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TEMPERATURE DEPENDENCE OF THE ELASTIC MODULUS IN THREE-DIMENSIONAL GENERALIZED THERMOELASTICITY WITH DUAL-PHASE-LAG EFFECTS

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A three-dimensional problem for a homogeneous isotropic thermoelastic half-space solids with temperature-dependent mechanical properties subject to a time-dependent heat sources on the boundary of the half-space which is traction free is considered in the context of the generalized thermoelasticity with *dual-phase-lag* effects. The *normal mode analysis* and *eigenvalue approach* techniques are used to solve the resulting non-dimensional coupled field equations. Numerical results for the temperature, thermal stresses and displacement distributions are represented graphically and discussed. A comparison is made with the result obtained in the absence of the temperature dependent elastic modulus. Various problems of generalized thermoelasticity and conventional coupled dynamical thermoelasticity are deduced as special cases of our problem.

Keywords: Generalized thermoelasticity, Dual-phase-lag model, L-S theory, Temperature-dependent properties, Normal mode analysis, Eigenvalue approach.

1. Introduction

Biot [1] formulated the conventional coupled dynamical thermoelasticity (CCTE) theory to eliminate the paradox inherent in the classical uncoupled thermoelasticity theory that 'elastic changes have no effect on the temperature field'. The heat equations for both theories, however, are of the diffusion type predicting infinite speeds of propagation for heat waves contrary to physical observations. Predications based on the parabolic heat equation can become measurably false at very low temperature [2]. To eliminate the phenomena of infinite speeds for thermal signals, various modified dynamic thermoelasticity theories were proposed by Lord and Shulman [3] (L-S model), Green and Lindsay [4] (G-L model) and Green and Naghdi [5, 6, 7] (G-N I, G-N II and G-N III model respectively) based on "second sound" effects i.e., propagation of heat as a wave like phenomenon.

Lord and Shulman [3] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. In this theory a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. Green and Lindsay [4] proposed the theory of generalized thermoelasticity with two relaxation times and they modified both the energy and constitutive equations. This model admits second sound without violating the Fourier's law. Both the theories are structurally different and one can not be obtained as a particular case of the other.

There are some engineering materials (such as metals) which are not suitable for use in experiments concerning second sound propagation because they possess a relatively high rate of thermal damping. But given the state of recent advances in material science, it may be possible in the foreseeable future to identify (or even manufacture for laboratory purposes) an idealized material for the purpose of studying the propagation of thermal waves at finite speed. The relevant theoretical developments on the subject are due to Green and Naghdi [5, 6, 7] and provide sufficient basic modifications in the constitutive equations that permit treatment of a much wider class of heat flow

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problems, labeled as types G-N I, G-N II, G-N III model respectively. Among these models, G-N I model is the same as the CCTE model. In the G-N II model, the internal rate of production of entropy is taken to be identically zero which implies that there is no dissipation of thermal energy. This model admits undamped thermoelastic waves in thermoelastic solid. In G-N III model, Green and Naghdi replaced the Fourier's heat law by a generalized form of the equation $\overrightarrow{q}(P,t) = -[K\overrightarrow{\nabla}T(P,t) + K^*\overrightarrow{\nabla}\upsilon(P,t)]$, where υ is the thermal displacement, satisfying $\dot{\upsilon} = T$ and the two material constants K, K^* are the *thermal conductivity* and the *conductivity rate* respectively. An important feature of type–II model which is not present in type-I or type-III model is that it does not accommodate dissipation of thermal energy whereas type-III model accommodate dissipation of energy. The entropy flux vector in type-II (thermoelasticity without energy dissipation (TEWOED)) and type-III (thermoelasticity with energy dissipation (TEWED)) theories are determined in terms of the potential that also determines stresses.

