DOI 10.1007/s10598-018-9395-4

Computational Mathematics and Modeling, Vol. 29, No. 2, April, 2018

I. MATHEMATICAL MODELING

MAGNETOTELLURIC SOUNDING OF A LAYERED MEDIUM CONTAINING THIN NONHOMOGENEOUS LAYERS

V. I. Dmitriev

UDC 517.958

The article investigates the inverse sounding problem for a nonhomogeneous thin layer given the field on the surface of the half-space. A uniqueness theorem is proved for the solution of the inverse problem.

Keywords: differential equations, integral equations, electromagnetic sounding, inverse problems.

Introduction

The method of [1, 2] relies on the investigation of the forward and the inverse problem of magnetotelluric sounding (MTS) of layered media. In this setting, the forward problem reduces to solving the Riccati equation with the right-hand side equal to the distribution of the electrical conductivity over depth $\sigma(z)$. For the inverse problem, where $\sigma(z)$ is determined from the given frequency characteristic of the magnetotelluric field, the solution is unique if $\sigma(z)$ is piecewise-analytical [3]. The inverse problem is solved as an ill-posed problem [2] using the regularization method [4]. We may assume that the solution of the homogeneous inverse MTS problem has been fully developed.

Further application of the magnetotelluric sounding method requires solving more complex three-dimensional sounding problems for nonhomogeneous media. In this setting, the basic model of the medium is a nonhomogeneous three-dimensional zone with conductivity $\sigma(x, y, z)$ embedded in a layered medium. This model leads to certain difficulties even for the forward problem. When the forward problem is solved by the integral equation method, we obtain a linear algebraic system with more than 10^6 unknowns. When solving by the finite-element method or the finite-difference method, the dimension of the system is usually increased by a factor of 100. Three-dimensional forward problems thus have to solved by parallel computation schemes on supercomputers.

The solution of the inverse problem dramatically raises the resource requirements, as the forward problem has to be solved repeatedly. The difficulties are additionally exacerbated by the instability of the inverse problem, as instability grows with the increase in the number of unknowns.

It is therefore relevant to develop structural models of the medium that permit applying simpler and more efficient solution methods. Such models include, in particular, the model of a layered medium with thin nonhomogeneous layers. In what follows, we examine the solution of forward and inverse problems for such structural models.

Faculty of Computational Mathematics and Cybernetics, Moscow State University, Moscow, Russia; e-mail: dmitriev@cs.msu.ru.

Translated from Prikladnaya Matematika i Informatika, No. 55, 2017, pp. 5–11.

1. Statement of the Problem

Consider a layered medium in which conductivity is a function of depth only:

$$\sigma(z) = \begin{cases} \sigma_0 & \text{for} \quad z < 0, \\ \sigma_1(z) & \text{for} \quad z \in (0, h), \\ \sigma_c & \text{for} \quad z \in (h, H), \\ \sigma_2(z) & \text{for} \quad z > H. \end{cases}$$

The layer $z \in (h, H)$ contains a nonhomogeneous zone with conductivity

$$\sigma_c(M) = \begin{cases} \sigma_H(x, y) & \text{for} \quad M \in V_H, \\ \sigma_c = \text{const} & \text{for} \quad M \notin V_H. \end{cases}$$

The conductivity in the layer $z \in (h, H)$ is independent of z. This implies that the thickness of this layer is much less than the wavelength in the layer:

$$(H - h)\sqrt{\omega\mu\sigma_m} \ll 1, \quad \text{where} \quad \sigma_m = \max \sigma_H(x, y).$$

Furthermore, the thickness of the layer is much less than its depth below the surface, i.e., $(H - h) \ll h$. Under these conditions, we may assume that the electric field $\boldsymbol{E}$ in the layer (h, H) is virtually independent of z, i.e.,

$$\frac{\partial \boldsymbol{E}(x, y, z)}{\partial z} \approx 0 \quad \text{for} \quad z \in (h, H).$$

This property significantly simplifies the solution of the problem.

