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II. MATHEMATICAL MODELING

NUMERICAL SOLUTION OF THE INVERSE SCATTERING PROBLEM FOR THE ACOUSTIC EQUATION IN AN ABSORPTIVE LAYERED MEDIUM

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We consider a nonlinear ordinary differential equation associated with a number of inverse scattering problems in acoustic and seismic sounding in which acoustic impedance and an impedance-dependent unknown damping coefficient are the unknown function. We prove that the Cauchy problem is uniquely solvable when the derivative is treated as a generalized function. It is established that the inverse scattering problem in a layered dissipative medium simultaneously determines the acoustic impedance and the damping coefficient. A regularized numerical algorithm is proposed and numerical results are reported.

Keywords: equations of acoustics, dissipative medium, acoustic impedance, Dirac system, generalized function, regularization method.

Various applied problems [1–8, 15–17, 25–26], including inverse scattering problems, require solving a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \varphi(y(x)) = f(x), \quad x \in [0, a], \tag{1}$$

where $y(x)$ is the sought solution, which in general is not continuously differentiable, $\varphi(y)$ is continuous on $(-\infty, \infty)$, $f(x)$ is a given right-hand side. Equations of this kind are typical for seismic data processing, when the function $y(x)$ describes the sought elasto-density properties of the layered medium and $f(x)$ are the seismic oscillations recorded by the instruments (the seismogram) inside the borehole or on the daylight surface.

The independent variable x in this case is the spatial variable with the dimension of time, because it stands for the eikonal, i.e., the propagation time of the seismic signal from the current point to the daylight surface. The functional term in the left-hand side of (1) characterizes the dissipative properties of the medium and in most mathematical models is specified a priory [5–8].

In the general formulation, the problem of simultaneous determination of the elasto-density and dissipative properties of a layered medium, i.e., the functions $y(x)$ and $\varphi(y)$, clearly does not have a unique solution in the framework of the Cauchy problem for (1). However, the nature of the geological origin of the upper strata of the Earth's crust makes it possible to propose a reasonable mathematical specification of such a problem. The Earth's crust consists of homogeneous layers of various minerals with their thickness and stiffness appearing significantly different in seismic observations. As a result, seismic sounding usually focuses on finding layer boundaries and the coefficients of reflection from the boundaries, which corresponds to the determination of discontinuities of the function $y(x)$ or, equivalently, the constancy intervals of this function.

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There is another weighty reason to abandon the classical view of (1) as an equation that holds in all points of $[0, a]$. This is so because the recording and processing of observations, and also the construction of the numerical algorithms, typically employ a discrete approach. The discretization interval is identified with a thin layer and the reflection coefficient at the layer boundary is defined as the difference derivative. An attempt to determine simultaneously the grid functions corresponding to $y(x)$ and $\varphi(y)$ obviously does not produce a meaningful result.

The functions $y(x)$ and $\varphi(y)$ cannot be determined simultaneously because in the classical definition the derivative is the limit of a ratio of differences. Indeed, let the function $y(x)$ be of bounded variation on $[0, a]$ and its derivative is zero almost everywhere [9], i.e., $\mathrm{d}y/\mathrm{d}x = 0$ a.e. on $[0, a]$. Then

$$\varphi(y(x)) = f(x) \quad \text{a.e. on} \quad [0, a],$$

which certainly does not allow unique determination of the sought functions. Thus, the examination of (1) on the class of bounded-variation functions with the classical definition of the derivative does not solve the problem of simultaneous determination of the functions $y(x)$ and $\varphi(y)$.

The goal of this study is to pose and solve the Cauchy problem for Eq. (1) in the framework of the theory of generalized functions (distributions) [10–12]. As a substantive application of the proposed approach and the resulting numerical algorithm we consider the inverse scattering problem for plane seismic waves in a layered medium. In this case the scattering problem reduces to an acoustic equation for longitudinal or respectively transverse elastic waves. This ensures unique determination of the acoustic impedance and the absorption coefficient as functions of the eikonal and also their functional relationship along the single-reflections seismic path for the corresponding wave types [6–7].

From practical considerations, the possibility of determining the absorption coefficient and especially anomalously large values of the coefficient suggests in the framework of geophysical interpretation the presence of highly porous layers [13]. This, in turn, makes it possible to produce, in the geological interpretation stage, valid predictions of the presence of natural hydrocarbon collectors in the given horizon [14].

1. The Inverse Scattering Problem

Consider the following hyperbolic system of equations describing the propagation of plane waves in an acoustic or elastic medium for $z > 0$:

$$\rho(z)s_t = -p_z - 2\nu(z)s, \qquad p_t = -\lambda(z)s_z, \tag{2}$$

where $s(z, t)$ is the velocity of small displacements of the medium, $p(z, t)$ the pressure, $\rho(z)$ the density, $\lambda(z)$ the elastic parameter, $\nu(z)$ the dynamic viscosity, z the spatial variable, t the physical time. By geophysical convention, z is depth and $z = 0$ is the daylight surface. All the properties of the medium thus depend only on depth, i.e., we are dealing with a layered vertically nonhomogeneous medium.