Latter on, Tzou [8, 9] and Chandrasekhariah [10] developed the theory of thermoelasticity with dual-phase-lags (DPL model) which describes the interactions between *phonons* and *electrons* on the microscopic level as retarding sources causing a delayed response on the macroscopic scale. For macroscopic formulation, it would be convenient to use the DPL model to investigate of the micro-structural effect on the behavior of heat transfer. The physical meanings and the applicability of the DPL model have been supported by the experimental results [11]. In DPL model, Tzou [11] replaced the classical Fourier's law $\overrightarrow{q}(P,t) = -K\overrightarrow{\nabla}T(P,t)$ by $\overrightarrow{q}(P+\tau_q,t) = -[K\overrightarrow{\nabla}T(P,t+\tau_T)]$, where the temperature gradient $\overrightarrow{\nabla}T$ at a point P of the thermoelastic solid at time $t+\tau_q$ corresponds to the heat flux vector $\overrightarrow{q}$ at the same point at time $t+\tau_q$. The delay time τ_T is the *phase-lag of temperature gradient* that is interpreted as the delay time caused by the micro-structural interactions (a small scale effects of heat transport in space such as phonon-electron interaction or phonon scattering) whereas the other delay time τ_q is interpreted as the relaxation time due to the fast transient effects of thermal inertia and is called the *phase-lag of the heat flux*. The model transmit thermoelastic disturbances in a wave-like manner if the approximation is linear with respect to τ_q and τ_T, and $0 \leq \tau_T < \tau_q$ or quadratic in τ_q and linear in τ_T, with $\tau_q > 0$ and $\tau_T > 0$. Chandrasekharaiah [10] proposed a parabolic as well as a hyperbolic thermoelastic model with dual-phase-lags by extending the dual-phase-lag heat conduction law [9] using a Taylor series expansion of the same. Quintanilla and Racke [12] discussed the stability of dual-phase-lag heat conduction equation and Horgan and Quintanilla [13] studied the spatial behavior of the solution of dual-phase-lag heat equation. Roychoudhuri [14] studied a one-dimensional thermoelastic wave propagation in an elastic half-space using the dual-phase-lag heat conduction law and Prasad et al. [15] worked on the propagation of harmonic plane waves under thermoelasticity with dual-phase-lags. The exact solutions of one-dimensional initial boundary value problem on the basis of two-temperature thermoelasticity with dual-phase-lag effects [16] was studied by Quintanilla and Jordan [17].

Recently, Roychoudhuri [18] introduced another model of thermoelasticity with three-phase-lags by replacing the dual-phase-lag heat conduction law [9] by $\overrightarrow{q}(P+\tau_q,t) = -[K\overrightarrow{\nabla}T(P,t+\tau_T) + K^*\overrightarrow{\nabla}\upsilon(P,t+\tau_\upsilon)]$, where $\overrightarrow{\nabla}\upsilon$ is the thermal displacement gradient and τ_υ is the phase-lag for thermal displacement gradient. Three-phase-lag thermoelastic model is very useful in the problems of nuclear boiling, exothermic catalytic reactions, phonon-electron interactions, phonon-scattering etc., where the delay time τ_q captures the thermal wave behavior (a small scale response in time), the phase-lag τ_T captures the effect of phonon-electron interactions (a microscopic response in space) and the other delay time τ_υ is effective since in the three-phase-lag model the thermal displacement gradient is considered as a constitutive variable whereas in the CCTE theory temperature gradient is considered as a constitutive variable. Kar and Kanoria [19] worked on a thermo-visco-elastic problem of a spherical shell using the three-phase-lag model and Kar and Kanoria [20] studied the analysis of thermoelastic response in a fiber reinforced thin annular disc with three-phase-lag effect.

Many problems in engineering practice involve the determination of stresses and/or displacements in bodies that are three-dimensional. Exact analytical solutions are available only for a few three-dimensional problems [21, 22, 23, 24] with simple geometries and/or loading conditions. Hence numerical or experimental analysis are generally required in solving such problems. In solving three-dimensional problems of generalized thermoelasticity, many authors generally use the Laplace-Fourier transform method or other methods such as finite difference

Nomenclature

λ, μ	Lame's constant
ρ	constant mass density of the medium
C_E	specific heat of the solid at constant strain
σ_{ij}	components of the stress tensor
e_{ij}	components of the strain tensor
u_i	components of the displacement vector
e_{kk}	$= e$, cubical dilatation
t	time variable
x, y, z	space variables
$\tau_T,\ \tau_q$	the phase-lags of the temperature gradient and of heat flux respectively, such that $\tau_T < \tau_q$
T	absolute temperature
T_0	the temperature of the medium in it's natural state, assumed to be such that $\left\vert\frac{(T-T_0)}{T_0}\right\vert << 1$
γ	$= (3\lambda + 2\mu)\alpha_T$, a material constant characteristic of the theory
α_T	coefficient of linear thermal expansion
k	thermal conductivity
$E(T)$	modulus of elasticity at temperature T
E_0	constant (modulus of elasticity at $\alpha^* = 0$)
ν	Poisson's ratio
λ_0	$= \frac{\nu}{(1+\nu)(1-2\nu)}$
μ_0	$= \frac{1}{2(1+\nu)}$
γ_0	$= \frac{\alpha_T}{(1-2\nu)}$
α^*	empirical material constant $\left[\frac{1}{K}\right]$
α	$= \frac{1}{(1-\alpha^* T_0)}$
c_0^2	$= \frac{E_0(\lambda_0+2\mu_0)}{\rho}$
δ_0	non–dimensional constant
T_0	$= \frac{\delta_0 \rho c_0^2}{E_0\gamma_0} = \left(\frac{\delta_0}{\alpha_T}\right)\left(\frac{1-\nu}{1+\nu}\right)$
ε_1	$= \frac{E_0\gamma_0}{\rho C_E}$
ε	$= \varepsilon_1 \delta_0$
ε_0	$= \frac{\varepsilon}{\alpha}$
β	$= \frac{\mu_0}{(\lambda_0+2\mu_0)} = \frac{(1-2\nu)}{2(1-\nu)}$
δ	$= \frac{(3-4\beta)}{3}$
∇^2	$\equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$
D	$\equiv \frac{d}{dx}$