The forward MTS problem involves determining the electric and magnetic fields $\boldsymbol{E}(M)$, $\boldsymbol{H}(M)$, which satisfy the Maxwell's equations

$$\operatorname{rot} \boldsymbol{H} = \sigma(M)\boldsymbol{E}; \qquad \operatorname{rot} \boldsymbol{E} = i\omega\mu\boldsymbol{H}. \tag{1}$$

The tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ should be continuous on the discontinuity boundaries of the conductivity $\sigma(M)$. At infinity, we should have the radiation condition for $\boldsymbol{E} - \boldsymbol{E}^0$ and $\boldsymbol{H} - \boldsymbol{H}^0$, where $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ is the primary field of the plane wave normally incident on the layered medium. The primary field $\boldsymbol{E}^0(z)$ and $\boldsymbol{H}^0(z)$ is the solution of the Maxwell equations

$$\operatorname{rot} \boldsymbol{H}^0 = \sigma(z)\boldsymbol{E}^0, \qquad \operatorname{rot} \boldsymbol{E}^0 = i\omega\mu\boldsymbol{H}^0 \tag{2}$$

and is easily evaluated for a piecewise-continuous conductivity $\sigma(z)$.

The magnetotelluric field is representable as the sum of primary and anomalous fields:

$$\boldsymbol{E}(M) = \boldsymbol{E}^0(z) + \boldsymbol{E}^a(M); \qquad \boldsymbol{H}(M) = \boldsymbol{H}^0(z) + \boldsymbol{H}^a(M).$$

The anomalous fields satisfy the Maxwell equations

$$\operatorname{rot} \boldsymbol{H}^a = \sigma(z)\boldsymbol{E}^a + \boldsymbol{J}_s, \qquad \operatorname{rot} \boldsymbol{E}^a = i\omega\mu\boldsymbol{H}^a, \tag{3}$$

where $\boldsymbol{J}_s = \left(\sigma(M) - \sigma(z)\right)\boldsymbol{E}$ is the surplus current generated in the nonhomogeneity.

2. Lorentz Lemma

The Lorentz lemma links the fields of two distinct sources in the same medium. Consider the Maxwell equations for two distinct sources $\boldsymbol{j}^{(1)}$ and $\boldsymbol{j}^{(2)}$ inside the body V_0 bounded by the surface S_0, with complex electrical conductivity $\sigma(M)$ and magnetic permeability μ.

Consider the expression

$$W = \operatorname{div}\left(\left[\boldsymbol{E}^{(1)} \times \boldsymbol{H}^{(2)}\right] - \left[\boldsymbol{E}^{(2)} \times \boldsymbol{H}^{(1)}\right]\right)$$

$$= \boldsymbol{H}^{(2)} \operatorname{rot} \boldsymbol{E}^{(1)} - \boldsymbol{E}^{(1)} \operatorname{rot} \boldsymbol{H}^{(2)} - \boldsymbol{H}^{(1)} \operatorname{rot} \boldsymbol{E}^{(2)} + \boldsymbol{E}^{(2)} \operatorname{rot} \boldsymbol{H}^{(1)}. \tag{4}$$

Substituting the curls from Maxwell equations (2), we obtain a differential form of the Lorentz lemma:

$$\operatorname{div}\left(\left[\boldsymbol{E}^{(1)} \times \boldsymbol{H}^{(2)}\right] - \left[\boldsymbol{E}^{(2)} \times \boldsymbol{H}^{(1)}\right]\right) = \boldsymbol{E}^{(2)} \cdot \boldsymbol{j}^{(1)} - \boldsymbol{E}^{(1)} \cdot \boldsymbol{j}^{(2)}. \tag{5}$$

Applying to (5) the Gauss divergence formula, we obtain an integral form of the Lorentz lemma:

$$\int\limits_{S_0} \left(\left[\boldsymbol{E}^{(1)} \times \boldsymbol{H}^{(2)}\right] - \left[\boldsymbol{E}^{(2)} \times \boldsymbol{H}^{(1)}\right]\right) \cdot \boldsymbol{n}\, ds = \int\limits_{V_0} \left(\boldsymbol{E}^{(2)} \cdot \boldsymbol{j}^{(1)} - \boldsymbol{E}^{(1)} \cdot \boldsymbol{j}^{(2)}\right) dv, \tag{6}$$

where $\boldsymbol{n}$ is the outer normal to the surface S_0.

