of the excess currents* $J_n(y)$ *leads to a change of the electric field* $E_x(y, z=0)$.

Proof. We first determine the spectrum of the Green's function $S_g^{(n)}(\nu)$. Fourier-transforming $\tilde{g}_n(y, z=0)$ as defined by (22) and (18), we obtain

$$S_g^{(n)}(\nu) = \frac{1}{(\eta_1 + |\nu|)} \int_{z_n}^{z_n+h_n} e^{-\eta_1 z_0}\, dz_0 = \frac{e^{-\eta_1 z_n}(1 - e^{-\eta_1 h_n})}{\eta_1(\eta_1 + |\nu|)},$$

where $\eta_1 = \sqrt{\nu^2 - i\omega\mu\sigma^*}$, $\operatorname{Re}\eta_1 > \nu + \dfrac{(\omega\mu\sigma^*)^2}{8\nu^3}$. Then for large ν we obtain from (24)

$$\frac{i\omega\mu}{2\nu^2} \sum_{n=1}^{N} S_J^{(n)} e^{-\eta z_n} = S_E(\nu). \tag{25}$$

Since $z_{n+1} - z_n \geq h_n$, changing, say, $S_J^{(n)}$ with $n \in [k, N]$, we obtain

$$\|\Delta S_E\| = \frac{\omega\mu}{2\nu^2} \left\|\Delta S_J^k\right\| e^{-\eta z_k} + O\left(e^{-\eta(z_k+h_k)}\right)$$

Thus, with $\left\|\Delta S_J^{(k)}\right\| \neq 0$ we have $\|\Delta S_E\| \neq 0$.

Q.E.D.

From Lemma 1 and Theorem 1 we now obtain Theorem 2.

Theorem 2. *The inverse problem to find the conductivity* $\sigma_n(y)$ *of layer* $n \in [1, N]$ *given the electric field on the plane* $z = 0$ *with* E*-polarized primary plane-wave field has a unique solution.*

Proof. Assume that the same electric field in the plane $z = 0$, $E_x(y, z=0)$, corresponds to two different layer conductivity distributions $\sigma_n(y)$, $n \in [1, N]$. Then, by Lemma 1, the same excess current $J_n(y)$ corresponds to different $\sigma_n(y)$. Hence, by Theorem 1, the electric fields at $z = 0$ should be different. A contradiction. Thus, a unique distribution $\sigma_n(y)$ should correspond to the given $E_x(y, z)$.

Note that in our investigation of the inverse problem we have considered nonhomogeneous layers embedded in a homogeneous conducting half-space. The result is easily generalized if the homogeneous half-space is replaced with a layered half-space. In this case, we have to take the Green's function for a layered half-space. The spectrum of this Green's function is the same as in the case of a homogeneous half-space as $N \to \infty$.

Inverse Sounding Problem for a Nonhomogeneous Region

Consider the inverse sounding problem for the case when a region S with an arbitrary conductivity distribution $\sigma(y, z)$ is embedded in a homogeneous half-space with conductivity σ^*. The problem for the electric field reduces to solving an integral equation for the excess current in the nonhomogeneity

$$J(y,z) = \left(\sigma^* - \sigma(y,z)\right) E_x(y,z) \quad \text{for} \quad M = \{x,y\} \in S.$$

The anomalous field $E_x^s(y,z) = E_x(y,z) - E_x^0(z)$, where $E_x^0(z)$ is the primary field, is representable in the entire plane in the form

$$E_x^s(y,z) = i\omega\mu \int_S J(y,z_0) g(y-y_0, z, z_0)\, dy_0\, dz_0. \tag{26}$$

Representation (26) can be discretized by stratifying the region S into z-layers with uniform spacing h by the planes $z = z_n = h_1 + (n-1)h$, $n \in [1,N]$, where h_1 is the depth of the uppermost point of S. Setting $J(y,z)$ in the layer equal to the z-average value $J(y,z) = J_n$ for $z \in [z_n, z_{n+1}]$, we obtain

$$E_x^s(y,z) = i\omega\mu \sum_{n=1}^{N} \int_{-l_n}^{l_n} J_n(y_0)\, dy_0 \int_{z_n}^{z_{n+1}} g(y-y_0, z_0, z)\, dz_0, \tag{27}$$

here $y \in [-l_n, l_n]$ is the interval in layer n where $J_n(y)$ does not vanish. For $z = 0$ we obtain from (27) the condition for the inverse problem

$$i\omega\mu \sum_{n=1}^{N} \int_{-l_n}^{l_n} J_n(y_0) \tilde{g}_n(y-y_0)\, dy_0 = E_x^s(y, z=0), \tag{28}$$

where

$$\tilde{g}_n(y-y_0) = \int_{z_n}^{z_{n+1}} g(y-y_0, z_0, z=0)\, dz_0$$

Applying representation (18), we find

$$\tilde{g}_n(y-y_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda(y-y_0) - \eta z_n} \left(1 - e^{-\eta h}\right) \frac{d\lambda}{\eta(\eta + |\lambda|)}. \tag{29}$$