method, finite element method, weighted residuals method and boundary element method, etc. Recently, Sarkar and Lahiri [21] applied *normal mode analysis* to study a three-dimensional thermoelastic problem for a half-space without energy dissipation.

The present paper concerned with a three-dimensional problem for a homogeneous isotropic thermoelastic half-space solid with temperature-dependent mechanical properties subject to a time-dependent heat source on the boundary of the space which is traction free is considered in the context of the generalized thermoelasticity with *dual-phase-lag* effects. The *normal mode analysis* [21, 26, 27, 28] and *eigenvalue approach* [21, 22, 23, 25] techniques are used to solve the resulting non-dimensional coupled field equations. Numerical results for the temperature, thermal stresses and displacement distributions are represented graphically and discussed. A comparison is made with the results obtained in absence of the temperature independent modulus of elasticity. Various problem of generalized thermoelasticity and coupled thermoelasticity are deduced as special cases of our problem.

2. Governing Equations

For a homogeneous isotropic elastic solid, the basic equations for the linear theory of generalized thermoelasticity with *dual-phase-lags* and reference temperature-dependent mechanical properties [8, 9, 10] in the absence of body forces and heat sources are:

Equations of motion:

$$\sigma_{ij,j} = \rho \ddot{u}_i. \tag{1}$$

Heat conduction equation:

$$k\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla^2 T = \left(1+\tau_q\frac{\partial}{\partial t}+\frac{n\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_E\dot{T}+\gamma T_0\dot{e}\right), \quad n=0,1. \tag{2}$$

Constitutive relation:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma\left(T-T_0\right)\delta_{ij}, \tag{3}$$

where

$$e_{ij} = \frac{1}{2}\left(u_{i,j}+u_{j,i}\right)$$

and $i, j = x, y, z$ referee to a general coordinates. The *comma* notation is used for derivative with respect to space variables and *superimposed dot* represents differentiation with respect to time t.

Our main interest is to study the effects of *dual-phase-lag* and the *temperature dependency of modulus of elasticity* on the variations of different field quantities inside the three-dimensional homogenous isotropic thermoelastic half-space keeping the other elastic and thermal parameters constants, therefore we assume that

$$E = E_0 f(T),\ \lambda = E_0\lambda_0 f(T), \quad \mu = E_0\mu_0 f(T), \quad \gamma = E_0\gamma_0 f(T), \tag{4}$$

where $E_0,\ \lambda_0,\ \mu_0,\ \gamma_0$ are considered to be constants and $f(T)$ is a given non-dimensional function of temperature. In the case of temperature-independent modulus of elasticity, $f(T)=1$ and $E=E_0$.

In generalized thermoelasticity as well as coupled theory only the infinitesimal temperature deviations from the reference temperature T_0 are considered. Therefore, we can consider $f(T)$ in the form $f(T) = (1-\alpha^* T_0)$, where α^* is an *empirical material constant* $\left[\frac{1}{K}\right]$.

3. Formulation of the Problem

We consider a homogenous isotropic thermoelastic half-space in three-dimensional space which fills the region $\Omega = \{(x, y, z) : 0 \leq x < \infty, \infty < y < \infty, \infty < z < \infty\}$ subject to a time dependent heat sources on the bounding plane to the surface $x = 0$. The body is initially at rest and the surface $x = 0$ is assumed to be traction free. We use the Cartesian co-ordinates (x, y, z). In this case the components of the displacement vector are $u_i = (u, v, w)$. Thus the governing equations can be written in the context of the *dual-phase-lag* model of generalized thermoelasticity [8, 9, 10] as follows:

Equations of motion are:

$$\rho\ddot{u} = E_0 f(T)[(\lambda_0 + 2\mu_0)u_{,xx} + \mu_0 (u_{,yy} + u_{,zz}) + (\lambda_0 + \mu_0) (v_{,xy} + w_{,xz}) - \gamma_0 T_{,x}], \tag{5}$$

$$\rho\ddot{v} = E_0 f(T)[(\lambda_0 + 2\mu_0)v_{,yy} + \mu_0 (v_{,xx} + v_{,zz}) + (\lambda_0 + \mu_0) (u_{,xy} + w_{,yz}) - \gamma_0 T_{,y}], \tag{6}$$

$$\rho\ddot{w} = E_0 f(T)[(\lambda_0 + 2\mu_0)w_{,zz} + \mu_0 (w_{,xx} + w_{,yy}) + (\lambda_0 + \mu_0) (u_{,xz} + v_{,yz}) - \gamma_0 T_{,z}]. \tag{7}$$

The heat conduction equation is:

$$k\left(1 + \tau_T \frac{\partial}{\partial t}\right) \nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_q \frac{\partial^2}{\partial t^2} + \frac{n\tau_q^2}{2} \frac{\partial^3}{\partial t^3}\right) [\rho c_E T + E_0 \gamma_0 f(T) T_0 e]. \tag{8}$$

The constitutive equations are:

$$\sigma_{xx} = E_0 f(T) [2\mu_0 u_{,x} + \lambda_0 e - \gamma_0 (T - T_0)], \tag{9}$$

$$\sigma_{yy} = E_0 f(T) [2\mu_0 v_{,y} + \lambda_0 e - \gamma_0 (T - T_0)], \tag{10}$$

$$\sigma_{zz} = E_0 f(T) [2\mu_0 w_{,z} + \lambda_0 e - \gamma_0 (T - T_0)], \tag{11}$$

$$\sigma_{xy} = E_0 \mu_0 f(T) (u_{,y} + v_{,x}), \tag{12}$$

$$\sigma_{xz} = E_0 \mu_0 f(T) (u_{,z} + w_{,x}), \tag{13}$$

$$\sigma_{yz} = E_0 \mu_0 f(T) (v_{,z} + w_{,y}), \tag{14}$$

where

$$e = (u_{,x} + v_{,y} + w_{,z}). \tag{15}$$

To transform the above equations in non-dimensional forms, we define the following non-dimensional variables

$$(x', y', z') = c_0\eta(x, y, z), \;\; (u', v', w') = c_0\eta(u, v, w), \;\; (t', \tau_T', \tau_q') = c_0^2\eta(t, \tau_T, \tau_q),$$

$$\theta = \frac{E_0\gamma_0}{\rho c_0^2}(T - T_0), \;\; \sigma_{ij}' = \frac{\sigma_{ij}}{\rho c_0^2}, \;\; \eta = \frac{\rho c_E}{k}.$$

Eqs. (5)–(14) in the non-dimensional forms then reduce to (omitting the primes for convenience)

$$\alpha\ddot{u} = u_{,xx} + \beta\left(u_{,yy} + u_{,zz}\right) + (1-\beta)\left(v_{,xy} + w_{,xz}\right) - \theta_{,x}, \tag{16}$$

$$\alpha\ddot{v} = v_{,yy} + \beta\left(v_{,xx} + v_{,zz}\right) + (1-\beta)\left(u_{,xy} + w_{,yz}\right) - \theta_{,y}, \tag{17}$$

$$\alpha\ddot{w} = w_{,zz} + \beta\left(w_{,xx} + w_{,yy}\right) + (1-\beta)\left(u_{,xz} + v_{,yz}\right) - \theta_{,z}, \tag{18}$$

$$\left(1 + \tau_T \frac{\partial}{\partial t}\right)\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau_q \frac{\partial^2}{\partial t^2} + \frac{n\tau_q^2}{2}\frac{\partial^3}{\partial t^3}\right)\left(\theta + \varepsilon_0 e\right), \tag{19}$$

$$\alpha\sigma_{xx} = 2\beta u_{,x} + (1-2\beta)e - \theta, \tag{20}$$

$$\alpha\sigma_{yy} = 2\beta v_{,y} + (1-2\beta)e - \theta, \tag{21}$$

$$\alpha\sigma_{zz} = 2\beta w_{,z} + (1-2\beta)e - \theta, \tag{22}$$

$$\alpha\sigma_{xy} = \beta\left(u_{,y} + v_{,x}\right), \tag{23}$$

$$\alpha\sigma_{xz} = \beta\left(u_{,z} + w_{,x}\right), \tag{24}$$

$$\alpha\sigma_{yz} = \beta\left(v_{,z} + w_{,y}\right). \tag{25}$$

Using Eq. (15), Eqs. (16)–(18) can be re-written in the following forms

$$\beta\nabla^2 u_{,x} + (1-\beta)e_{,xx} - \theta_{,xx} = \alpha\ddot{u}_{,x}, \tag{26}$$