Change the spatial variable in (2) to the eikonal $x(z) = \int_0^z \mathrm{d}\zeta/c(\zeta)$, where $c(z) = \sqrt{\lambda/\rho}$ is the propagation velocity of the oscillations in the medium. Eliminating p, we obtain an acoustic equation with a dissipative term:

$$w_{tt} = w_{xx} + \frac{\sigma'(x)}{\sigma(x)}\, w_x - 2\mu(x)w_t, \tag{3}$$

where $w(x, t) = s(z(x), t)$, $\mu(x) = \nu(z(x))/\rho(z(x))$ is the damping coefficient (kinematic viscosity), $\sigma(x) = c(z(x))\rho(z(x))$. The function $\sigma(x)$ is the acoustic impedance or stiffness of the medium and

$$k(x) = -0.5\,\sigma'(x)/\sigma(x)$$

is the coefficient of reflection in the nonhomogeneous medium. In what follows we assume that $\sigma(0)$ is known and without loss of generality take $\sigma(0) = 1$.

With Eq. (3) we associate in what follows an equivalent canonical system in Riemannian invariants (a nonstationary Dirac system). To this end we write (3) in the form

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)w = -2k(x)w_x - 2\mu(x)w_t,$$

and introduce the Riemannian invariants $\hat{v}(x,t),\ \hat{u}(x,t)$:

$$\hat{v} = w_t - w_x, \qquad \hat{u} = w_t + w_x.$$

We obtain

$$\hat{v}_t + \hat{v}_x + k(x)(\hat{u} - \hat{v}) + \mu(x)(\hat{u} + \hat{v}) = 0,$$

$$\hat{u}_t - \hat{u}_x + k(x)(\hat{u} - \hat{v}) + \mu(x)(\hat{u} + \hat{v}) = 0.$$

Making the last simplifying change of variable

$$\hat{v} = v \exp\left(\int_0^x (k(\xi) - \mu(\xi))\,\mathrm{d}\xi\right), \qquad \hat{u} = u \exp\left(\int_0^x (k(\xi) + \mu(\xi))\,\mathrm{d}\xi\right),$$

we finally obtain the following hyperbolic system:

$$v_t + v_x + (k(x) + \mu(x))u = 0, \qquad u_t - u_x - (k(x) - \mu(x))v = 0. \tag{4}$$

We further adopt a key physical conjecture according to which $\mu(x)$ and $\sigma(x)$ are functionally related and $\mu = \widetilde{\varphi}(\sigma)$. This conjecture relies on the existence of a natural argument on which depend all the previously considered properties of the nonhomogeneous (acoustic) medium. For this unique argument we take the material composition of the layered medium, which can be easily represented by a numerical value.

We now formulate a mixed problem (the forward scattering problem) for system (5) with $x, t > 0$ with the following initial and boundary conditions:

$$v(x,0) = u(x,0) = 0, \quad x > 0, \qquad v(0,t) = v_0(t), \quad t \ge 0, \tag{5}$$

where $v_0(t)$ is the source in the boundary condition, and let $v_0 \in C^1[0,\infty)$.

The inverse scattering problem for (4), (5) in T-local form involves finding the coefficients $k(x)$ and $\mu(x)$, $x \in [0,T]$, of system (4) from the scattered wave field given for $x = 0$, i.e., from the trace of the solution $u(0,t) = u_0(t)$ for $t \in [0,2T]$. The problem of determining two coefficients obviously does not have a unique solution in the general case. However, as we well know [5–8], one unknown coefficient (with the second known) can be uniquely determined in this situation under certain conditions on the function $v_0(t)$ in the neighborhood of zero, e.g., $v_0(0) \neq 0$.

The forward scattering problem stated above has been studied in detail also in generalized form [5–8, 18–19]. Our goal is to simultaneously determine the coefficients $k(x)$ and $\mu(x)$, as well as the function $\widetilde{\varphi}(\sigma)$, and we accordingly apply the known results for the solution of the inverse problem requiring reconstruction of a single coefficient.

Thus, if the source in the boundary condition in (5) is a Dirac delta-function, i.e., $v_0(t) = \delta(t)$, then the trace of the solution $u_0(0,t) = -f(t)$ is continuous when $k(x)$ and $\mu(x)$ are continuous. We moreover have a Volterra integro-functional equation of second kind:

$$k(t) - \mu(t) + 2\int_0^t \bar{v}(x, 2t - x)(k(x) + \mu(x))\,\mathrm{d}x = -2f(2t), \quad t \in [0, T], \tag{6}$$

where $\bar{v}(x,t)$ is a regular function, i.e., a continuous component of the solution of the forward problem representable as the following decomposition in smoothness:

$$v(x,t) = \delta(t - x) + \bar{v}(x,t).$$

Note that $\bar{v}$ functionally depends on k and μ by system (4).