Theorem 3. *The two-dimensional inverse problem to find the conductivity distribution* $\sigma(y,z)$ *inside the local nonhomogeneity* S *given the electric field* $E_x^s(y, z=0)$ *has a unique solution.*

Proof. We have to show that different anomalous electric fields on the plane $z = 0$ correspond to different conductivity distributions $\sigma^{(1)}(y,z)$ and $\sigma^{(2)}(y,z)$. From (27) we obtain that for different $\sigma^{(1)}(y,z)$ and $\sigma^{(2)}(y,z)$ the excess currents $J_n^{(1)}(y)$ and $J_n^{(2)}$ should be different in at least one of the layers partitioning the nonhomogeneity S. This assertion follows from the unique solvability of the forward problem, according to which different anomalous fields $E_x^s(y,z)$ correspond to different conductivity distribution in S. By (27), at least one $J_n^{(2)}(y)$ is not equal to $J_n^{(1)}(y)$ in this case.

We will now show that if at least one $J_n^{(2)}(y)$ is not equal to $J_n^{(1)}(y)$, then the anomalous fields are also not equal at $z=0$, $E_x^{s1}(y,z=0) \neq E_x^{s2}(y,z=0)$. To this end we Fourier-transform relationship (24). Since $J_n(y)=0$ for $|y|>l_n$, we have

$$i\omega\mu \sum_{n=1}^{N} S_J^{(n)}(\nu) S_g^{(n)}(\nu) = S_E(\nu). \tag{30}$$

This relationship is similar to (24). From (29) we obtain

$$S_g^{(n)} = \frac{e^{-\eta_1 z_n}}{\eta_1(\eta_1+|\nu|)}(1-e^{-\eta_1 h}), \qquad \eta_1 = \sqrt{\nu^2 - i\omega\mu\sigma^*}, \tag{31}$$

where $z_n = h_1 + (n-1)h$, $n \in [1, N+1]$.

We have previously shown that for different $\sigma^{(1)}(y,z)$ and $\sigma^{(2)}(y,z)$ at least one $J_n^{(2)}(y)$ is not equal to $J_n^{(1)}(y)$, $n \in [1,N]$. Thus, at least one Fourier transform $S_J^{(n)2}$ is not equal to $S_J^{(n)1}$. Substituting (31) in (30), we obtain

$$\frac{i\omega\mu e^{-\eta_1 h_1}(1-e^{-\eta_1 h_1})}{\eta_1(\eta_1+|\nu|)} \sum_{n=1}^{N} S_J^{(n)}(\nu) e^{-\eta_1 (n-1)h}) = S_E(\nu). \tag{32}$$

Note that $\operatorname{Re}\eta_1 = \sqrt{\dfrac{\nu + \nu\sqrt{\nu^2 + (\omega\mu\sigma^*)^2}}{2}} > \varepsilon > 0$. Then from (32), it follows that different $S_E(\nu)$ correspond to different $S_J^{(n)}$. Assume that the first $S_J^{(n)}$, $n \in [1, k-1]$ are equal for different $\sigma(y,z)$, and $S_J^{(k)2} \neq S_J^{(k)1}$. Then there exists ν_0 such that for $\nu \geq \nu_0$ we have $S_E^{(2)} \neq S_E^{(1)}$. Since the Fourier transforms are unequal, the fields at $z=0$ are also unequal. We have thus shown that different fields $E_x^s(y,z=0)$ on the plane $z=0$ correspond to different distributions, with $M(x,y) \in S$. This implies that equal $\sigma(y,z)$ correspond to equal fields $E_x^s(y,z=0)$.

Q.E.D.