The Lorentz lemma (6) has been derived for a region V_0 bounded by the surface S_0. It can be easily generalized to an infinite layered medium with piecewise-constant conductivity $\sigma(z)$. To this end, we apply the lemma (6) to each layer and add up the resulting expressions. The integrals over the layer boundaries cancel out. The boundary in the upper and the lower half-spaces is taken as a sphere S_R of radius $R \to \infty$. The radiation conditions for the electromagnetic field yield

$$\lim_{R \to \infty} \int\limits_{S_R} \left(\left[\boldsymbol{E}^{(1)} \times \boldsymbol{H}^{(2)}\right] - \left[\boldsymbol{E}^{(2)} \times \boldsymbol{H}^{(1)}\right]\right) \cdot \boldsymbol{n}\, ds = 0.$$

All the boundary integrals in the Lorentz lemma vanish and we obtain an integral identity

$$\int\limits_V \boldsymbol{E}^{(1)}(M)\cdot \boldsymbol{j}^{(2)}(M)\,dv_M = \int\limits_V \boldsymbol{E}^{(2)}(M)\cdot \boldsymbol{j}^{(1)}(M)\,dv_M , \tag{7}$$

where V is the entire infinite layered space.

3. Derivation of the Integral Equation

The Lorentz lemma (7) easily reduces the forward MTS problem (3) to an integral equation. To this end, set in (7)

$$\boldsymbol{E}^{(1)} = \boldsymbol{E}^a ,\ \ \boldsymbol{j}^{(1)} = \boldsymbol{J}_s$$

and for $\boldsymbol{E}^{(2)}$ take the field $\boldsymbol{E}^{(P)}$ of a point unit dipole $\boldsymbol{j}^{(2)} = \boldsymbol{p}\,\delta\left(r_{MM_0}\right)$, where $\boldsymbol{p}$ is an arbitrary vector, $\delta\left(r_{MM_0}\right)$ a three-dimensional Dirac function. Then

$$\boldsymbol{p}\cdot\boldsymbol{E}^a(M_0) = \int\limits_V \boldsymbol{E}^{(P)}(M,M_0)\cdot \boldsymbol{J}_s(M)\,dv_M . \tag{8}$$

The auxiliary fields $\boldsymbol{E}^{(x)}(M,M_0)$, $\boldsymbol{E}^{(y)}(M,M_0)$, $\boldsymbol{E}^{(z)}(M,M_0)$ for $\boldsymbol{p}_1 = (1,0,0)$, $\boldsymbol{p}_2 = (0,1,0)$, $\boldsymbol{p}_3 = (0,0,1)$ form the Green's electric tensor for the Maxwell equations:

$$\hat{\boldsymbol{G}}_E(M,M_0) = \left(\boldsymbol{E}^{(x)}(M,M_0),\boldsymbol{E}^{(y)}(M,M_0),\boldsymbol{E}^{(z)}(M,M_0)\right) \tag{9}$$

Using the Green's electric tensor, we express by (8) the anomalous electric field at any point of the space in terms of the surplus current in the nonhomogeneity

$$\boldsymbol{E}^a(M) = \int\limits_{V_H} \hat{\boldsymbol{G}}_E(M,M_0)\cdot \boldsymbol{J}_s(M_0)\,dv_{M_0} , \tag{10}$$

while the anomalous magnetic field is expressed by (3) as

$$\boldsymbol{H}^a(M) = \frac{1}{i\omega\mu}\,\mathrm{rot}\int\limits_{V_H} \hat{\boldsymbol{G}}_E(M,M_0)\cdot \boldsymbol{J}_s(M_0)\,dv_{M_0} . \tag{11}$$

Multiplying (10) by $\delta\sigma = \left(\sigma_H(x,y) - \sigma_c\right)$, we obtain an integral equation for the surplus current

$$\boldsymbol{J}_s(M) - \delta\sigma\int\limits_{V_H} \hat{\boldsymbol{G}}_E(M,M_0)\cdot \boldsymbol{J}_s(M_0)\,dv_{M_0} = \delta\sigma\boldsymbol{E}^0(M_0), \qquad M\in V_H . \tag{12}$$

Determining the surplus current $\boldsymbol{J}_s(M)$ from the integral equation (12), we apply (10)-(11) to find the electromagnetic field at every point in the space.