$$\beta\nabla^2 v_{,y} + (1-\beta)e_{,yy} - \theta_{,yy} = \alpha\ddot{v}_{,y}, \tag{27}$$

$$\beta\nabla^2 w_{,z} + (1-\beta)e_{,zz} - \theta_{,zz} = \alpha\ddot{w}_{,z}. \tag{28}$$

Adding Eqs. (26)–(28) and using Eq. (15), we get

$$\nabla^2 e - \nabla^2\theta = \alpha\ddot{e}. \tag{29}$$

We shall now consider the *mean stress* σ as follows:

$$\sigma = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}. \tag{30}$$

Adding the Eqs. (20)–(22) and using Eqs. (15) and (30), we get

$$\alpha\sigma = \delta e - \theta. \tag{31}$$

Eliminating e from Eqs. (19), (29) and (31), we obtain after some simple manipulations

$$\nabla^2\sigma + \frac{(1-\delta)}{\alpha}\nabla^2\theta = \ddot{\theta} + \alpha\ddot{\sigma}. \tag{32}$$

4. Normal Mode Analysis

The solutions of the physical variables can be decomposed in terms of normal modes in the following forms [21]:

$$[u, v, w, e, \theta, \sigma, \sigma_{ij}]\,(x, y, z, t) = \left[u^*, v^*, w^*, e^*, \theta^*, \sigma^*, \sigma^*_{ij}\right](x)\exp\left[\omega t + i(ay + bz)\right], \tag{33}$$

where $u^*(x)$ etc. are the amplitude of the function $u(x, y, t)$ etc., i is the imaginary unit, ω (complex) is the angular frequency and $a,\ b$ are the wave number in the y and z-direction respectively.

Using Eq. (33), we can obtain the following equations from Eqs. (19) and (32) respectively

$$D^2\theta^* = C_1\theta^* + C_2\sigma^*, \tag{34}$$

$$D^2\sigma^* = D_1\theta^* + D_2\sigma^*, \tag{35}$$

where

$$C_1 = \left[a^2 + b^2 + \frac{A(\varepsilon_0 + \delta)}{\delta}\right], \quad C_2 = \frac{A\varepsilon_0}{\delta}, \quad D_1 = \left[\omega^2 - \frac{A(\varepsilon_0 + \delta)(1 - \delta)}{\alpha\delta}\right],$$

$$D_2 = \left[\alpha\omega^2 + a^2 + b^2 - \frac{A\varepsilon_0(1 - \delta)}{\alpha\delta}\right], \quad A = \left[\frac{\omega + \tau_q\omega^2 + \frac{n}{2}\tau_q^2\omega^3}{1 + \omega\tau_T}\right].$$

Eqs. (34) and (35) can be written in a *matrix-differential equation* as follows [21]:

$$D\mathcal{V}(x) = \mathcal{A}(\omega, a, b)\mathcal{V}(x), \tag{36}$$

where

$$\mathcal{V} = \begin{pmatrix}\theta^* & \sigma^* & D\theta^* & D\sigma^*\end{pmatrix}^T, \quad \mathcal{A}(\omega, a, b) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ C_1 & C_2 & 0 & 0 \\ D_1 & D_2 & 0 & 0 \end{pmatrix}.$$

5. Eigenvalue Approach

Following the solution methodology through eigenvalue approach [21, 22, 23, 25], we now proceed to solve the *matrix-differential equation* (36). The characteristic equation of the matrix $\mathcal{A}$ can be written as

$$\lambda^4 - (C_1 + D_2)\lambda^2 + (C_1D_2 - C_2D_1) = 0. \tag{37}$$

Suppose λ_1^2 and λ_2^2 be the roots of the above characteristic equation with positive real parts. Then all the four roots of the characteristic Eq. (37) which are also the eigenvalues of the matrix $\mathcal{A}$ are of the form:

$$\lambda = \pm\lambda_1, \pm\lambda_2,$$

where

$$\lambda_j = \sqrt{\frac{C_1 + D_2 + (-1)^{j+1}\sqrt{(C_1 - D_2)^2 + 4C_2D_1}}{2}}, \quad j = 1, 2.$$