Many applied seismic-sounding problems examine a so-called Born (linearized) approximation, i.e., a solution of the scattering problem that allows only for single reflections. This is an approximate solution of Eq. (6) in the form

$$k(x) - \mu(x) = -2f(2x), \quad x \in [0, T], \tag{7}$$

corresponding to small values of $k(x)$ and μ (in seismic prospecting, we typically have $|k| < 0.1$ and $\mu \ll 0.1$.

We will show that the last equation is of type (1). Indeed, let

$$k(x) = -y'(x), \qquad y(x) \equiv \ln\sqrt{\sigma(x)},$$

$$\mu(x) = \widetilde{\varphi}\big(\exp(2y(x))\big) \equiv \varphi(y(x)).$$

From these expressions we obtain

$$y'(x) + \varphi(y(x)) = 2f(2x).$$

In the next section we prove how by examining this equation in the class of bounded-variation functions $y(x)$ we can uniquely determine the functions $\sigma(x)$ and $\mu(x)$, as well as $\widetilde{\varphi}(\sigma)$ from scattering observations in the Born approximation framework.

Furthermore, it has been shown in [15–16] that equality (7) holds (in the sense of generalized function theory) also for a piecewise-continuous function $\sigma(x)$ with finitely many discontinuities of the first kind on $[0, T]$. Therefore, Eq. (1) may also contain singular terms with delta functions corresponding to the jumps in the function $y(x)$. Since such terms differ by smoothness, their presence does not affect the generality of our analysis.

2. Uniqueness of Generalized Solution

Consider Eq. (1) in the class of bounded-variation functions $y(x)$ such that their classical derivative $y'(x) = \mathrm{d}y/\mathrm{d}x$ equals zero almost everywhere on $[0, a]$. We assume that $y(x)$ may have finitely many discontinuities of the first kind inside $[0, a]$, which in general are not known. The remaining points in $[0, a]$ are thus points of continuity and the function $y(x)$ itself is constant almost everywhere on $[0, a]$. We also assume that $y(x)$ and $f(x)$ can be continuously extended as constants from the closed interval $[0, a]$ to the entire line $(-\infty, \infty)$ and the value $y(0) = b$ is known.

By the theory of functions of a real variable, $y(x)$ is representable in the form

$$y(x) = H(x) + \bar{y}(x) = \sum_{j=1}^{n} H_j \theta(x - x_j) + \bar{y}(x), \quad 0 < x_j < a, \tag{8}$$

where $\theta(x - x_j)$ is the unit jump function at the point x_j (the Heaviside function) and $\bar{y}(x)$ is a continuous function such that almost everywhere on $(-\infty, \infty)$ we have the equality

$$\bar{y}'(x) = 0 \quad \text{a.e. on} \quad (-\infty, \infty).$$

Such continuous and a.e. constant functions are called *singular* in the theory of functions of a real variable [9]. A typical example of such a function is the Cantor staircase.

We use the concept of generalized derivative (in the sense of distributions) [10–12]. In this case, $y(x)$ is an element of the space $L_{\text{loc}}(-\infty, \infty)$ of locally Lebesgue-integrable functions. Here $y(x)$ uniquely defines a generalized function y — a linear continuous functional (y, ψ) on the space $\mathcal{D}$ of the test functions $\psi(x) \in C^\infty(-\infty, \infty)$ with a compact support. The space of generalized functions on $\mathcal{D}$ is denoted by $\mathcal{D}'$. In what follows to every function $y(x)$ of a real variable from this class we associate a generalized function y defined by the equality $(y, \psi) = \int_{-\infty}^{\infty} y(x)\psi(x)\,\mathrm{d}x$. Generalized functions defined in this way are called *regular* in the theory of generalized functions [9, 11].

The generalized function y has a derivative, which is also a generalized function. From the definition of the derivative y' of the generalized function y we have

$$(y', \psi) = -(y, \psi') = -\int_{-\infty}^{\infty} y(x)\psi'(x)\,\mathrm{d}x.$$

We know that in the generalized sense $\theta'(x - x_j) = \delta(x - x_j)$, where $\delta(x - x_j)$ is the Dirac delta function, i.e., a singular generalized function concentrated at the point x_j. Therefore from (8) we directly obtain

$$y' = H' + \bar{y}', \qquad H' = \sum_{j=1}^{n} H_j \delta(x - x_j).$$

Associate with (1) the following differential equation written in terms of generalized functions:

$$y' + \varphi[y] = f, \tag{9}$$

where $\varphi[y]$ is a regular generalized function defined in the following way. Since

$$y(x) \in L_{\text{loc}}(-\infty, \infty) \qquad \text{and} \qquad \varphi(y) \in C(-\infty, \infty),$$

the following functional is defined on $\mathcal{D}$:

$$(\varphi[y], \psi) = \int_{-\infty}^{\infty} \varphi(y(x))\psi(x)\,\mathrm{d}x.$$

This functional is clearly linear, continuous, and by definition it is a regular generalized function.