4. Solution of the Integral Equation

Since σ_H is independent of z and the electric field in the thin layer is also virtually independent of z, the solution of the integral equation $\boldsymbol{J}_s$ is independent of z. Then Eq. (12) may be written as

$$\boldsymbol{J}_s(x,y) - \delta\sigma_c(x,y) \int_{S_H} \hat{\boldsymbol{K}}(x-x_0, y-y_0) \cdot \boldsymbol{J}_s(x_0,y_0)\, dx_0\, dy_0$$

$$= \delta\sigma_c(x,y)\, \boldsymbol{E}^0(x,y), \qquad (x,y) \in S_H, \tag{13}$$

where S_H is the nonhomogeneity region (section V_H), and

$$\hat{\boldsymbol{K}}(x-x_0, y-y_0) = \int_h^H \hat{\boldsymbol{G}}(x-x_0, y-y_0, z=z_c, z_0)\, dz_0, \qquad \text{where} \qquad z_c = \frac{H+h}{2}. \tag{14}$$

We have thus obtained a two-dimensional integral equation with a kernel that depends on a difference of arguments. This essentially simplifies the solution.

From (13) find the surplus current $\boldsymbol{J}_s(x,y)$ and then apply (10)–(11) to find the anomalous electromagnetic field on the Earth's surface at $z=0$:

$$\boldsymbol{E}^a(x,y,z=0) = \int_{S_H} \hat{\boldsymbol{K}}_E(x-x_0, y-y_0) \cdot \boldsymbol{J}_s(x_0,y_0)\, dx_0\, dy_0, \tag{15}$$

$$\boldsymbol{H}^a(x,y,z=0) = \int_{S_H} \hat{\boldsymbol{K}}_H(x-x_0, y-y_0) \cdot \boldsymbol{J}_s(x_0,y_0)\, dx_0\, dy_0, \tag{16}$$

where

$$\hat{\boldsymbol{K}}_E(x-x_0, y-y_0) = \int_h^H \hat{\boldsymbol{G}}(x-x_0, y-y_0, z=0, z_0)\, dz_0, \tag{17}$$

$$\hat{\boldsymbol{K}}_H(x-x_0, y-y_0) = \int_h^H \frac{1}{i\omega\mu} \operatorname{rot} \hat{\boldsymbol{G}}(x-x_0, y-y_0, z=0, z_0)\, dz_0. \tag{18}$$

5. The Inverse MTS Problem

The inverse MTS problem involves determining the distribution of the conductivity $\sigma(M)$ in the half-space $z>0$ given the impedance $\hat{\boldsymbol{Z}}$ at $z=0$ as a function of the coordinates x, y and the frequency ω, where the impedance tensor $\hat{\boldsymbol{Z}}$ relates the electromagnetic fields on the Earth's surface $z = 0$:

$$\boldsymbol{E}_\tau(x,y) = \hat{\boldsymbol{Z}}(x,y)\boldsymbol{H}_\tau(x,y), \tag{19}$$

where

$$\hat{\boldsymbol{Z}} = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix}, \quad \boldsymbol{E}_\tau = \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad \boldsymbol{H}_\tau = \begin{pmatrix} H_x \\ H_y \end{pmatrix}.$$

The fields $\boldsymbol{E}_\tau$, $\boldsymbol{H}_\tau$ depend on the field frequency ω and the impedance tensor $\hat{\boldsymbol{Z}}$ thus also depends on ω. As we move away from the nonhomogeneity, the impedance tensor goes to the impedance tensor for a layered medium, i.e.,

$$\hat{\boldsymbol{Z}}(x,y,\omega) \to \hat{\boldsymbol{Z}}^0(\omega) = \begin{pmatrix} 0 & Z_0 \\ -Z_0 & 0 \end{pmatrix} \quad \text{as} \quad \sqrt{x^2+y^2} \to \infty, \tag{20}$$

where $Z_0(\omega)$ is the scalar impedance of the layered medium.

By Tikhonov's theorem [3], the impedance frequency characteristic $Z_0(\omega)$ uniquely determines the conductivity of the layered medium $\sigma(z)$. Various stable methods have been developed for solving the one-dimensional inverse MTS problem; they all rely on regularization of unstable (ill-posed) problems [4]. Therefore, in our further studies of the inverse MTS problem for a nonhomogeneous thin layer we start with the assumption that the background conductivity $\sigma(z)$ of the layered medium is known.