Inverse Sounding Problem in a Nonhomogeneous Region with *H*-Polarized Field

Consider an H-polarized electromagnetic field in the half-space $z>0$ with conductivity σ^* that includes a nonhomogeneity with a piecewise-continuous conductivity distribution $\sigma(y,z)$. The nonhomogeneity $M(y,z) \in S_H$ is local in the sense that $y \in [-L, L]$, $z \in [h_1, H]$ for $M(y,z) \in S_H$. The forward problem involves the determination of $E_y(y,z)$, $E_z(y,z)$, which are the solutions of Eqs. (9) with boundary conditions (10). The primary field, in this case, is given by

$$E_y^0(z) = \sqrt{\frac{i\omega\mu}{\sigma^*}} e^{-\sqrt{i\omega\mu\sigma^*} z}, \qquad E_z = 0. \tag{33}$$

We have $H_x(y, z=0) = 1$. Introduce a secondary (anomalous) field induced by the nonhomogeneity in the medium,

$$E_y^s(y,z) = E_y(y,z) - E_y^0(z), \qquad E_z^s(y,z) = E_z(y,z). \tag{34}$$

Then for the secondary field, we obtain the following problem: for $z > 0$,

$$\begin{cases} \dfrac{\partial}{\partial z}\left(\dfrac{\partial E_z^s}{\partial y} - \dfrac{\partial E_y^s}{\partial z}\right) = i\omega\mu\left(\sigma(y,z) - \sigma^*\right)(E_y^0 + E_y^s) + i\omega\mu\sigma^* E_y^s, \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial E_y^s}{\partial z} - \dfrac{\partial E_z}{\partial y}\right) = i\omega\mu(\sigma - \sigma^*)E_z^s + i\omega\mu\sigma^* E_z^s, \end{cases} \tag{35}$$

with the following boundary conditions at $z = 0$:

$$E_z^s(y, z=0) = 0, \qquad \frac{\partial E_y^s(y, z=0)}{\partial z} = 0. \tag{36}$$

The solution of system (35) is expressed in terms of the tensor Green's function in the form

$$\bar{E}^s(y,z) = i\omega\mu\int_S \hat{g}\left((y - y_0), z, z_0\right)\bar{J}(y_0, z_0)\,dy_0\,dz_0, \tag{37}$$

where

$$\bar{E}^s = \begin{pmatrix} E_y^s \\ E_z^s \end{pmatrix}, \quad \bar{J} = \begin{pmatrix} J_y \\ J_z \end{pmatrix}, \quad J_y = (\sigma - \sigma^*)(E_y^s + E_y^0), \quad J_z = (\sigma - \sigma^*)E_z^s,$$

are the excess currents in the nonhomogeneity.

The tensor Green's function

$$\hat{g}(y - y_0, z, z_0) = \begin{pmatrix} g_{yy} & g_{yz} \\ g_{zy} & g_{zz} \end{pmatrix}$$

is a Fourier transform of a matrix spectral function

$$\hat{g}(y - y_0, z, z_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(y - y_0)}\hat{S}(\lambda, z, z_0)\,d\lambda. \tag{38}$$

For a homogeneous half-space, the matrix spectral function has the form

$$S_{yy} = \frac{\eta}{2k^2}\left(e^{-\eta|z-z_0|} + e^{-\eta(z+z_0)}\right), \qquad \eta = \sqrt{\lambda^2 - i\omega\mu\sigma^*}\,,$$

$$S_{zy} = \frac{i\lambda}{2k^2}\left(\frac{z-z_0}{|z-z_0|}e^{-\eta|z-z_0|} + e^{-\eta(z+z_0)}\right),$$

$$S_{yz} = \frac{1}{2i\lambda}\left(\frac{z-z_0}{|z-z_0|}e^{-\eta|z-z_0|} - e^{-\eta(z+z_0)}\right),$$

$$S_{zz} = \frac{1}{2\eta}\left(\frac{z-z_0}{|z-z_0|}e^{-\eta|z-z_0|} + e^{-\eta(z+z_0)}\right). \tag{39}$$

From the representation of the secondary field (37), it follows that different excess currents $\bar{J}(y,z)$ correspond to different conductivity distributions in the nonhomogeneity. This assertion follows from the uniqueness theorem for the forward problem, because equal excess currents generate, by (37), equal secondary fields, which is impossible with different conductivities $\sigma(y,z)$. To prove uniqueness for the inverse problem, it remains to show that different electric fields on the plane $z=0$ correspond to different excess currents. To this end, we pass to a discrete model of the nonhomogeneity, partitioning it into layers of thickness h and assuming that inside each layer the conductivity varies only longitudinally. If $\omega\mu\sigma_{\max}h^2 << 1$, where $\sigma_{\max}$ is the maximum conductivity inside the nonhomogeneity, then the electric field may be regarded as independent of z inside the layer. Under these assumptions, the excess current in layer n, $\bar{J}^{(n)}(y)$, depends only on y. Then representation (37) may be written as

$$\bar{E}(y,z) = i\omega\mu\sum_{n=1}^{N}\int_{-l_n}^{l_n}\hat{g}^{(n)}(y-y_0,z)\bar{J}^{(n)}(y_0)\,dy_0\,, \tag{40}$$

where

$$\hat{g}^{(n)}(y-y_0,z) = \int_{z_n}^{z_n+h}\hat{g}(y-y_0,z,z_0)\,dz_0\,, \qquad z_n = h_1 + (n-1)h\,. \tag{41}$$