To the function $f(x)$ from (1) we associate the generalized function $f \in \mathcal{D}'$, such that the generalized function f is equal to a constant $f(0)$ in the region $\mathbb{R}_- = \{x \mid x < 0\}$ [9, 11].

To the initial condition $y(0) = b$ in the Cauchy problem for (1) we associate equality of the generalized function y to the constant b in the region $\mathbb{R}_-$. We also take by convention $y' = 0$ in $\mathbb{R}_-$, which leads to the equality $\varphi[y] = \text{const} = \varphi(b)$ in $\mathbb{R}_-$.

The next lemma establishes some properties of the generalized problem for (9) with the condition $\varphi[y] = f$ in $\mathbb{R}_-$. In the functional term in Eq. (9) set $y = y_0$, where $y_0 \in \mathcal{D}'$ is a given function, and consider the following problem in generalized functions:

$$y' + \varphi[y_0] = f, \qquad \varphi[y_0] = \varphi(b) = f \quad \text{in} \quad \mathbb{R}_-. \tag{10}$$

Lemma *Generalized problem (10) is uniquely solvable in $\mathcal{D}'$ for every $f \in \mathcal{D}'$, and $y = \text{const} = b$ in $\mathbb{R}_-$.*

Proof. Let $g = f - \varphi[y_0]$. Then for the equation $y' = g$ for every $\psi \in \mathcal{D}$ we have

$$(y', \psi) = (g, \psi) = -(y, \psi'). \tag{11}$$

The right-hand side of this equality defines a linear continuous functional on the subspace $\mathcal{D}_0$ of test functions from $\mathcal{D}$ each of which is the derivative of some test function. We further take $\psi_0(x) = -\psi'(x)$. It is easy to see that the test function $\psi(x)$ is in $\mathcal{D}_0$ if and only if $\int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 0$, i.e., $\mathcal{D}_0$ is the kernel of the functional $\int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x$ and thus the subspace $\mathcal{D}_0$ is of codimension 1.

Every test function from $\mathcal{D}$ is obviously representable in the form

$$\psi(x) = \psi_0(x) + \alpha\psi_1(x), \qquad \psi_0 \in \mathcal{D}_0, \quad \alpha \in \mathbb{R},$$

where $\psi_1(x)$ is a given test function not contained in $\mathcal{D}_0$. To this end, it suffices to take

$$\int_{-\infty}^{\infty} \psi_1(x)\,\mathrm{d}x = 1, \qquad \alpha = \int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x, \qquad \psi_0(x) = \psi(x) - \alpha\psi_1(x).$$

Equality (11) defines the value of the functional y on every test function $\psi_0 \in \mathcal{D}_0$:

$$(y, \psi_0) = -\left(g, \int_{-\infty}^{x} \psi_0(\xi)\,\mathrm{d}\xi\right).$$

Since every test function $\psi \in \mathcal{D}$ is representable in the form $\psi = \psi_0 + \alpha\psi_1$, by setting $(y, \psi_1) = 0$ we extend the definition of the functional y to the entire $\mathcal{D}$ in the following form:

$$(y, \psi) = (y, \psi_0) = -\left(g, \int_{-\infty}^{x} \psi_0(\xi)\,\mathrm{d}\xi\right).$$

This functional is clearly linear and continuous. Moreover, it satisfies conditions (10), because for the test function $\psi \in \mathcal{D}$ we have the equalities

$$(y', \psi) = -(y, \psi') = \left(g, \int\limits_{-\infty}^{x} \psi'(\xi)\, \mathrm{d}\xi \right) = (g, \psi),$$

and since $g = f - \varphi[y_0] = 0$ in $\mathbb{R}_-$, we have

$$(y', \psi)_{\mathbb{R}_-} = -(y, \psi')_{\mathbb{R}_-} = \left(g, \int\limits_{-\infty}^{x} \psi'(\xi)\, \mathrm{d}\xi \right)_{\mathbb{R}_-} = 0,$$

where $(y, \psi)_{\mathbb{R}_-}$ is a functional considered on the space of test functions from $\mathcal{D}$ with the support in $\mathbb{R}_-$.

Alongside with (9), we consider the equation for the primitives $y,\ \Phi[y],\ F$ of the generalized functions y', $\varphi[y],\ f$, respectively. Since these primitives are defined up to a constant, we choose them so that $\Phi[y] = F = \mathrm{const} = 0$ in $\mathbb{R}_-$ and thus $y = \mathrm{const} = b$ in $\mathbb{R}_-$. From representation (8) we have

$$H + \bar{y} + \Phi[H + \bar{y}] = F,$$

where the generalized function $\Phi[H + \bar{y}]$ is regular. This assertion is obtained from the following argument.

By definition $\Phi'[y] = \varphi[y]$. Thus,

$$\big(\Phi'[y], \psi\big) = -\big(\Phi[y], \psi'\big) = \big(\varphi[y], \psi\big)$$

for every test function ψ from $\mathcal{D}$. The last equality defines the value of the functional $\Phi[y]$ on all test functions $\psi_0 \in \mathcal{D}_0$.