The right eigenvector $\mathcal{X}_\lambda$ (say) corresponding to the eigenvalue λ can be written as

$$\mathcal{X}_\lambda = \left[(\lambda^2 - D_2),\ \ D_1,\ \ \lambda(\lambda^2 - D_2),\ \ \lambda D_1\right]^T. \tag{38}$$

From (38), we can easily calculate the eigenvector $\mathcal{X}_j$ $(i = 1, 2, 3, 4)$ corresponding to the eigenvalue $\pm\lambda_j$ $(j = 1, 2)$. For our further reference, we shall use the following notations:

$$\mathcal{X}_1 = [\mathcal{X}_\lambda]_{\lambda=\lambda_1}, \quad \mathcal{X}_2 = [\mathcal{X}_\lambda]_{\lambda=-\lambda_1}, \quad \mathcal{X}_3 = [\mathcal{X}_\lambda]_{\lambda=\lambda_2}, \quad \mathcal{X}_4 = [\mathcal{X}_\lambda]_{\lambda=-\lambda_2}. \tag{39}$$

Hence the solution of Eq. (36) can be written from [21] as:

$$\mathcal{V} = A_1\mathcal{X}_2 e^{-\lambda_1 x} + A_2\mathcal{X}_4 e^{-\lambda_2 x} \quad (x \geq 0), \tag{40}$$

where the terms containing exponentials of growing nature in the space variables x are discard due to the regularity condition of the solution at infinity and $A_1,\ A_2$ (depends only on $a,\ b$ and ω) are constants to be determined from the boundary conditions of the problem.

Thus the temperature field $\theta^*(x)$ and the stress $\sigma^*(x)$ can be written from Eqs. (38)-(40) for $x \geq 0$ as

$$\theta^*(x) = \sum_{j=1}^{2}(\lambda_j^2 - D_2)A_j e^{-\lambda_j x}, \tag{41}$$

$$\sigma^*(x) = D_1\sum_{j=1}^{2} A_j e^{-\lambda_j x}. \tag{42}$$

Substituting from Eqs. (41) and (42) in Eq. (30), the cubical dilatation $e^*(x)$ can be obtained as

$$e^*(x) = \frac{1}{\delta}\sum_{j=1}^{2}\left(\lambda_j^2 + D_1 - D_2\right)A_j e^{-\lambda_j x}. \tag{43}$$

6. Application

In order to determine the constants $A_1,\ A_2,$ the following boundary conditions in non–dimensional form are considered at the surfaces $x = 0$:

(i) Mechanical boundary condition: the surface $x = 0$ has no traction anywhere, so we have

$$\sigma(0, y, z, t) = \sigma_{xx}(0, y, z, t) = \sigma_{yy}(0, y, z, t) = \sigma_{zz}(0, y, z, t) = 0, \tag{44}$$

which gives on using the normal modes (33)

$$\sigma^*(x) = \sigma^*_{xx}(x) = \sigma^*_{yy}(x) = \sigma^*_{zz}(x) = 0 \quad \text{on} \quad x = 0. \tag{45}$$

(ii) The thermal boundary condition is

$$q_n + \nu\theta(x,y,z,t) = r(x,y,z,t) \quad \text{on} \quad x = 0, \tag{46}$$

where q_n denotes the normal component of the heat flux vector, ν is Biot's number, and $r(0,y,z,t)$ represents the intensity of the applied heat sources. In order to use the thermal boundary condition (46), we now make use of the generalized Fourier's law of heat conduction of *dual-phase-lag* model in the non-dimensional form, namely

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right) q_n = -\left(1 + \tau_T \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial n}. \tag{47}$$

From Eqs. (46), (47) and (33), we get

$$\nu\theta^*(x) - \tau D\theta^*(x) = r^*(a,b,\omega) \quad \text{on} \quad x = 0, \tag{48}$$

where $\tau = (1+\tau_T\omega)/(1+\tau_q\omega)$.

Using the boundary conditions (45) and (48) in Eqs. (42) and (41) respectively, we get

$$A_1(\tau\lambda_1 + \nu)(\lambda_1^2 - D_2) + A_2(\tau\lambda_2 + \nu)(\lambda_2^2 - D_2) = r^*, \tag{49}$$

$$A_1 + A_2 = 0. \tag{50}$$

Solving the above system of equations, we obtain

$$A_1 = \frac{r^*}{\Delta} \quad \text{and} \quad A_2 = \frac{-r^*}{\Delta}, \tag{51}$$

where

$$\Delta = \left[(\nu + \tau\lambda_1)(\lambda_1^2 - D_2) - (\nu + \tau\lambda_2)(\lambda_2^2 - D_2)\right].$$