Now consider the electric field on the plane $z=0$, where $E_z^s = 0$, $E_y^s(y,z=0)$ is a known field. By (38)–(39), $g_{zy}(z=0)=0$ and $g_{zz}(z=0)=0$, and so the field $E_y^s(y,z=0)$ is expressed in terms of the excess currents by (40) in the form

$$E_y^s(y,z=0) = i\omega\mu\sum_{n=1}^{N}\int_{-\infty}^{\infty}g_{yy}^{(n)}(y-y_0)J_y^{(n)}(y_0) + g_{yz}(y-y_0)\bar{J}_z^{(n)}(y_0)\,dy_0\,, \tag{42}$$

where we have used the identity $\bar{J}^{(n)}(y) \equiv 0$ for $|y| > l$.

Fourier-transforming (42), we obtain the following equality for the Fourier transforms:

$$S_E(\nu) = i\omega\mu \sum_{n=1}^{N} \left(S_{yy}^{(n)}(\nu) S_{J_y}^{(n)}(\nu) + S_{yz}^{(n)}(\nu) S_{J_z}^{(n)}(\nu) \right). \tag{43}$$

Different Fourier transforms $S_{J_y}^{(n)}(\nu)$ and $S_{J_z}^{(n)}(\nu)$ correspond to different $J_y^{(n)}(y)$ and $J_z^{(n)}(y)$. We have to show that this leads to different Fourier transforms for the field $E_y^s(y)$.

Consider the behavior of $S_{yy}^{(n)}(\nu)$ and $S_{yz}^{(n)}(\nu)$ for large ν:

$$\begin{aligned} S_{yy}^{(n)}(\nu) &= \frac{1}{k^2} e^{-\eta z_n}(1 - e^{-\eta h}), \quad k^2 = i\omega\mu\sigma^*, \\ S_{yz}^{(n)}(\nu) &= \frac{i}{\nu\eta} e^{-\eta z_n}(1 - e^{-\eta h}), \quad \eta = \sqrt{\lambda^2 - i\omega\mu\sigma^*}. \end{aligned} \tag{44}$$

Substituting (44) in (43) and noting that $z_n = h_1 + (n-1)h$, we obtain

$$S_E(\nu) = \frac{1}{k^2} e^{-\eta h_1}(1 - e^{-\eta h}) \sum_{n=1}^{N} \left(S_{J_y}^{(n)}(\nu) - \frac{\omega\mu\sigma^*}{\nu\eta} S_{J_z}^{(n)}(\nu) \right) e^{-\eta(n-1)h}, \tag{45}$$

From (45), we see that different $S_E(\nu)$ correspond to different $Q^{(n)} = S_{J_y}^{(n)}(\nu) - \dfrac{\omega\mu\sigma^*}{\nu\eta} S_{J_z}^{(n)}$ (this follows from the behavior as $\nu \to \infty$), and different $Q^{(n)}(\nu)$ correspond to different S_{J_y} and S_{J_z} (because $Q^{(n)}(\nu) \to S_{J(y)}^{(n)}(\nu)$ and $\left(Q^{(n)} - S_{J_y}^{(n)}(\nu) \right) \nu\eta \to S_{J_z}^{(n)} \omega\mu\sigma^*$ as $\nu \to \infty$).

We have thus proved that different electric fields $E_y^s(y, z = 0)$ correspond to different conductivity distributions $\sigma(y, z)$ in the nonhomogeneity. This means that a unique distribution $\sigma(y, z)$ corresponds to a given field $E_y^s(y, z = 0)$. In our proof, we have used the assumption that $\sigma(y, z)$ is piecewise-constant in z and piecewise-continuous in y. In other words,

$$\sigma(y, z) = \sigma_n(y) \quad \text{for} \quad z \in \left(h_1 + (n-1)h, h_1 + nh \right),$$

and $\sigma_n(y)$ is a piecewise-continuous function such that $\sigma_n(y) \equiv 0$ for $|y| > l_n$, $n \in [1, N]$, where l_n is a finite magnitude.

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