Let us now extend the definition of $\Phi[y]$ to the entire $\mathcal{D}$. We use the representation of an arbitrary element $\psi \in \mathcal{D}$ in the form $\psi = \psi_0 + \alpha\psi_1$. Setting $(\Phi[y], \psi_1) = c$, we thus define the functional $\Phi[y]$ on the entire $\mathcal{D}$:

$$(\Phi[y], \psi) = (\Phi[y], \psi_0) + \alpha(\Phi[y], \psi_1) = -\left(\varphi[y], \int\limits_{-\infty}^{x} \psi_0(\xi)\, \mathrm{d}\xi \right) + c(1, \psi). \tag{12}$$

This functional is obviously regular by continuity of the function $\varphi(y)$.

The representation of a generalized function as a sum of a singular and a regular term is by definition unique (up to const), and so from (8) we obtain that the generalized function H is uniquely defined as the primitive of H' when $H = 0$ in $\mathbb{R}_-$. Now taking H as an unknown function, we can pass to an equation of the form

$$\bar{y} + \Phi[H + \bar{y}] = F - H = \bar{F}, \tag{13}$$

where $\bar{y},\ \Phi[H + \bar{y}], \bar{F}$ are regular generalized functions. The following equation in generalized functions uniquely corresponds to Eq. (13):

$$\bar{y}' + \varphi[H + \bar{y}] = \bar{f}, \tag{14}$$

where $\bar{f} = \bar{F}'$. Excluding the δ-terms from (8), we continue the investigation of Eq. (13) in the framework of the theory of functions of a real variable.

Theorem (Uniqueness). *Generalized differential equation (8) with the condition $\bar{y} = \mathrm{const} = b$ in $\mathbb{R}_-$ has a unique solution that uniquely defines the function $\varphi(y)$ on the value set of the function $y(x) = H(x) + \bar{y}(x)$ for $x \in [0, a]$.*

Proof. The proof is by contradiction. Assume that two different generalized solutions exist, $\bar{y}_1$ and $\bar{y}_2$. By equivalence of (13) and (14), we obtain for the difference $\bar{y} = \bar{y}_1 - \bar{y}_2$

$$\bar{y} = \Phi_2[y_2] - \Phi_1[y_1] = \Phi_2[H + \bar{y}_2] - \Phi_1[H + \bar{y}_1].$$

This implies that for a locally integrable function $\bar{y}(x)$ and every $\psi \in \mathcal{D}$ we have the equality

$$\int_{-\infty}^{\infty} \bar{y}(x)\psi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \Big(\Phi_2(y_2(x)) - \Phi_1(y_1(x))\Big)\psi(x)\,\mathrm{d}x.$$

Let us use representation (12) to rewrite the last equality. For every test function $\psi(x)$ from $\mathcal{D}$ we thus obtain

$$\int_{-\infty}^{\infty} \bar{y}(x)\psi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \Big(\varphi_1\big(y_1(x)\big) - \varphi_2\big(y_2(x)\big)\Big) \int_{-\infty}^{x} \psi_0(\xi)\,\mathrm{d}\xi\,\mathrm{d}x + c,$$

and by the condition of the theorem $\bar{y}(0) = 0$, $\varphi_1(b) = \varphi_2(b)$. Thus, $c = 0$.

Integrating the last equality by parts, we obtain

$$\int_{-\infty}^{\infty} \bar{y}(x)\psi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \int_{0}^{x} \Big(\varphi_2\big(y_2(\xi)\big) - \varphi_1\big(y_1(\xi)\big)\Big)\,\mathrm{d}\xi\,\psi_0(x)\,\mathrm{d}x. \tag{15}$$

The choice of ψ_1 with $(1, \psi_1) = 1$ in the representation of the test function $\psi = \psi_0 + \psi_1$ is arbitrary. Choosing ψ_1 so that $\operatorname{supp}\psi_1 \subset \mathbb{R}_-$, we ensure that equality (15) holds on the entire $\mathcal{D}$. Hence,

$$\bar{y}(x) = \int_{0}^{x} \big(\varphi_2(H(\xi) + \bar{y}_2(\xi)) - \varphi_1(H(\xi) + \bar{y}_1(\xi))\big)\,\mathrm{d}\xi$$

almost everywhere on $(-\infty, \infty)$. Since $\bar{y}(x)$ is a singular continuous function constant almost everywhere on $[0, a]$, and the right-hand side of the equality is an absolutely continuous function of a real variable, then $\bar{y}(x) = \mathrm{const} = 0$, i.e., $\bar{y}_1(x) = \bar{y}_2(x) = \bar{y}(x)$.