To get the displacement component $u^*(x)$, we use Eqs. (33) and (41)-(42) in Eq. (26) to get

$$\left(D^2 - \lambda_u^2\right) u^*(x) = \sum_{j=1}^{2} L_j(\lambda_j^2 - \lambda_u^2)A_j e^{-\lambda_1 x}, \tag{52}$$

where

$$\lambda_u^2 = \left[a^2 + b^2 + \frac{\alpha\omega^2}{\beta}\right],\ L_j = \frac{\lambda_j\left[(1-\beta)D_1 + (1-\beta-\delta)(\lambda_j^2 - D_2)\right]}{\beta\delta(\lambda_j^2 - \lambda_u^2)}, \quad j = 1,2.$$

The solution of the ordinary differential equation (52) can be written as

$$u^*(x) = A_3 e^{-\lambda_u x} + \sum_{j=1}^{2} L_j A_j e^{-\lambda_j x}, \tag{53}$$

where $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_u^2$ and A_3 is a constant to be determined from the boundary conditions (45).

Using the normal modes (33) in Eq. (20), we get

$$\alpha\sigma^*_{xx}(x) = 2\beta Du^*(x) + (1-2\beta)\, e^*(x) - \theta^*(x). \tag{54}$$

Substituting from Eqs. (41), (43) and (53) in the relation (54), we obtain the stress component $\sigma^*_{xx}(x)$ as

$$\sigma^*_{xx}(x) = B_1 A_3 e^{-\lambda_u x} + \sum_{j=1}^{2} M_j A_j e^{-\lambda_j x}, \tag{55}$$

where

$$B_1 = \frac{-2\beta\lambda_u}{\alpha},\ M_j = \frac{1}{\alpha\delta}\left[(1-2\beta)D_1 + (1-2\beta-\delta)(\lambda_j^2 - D_2) - 2\beta\delta\lambda_j L_j\right], \quad j = 1, 2.$$

Now, applying the boundary condition (48) in Eq. (55), we get

$$A_3 = \frac{r^*(M_2 - M_1)}{B_1\Delta}. \tag{56}$$

7. Particular Cases

7.1. Hyperbolic generalized thermoelasticity *with* dual-phase-lag **(HDPL)** and temperature dependence of an elastic modulus can be obtained by setting $n = 1$.

7.2. Hyperbolic generalized thermoelasticity *with* dual-phase-lag **(HDPL)** and *without* temperature dependence of an elastic modulus can be deduced by setting $n = 1$ and $\alpha = 1$.

7.3. Parabolic generalized thermoelasticity *with* dual-phase-lag **(PDPL)** and *without* temperature dependence of an elastic modulus can be obtained by setting $n = 0$.

7.4. Parabolic generalized thermoelasticity *with* dual-phase-lag **(PDPL)** and *without* temperature dependence of an elastic modulus can be obtained by setting $n = 0$ and $\alpha = 1$.

7.5. Lord-Shulman theory **(L-S theory)** of generalized thermoelasticity *with* temperature dependence of an elastic modulus: Set $n = 0,\ \tau_T = 0$ and $\tau_q = \tau_0$.

7.6. Lord-Shulman theory **(L-S theory)** of generalized thermoelasticity *without* temperature dependence of an elastic modulus: Set $n = 0,\ \tau_T = 0,\ \tau_q = \tau_0$ and $\alpha = 1$.

7.7. Conventional coupled dynamical thermoelasticity **(CCTE)** *with* temperature dependence of an elastic modulus: Set $n = 0$ and $\tau_T = \tau_q = 0$.

7.8. Conventional coupled dynamical thermoelasticity **(CCTE)** *without* temperature dependence of an elastic modulus can be obtained by setting $n = 0,\ \tau_T = \tau_q = 0$ and $\alpha = 1$.

8. Numerical Example and Discussions

The copper material is chosen for the purpose of numerical example. Since we have $\omega = \omega_0 + i\zeta,\ e^{\omega t} = e^{\omega_0 t}(\cos\zeta t + i\sin\zeta t)$ and for small value of t, we can take $\omega = \omega_0$ (real). The numerical constants of the problem