For the anomalous fields $\boldsymbol{E}^a = \boldsymbol{E} - \boldsymbol{E}^0$, $\boldsymbol{H}^a = \boldsymbol{H} - \boldsymbol{H}^0$, noting that $\boldsymbol{E}_\tau^0 = \hat{\boldsymbol{Z}}^0\boldsymbol{H}_\tau^0$, we write boundary condition (19) in the form

$$\boldsymbol{E}_\tau^a = \hat{\boldsymbol{Z}}^0\boldsymbol{H}_\tau^0 + \hat{\boldsymbol{Z}}^a\left(\boldsymbol{H}_\tau^0 + \boldsymbol{H}_\tau^a\right), \quad \hat{\boldsymbol{Z}}^a = \hat{\boldsymbol{Z}} - \hat{\boldsymbol{Z}}^0. \tag{21}$$

Here $\hat{\boldsymbol{Z}}^a(x,y) \to 0$ as $\sqrt{x^2+y^2} \to \infty$.

The inverse problem with a given impedance can be reduced to an inverse problem with a given electromagnetic field on the Earth's surface $(z = 0)$. This transformation simplifies the analysis and solution of the inverse problem.

To pass to the inverse problem in this new form, we have to determine the field at $z = 0$ for a given impedance. The problem reduces to determining the field of a plane wave normally incident on a plane with the impedance boundary condition (21). This problem is easily solved by the integral equation method, which produces the tangential field components on the Earth's surface.

6. Unique Solvability of the Inverse Problem

The forward MTS problem requires solving integral equation (13) given the thin-layer conductivity distribution $\delta\sigma_c(x,y)$ and evaluating the anomalous electromagnetic field on the Earth's surface from (15)–(16) given the previously determined surplus current.

The inverse thin-layer MTS problem requires determining $\delta\sigma_c(x,y)$ using additional information (the electromagnetic field observed on the Earth's surface). Note that, given the surplus field, we can uniquely determine the distribution of $\delta\sigma_c$. To prove this assertion, we write the electric field from integral equation (13) in the form

$$\boldsymbol{E}(x,y) = \boldsymbol{E}^0(x,y) + \int\limits_{S_H} \hat{\boldsymbol{K}}(x-x_0, y-y_0)\cdot \boldsymbol{J}_s(x_0,y_0)\,dx_0\,dy_0. \quad (22)$$

The surplus current $\boldsymbol{J}_s$ equals

$$\boldsymbol{J}_s(x,y) = \delta\sigma_c(x,y)\,\boldsymbol{E}(x,y) \quad (23)$$

Given $\boldsymbol{J}_s(x,y)$, expressions (21)–(22) uniquely determine $\delta\sigma_c(x,y)$. The inverse problem thus reduces to finding $\boldsymbol{J}_s(x,y)$ given the surface field. The surface fields are linked with the surplus current by (13)–(14). These relationships with given fields are integral equations of the first kind with a kernel that depends on the difference of the arguments. A unique solution of these equations exists if a Fourier spectrum of the given anomalous field exists. For a local nonhomogeneity $\delta\sigma_c(x,y)$, the surface anomalous field is a function with compact support and its Fourier spectrum therefore exists. Thus, the surplus current $\boldsymbol{J}_s(x,y)$ is uniquely determined from (13)–(14), which, as noted above, uniquely produces the thin-layer conductivity $\delta\sigma_c(x,y)$.

REFERENCES

1. M. N. Berdichevskii and V. I. Dmitriev, *Magnetotelluric Sounding of Horizontally Nonhomogeneous Media* [in Russian], Nedra, Moscow (1992).
2. M. N. Berdichevsky and V. I. Dmitriev, *Magnetotellurics in the Context of the Theory of Ill-Posed Problems,* Society of Exploration Geophysicists, USA (2002).
3. A. N. Tikhonov, "Mathematical foundation of the electromagnetic sounding theory," *Zh. Vychisl. Matem. i Mat. Fiz.,* **5**, No. 3, 545–547 (1965).
4. A. N. Tikhonov and V. Ya. Arsenin, *Methods for Solving Ill-Posed Problems* [in Russian], Nauka, Moscow (1974).
5. M. N. Berdichevskii and V. I. Dmitriev, *Models and Methods in Magnetotellurics* [in Russian], Nauchnyi Mir, Moscow (2009).
6. V. I. Dmitriev and E. V. Zakharov, *Integral Equation Method in Computational Electrodynamics* [in Russian], MAKS Press, Moscow (2008).