Now, since $\bar{y}_1 = \bar{y}_2$, we have the equalities

$$\int_{0}^{x} \varphi_1\big(H(\xi) + \bar{y}(\xi)\big)\,\mathrm{d}\xi = \int_{0}^{x} \varphi_2\big(H(\xi) + \bar{y}(\xi)\big)\,\mathrm{d}\xi = \bar{F}(x) - \bar{y}(x) \equiv \Phi(x),$$

where the function $\Phi(x)$ is absolutely continuous on $[0, a]$ and is thus differentiable in the classical sense. Differentiating the last equalities, we obtain

$$\varphi_1(y(x)) = \varphi_2(y(x)) = \Phi'(x).$$

Thus, the function $\varphi(y)$ is unique on the value set of the function $y\colon [0, a] \to E(y)$.

3. Numerical Algorithm

We construct a numerical solution by the finite-difference method on a uniform grid $\{x_i = ih\}_{i=0}^N$ with the increment h such that $hN = a$. Introduce the grid functions

$$y = \{y_i\}_{i=0}^N, \qquad f = \{f_i\}_{i=0}^N, \qquad \mu = \{\mu_i\}_{i=0}^N, \qquad \varphi(y) = \big\{\varphi(y_i)\big\}_{i=0}^N$$

and define the difference derivative

$$y' = \{y_i'\} = \big\{(y_{i+1} - y_i)/h\big\}_{i=0}^N,$$

setting $y_{N+1} = y_N$. Associate with (1) the difference equation

$$y' + \mu = f, \quad \text{where} \quad \mu = \varphi(y), \tag{16}$$

and define the grid function f with a local relative error r, i.e., we know the function

$$f_\varepsilon = (1 + r)f, \qquad r = \{r_i\}_{i=0}^N, \quad |r_i| \le \varepsilon.$$

Given the function f_ε it is required to construct an approximate solution y_ε of Eq. (16) such that

$$\|y_\varepsilon - y\|_{C_h[0,a]} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Introduce the difference δ-function concentrated at the point x_j and the corresponding unit-jump function at the point x_j :

$$\delta_h(x_j) = \big\{\delta_{ij}/h\big\}_{i=0}^N, \qquad \theta_h(x_j) = \big\{\theta(x_i - x_j)\big\}_{i=0}^N,$$

where δ_{ij} is the Kronecker symbol. From (8) we directly obtain

$$y' = \sum_{j=1}^n H_j \delta_h(x_j) + \bar{y}'.$$

where the grid function $\bar{y}$ corresponds to $\bar{y}(x)$, which is continuous and a.e. constant. Denote the modulus of continuity of $\bar{y}(x)$ by $\omega(h)$. Then Eq. (16) can be formally expanded in a small parameter h as

$$\mathcal{O}(h^{-1}) + \mathcal{O}(h^{-1}\omega(h)) + \mathcal{O}(1) = (1 + \mathcal{O}(\varepsilon))f. \tag{17}$$

The function f is obviously representable by the same decomposition.

The algorithm for our problem separates the terms of the function f by the order of h. We first separate terms of order h^{-1} corresponding to jumps of the first kind of the function $y(x)$, then terms of order $h^{-1}\omega(h)$ corresponding to the continuous singular component, and this leaves us with a term of order 1. Computationally this process is realized by filtering through inequalities for the function f.

It is easy to see that the proposed algorithm is unstable, because f_ε contains terms of order εh^{-1} and $\varepsilon h^{-1}\omega(h)$, which may be of order greater than $\mathcal{O}(1)$. By the regularization conception [20–24], we should make

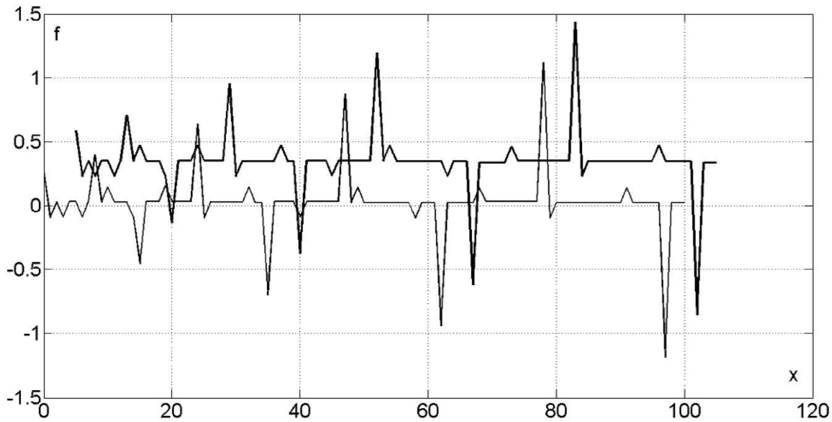

Fig. 1. Seismic traces of single reflections: input data (thin lines) and solution-based synthesized results.

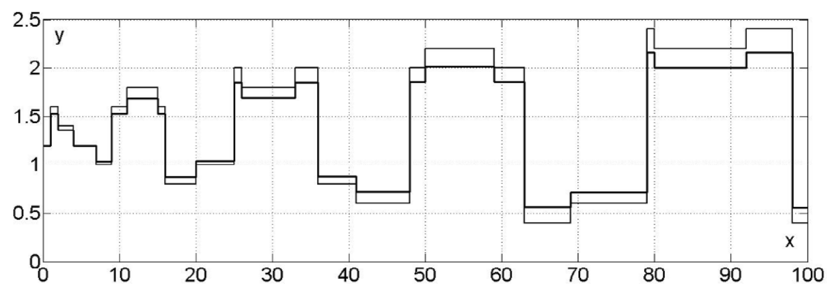

Fig. 2. Acoustic stiffness $y(x)$: section No. 1.

the method parameter h consistent with the input error ε. Passing to the limit in (17) we show that the proposed algorithm ensures convergence to the exact solution under the following conditions:

$$h(\varepsilon) \to 0, \qquad \varepsilon/h(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Note that this choice of the method parameter h is asymptotic and the algorithm therefore requires prior "fine tuning" when defining the second-level filter of order $h^{-1}\omega(h)$. For instance, to separate an arbitrary Cantor staircase, we need a filter of order $h^{-1/3}$, because for this function, as we know, $\omega(h) = \mathcal{O}(h^{2/3})$.

Let us now construct the function $\varphi(y)$ from the grid functions y and μ obtained above. The function $\varphi(y)$ is sought in parametric form

$$\varphi(y, \mathbf{b}) = b_0 + b_1 Y + \cdots + b_m Y^m, \qquad Y = 1/(\beta + y), \quad \beta > 0,$$

as a minimum over $\mathbf{b} = \{b_0, b_1, \ldots, b_m\}$ of the error functional

$$\Phi(\mathbf{b}) = \left\|\varphi(y, \mathbf{b}) - \mu\right\|^2_{L_{2,h}[0,a]}$$

on a compact set. This problem can be solved without difficulties [22–24]. The boundedness of the inverse map guarantees an error of order $\varepsilon/h(\varepsilon)$ in the solution.

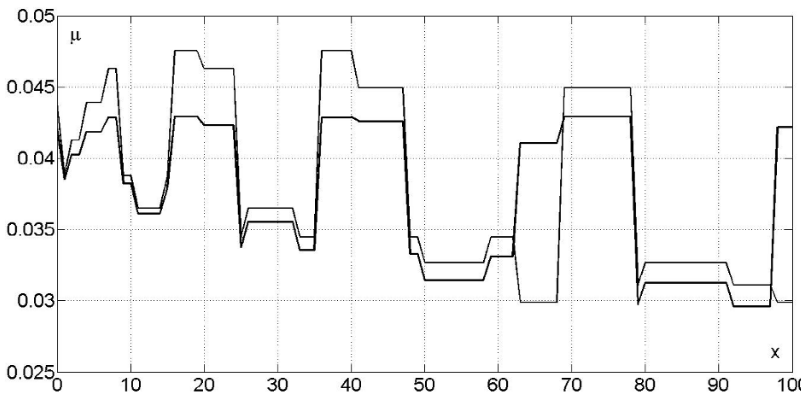

Fig. 3. Absorption coefficients $\mu(x)$: section No. 1.

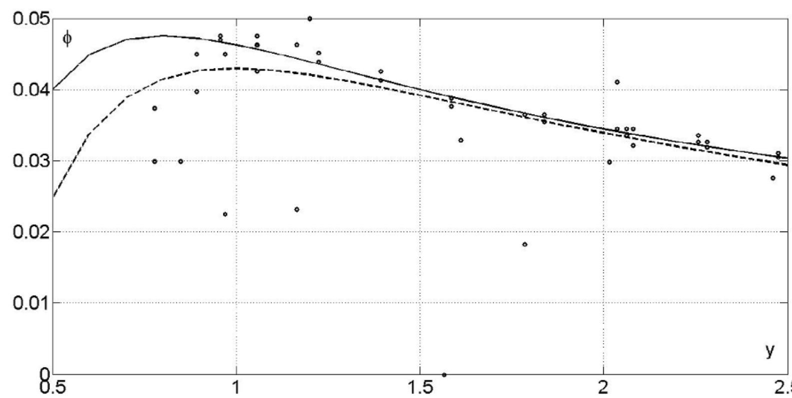

Fig. 4. The dependence $\varphi(y)$ of the absorption coefficient μ on acoustic stiffness y: section No. 1.

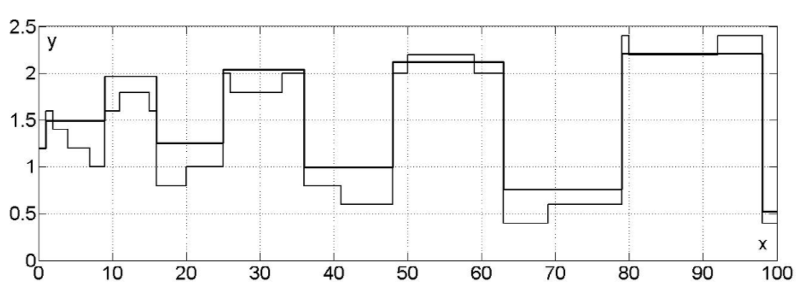

Fig. 5. Acoustic stiffness $y(x)$: section No. 1.

Figure 1 shows graphs of the function f: the original function (thin line) and the function calculated from the solution results (thick line) for section No. 1. Since the two graphs are very close, the second graph has been shifted upward and to the left. Such graphs are typical in seismic data processing: these are seismic traces (seismograms) of single reflections.

Figures 2–4 show the functions $y(x)$, $\mu(x)$, and $\varphi(y)$ for section No. 1. The original data for y, μ, φ are plotted by thin lines, the solution results for y, μ are plotted by thick lines, and those for φ are shown by a broken line (point-by-point approximation).

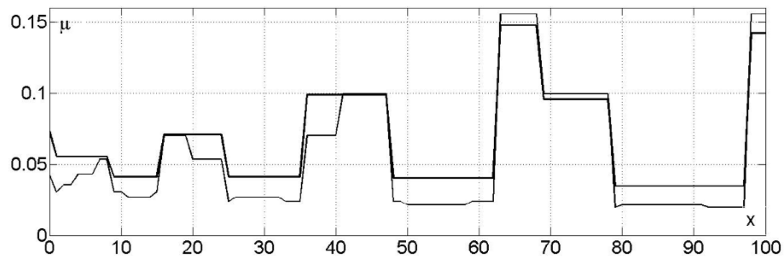

Fig. 6. Absorption coefficients $\mu(x)$: section No. 1.

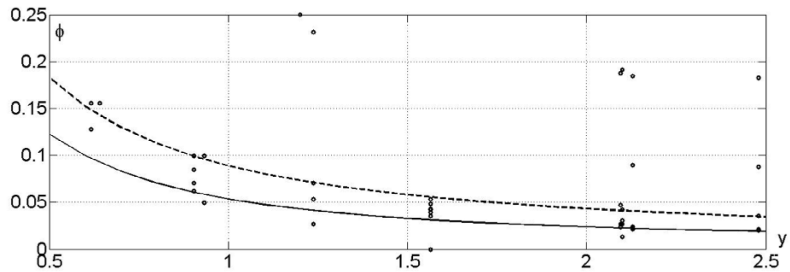

Fig. 7. The dependence $\varphi(y)$ of the absorption coefficient μ on acoustic stiffness y: section No. 1.

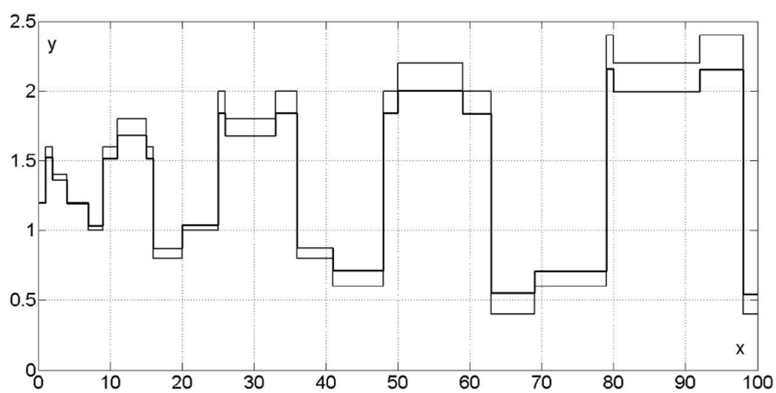

Fig. 8. Acoustic stiffness $y(x)$: section No. 1.

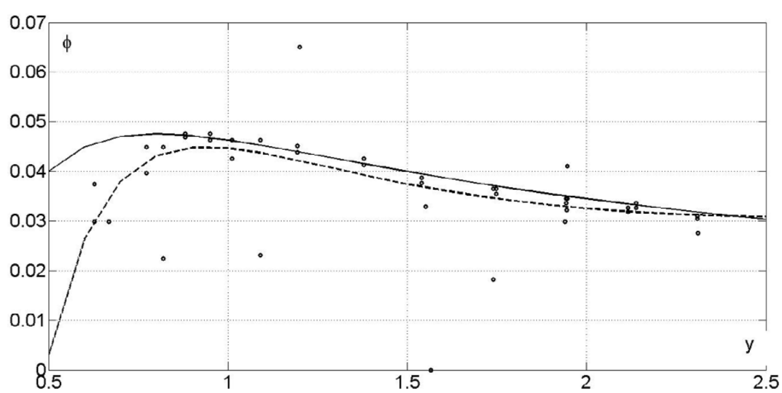

Fig. 9. The dependence $\varphi(y)$ of the absorption coefficient μ on acoustic stiffness y: section No. 1.

Figures 5–7 show the functions $y(x)$, $\mu(x)$, and $\varphi(y)$ for section No. 2 and Figs. 8–9 show the functions $y(x)$ and $\varphi(y)$ for section No. 3 (the legend for the curves is the same as in Figs. 2–4).

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