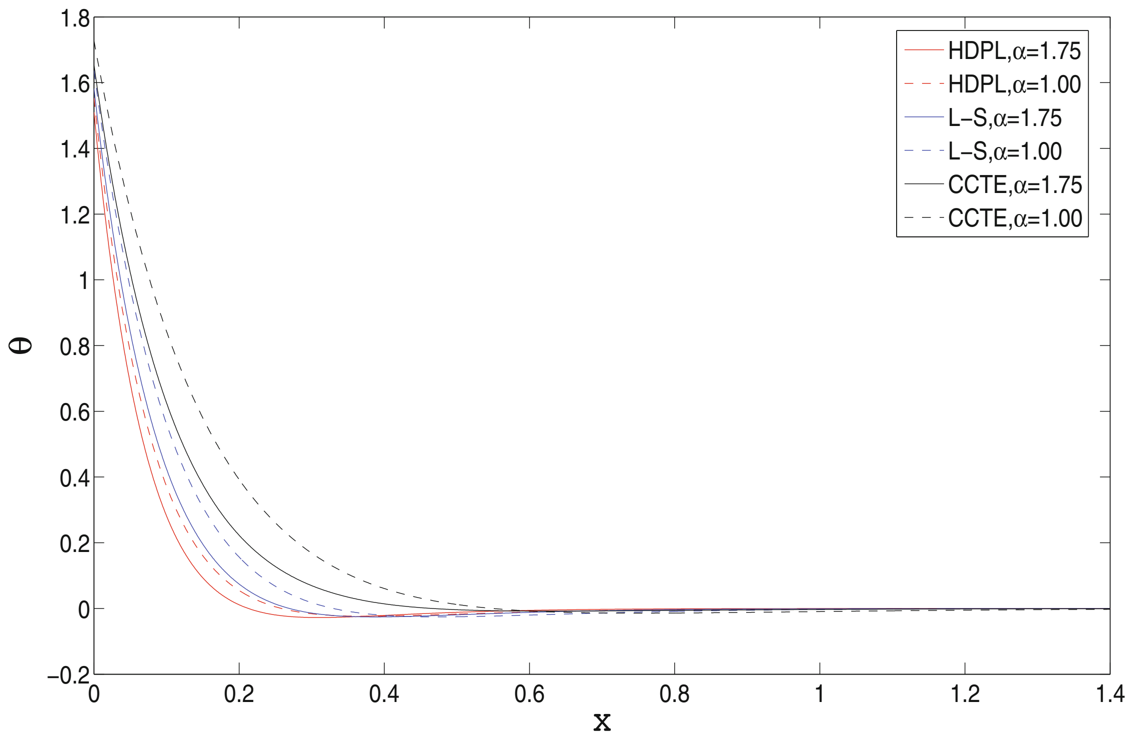

Fig. 1. The temperature distribution vs. distance x for different α at $t = 0.3$.

are taken as:

$$\varepsilon = 0.003, \delta_0 = 0.0199, \quad \beta = 0.25, \quad \tau_T = 0.015, \quad \tau_q = 0.6, \quad T_0 = 293K,$$

$$\alpha^* = 0.0005 \left[\frac{1}{K}\right], \quad \omega = 3, \quad a = 1.2, \quad b = 1.3, \quad \nu = 50, \quad r^* = 100.$$

Using the above constants, the numerical values of the real part of the temperature θ, mean stress σ, displacement component u and the stress σ_{xx} are computed at $(y, z, t) = (0.5, 0.5, 0.3)$. Comparisons of the above dimensionless physical quantities are made in four different cases:

(i) Figures 1–4 depict the variations of θ, σ, u and σ_{xx} with distance x for $(y, z, t) = (0.5, 0.5, 0.3)$ taking two values of α, namely $\alpha = 1.0$ (temperature-independent) and $\alpha = 1.75$ (temperature-dependent) in the cases of HDPL, L-S and CCTE theories.

(ii) The variations of the field variables θ, σ, u and σ_{xx} vs. distance x at $(y, z, t) = (0.5, 0.5, 0.3)$ for $\alpha = 1.0$ and $\alpha = 1.75$ for HDPL model at two values of the phase-lag of heat flux $\tau_q = 1.0$ and $\tau_q = 1.2$ are shown in Figs. 5-8.

(iii) Figures 9–12 exhibit the variations of θ, σ, u and σ_{xx} with distance x at $(y, z, t) = (0.5, 0.5, 0.3)$ for $\alpha = 1.0$ and $\alpha = 1.75$ for HDPL model at two values of the phase-lag of temperature gradient $\tau_T = 0.2$ and $\tau_T = 0.4$.

(iv) Figures 13–16 display the variations of the physical quantities θ, σ, u and σ_{xx} with distance x at $(y, z, t) = (0.5, 0.5, 0.1)$ and $(y, z, t) = (0.5, 0.5, 0.3)$ for $\alpha = 1.0$ and $\alpha = 1.75$ for HDPL model.

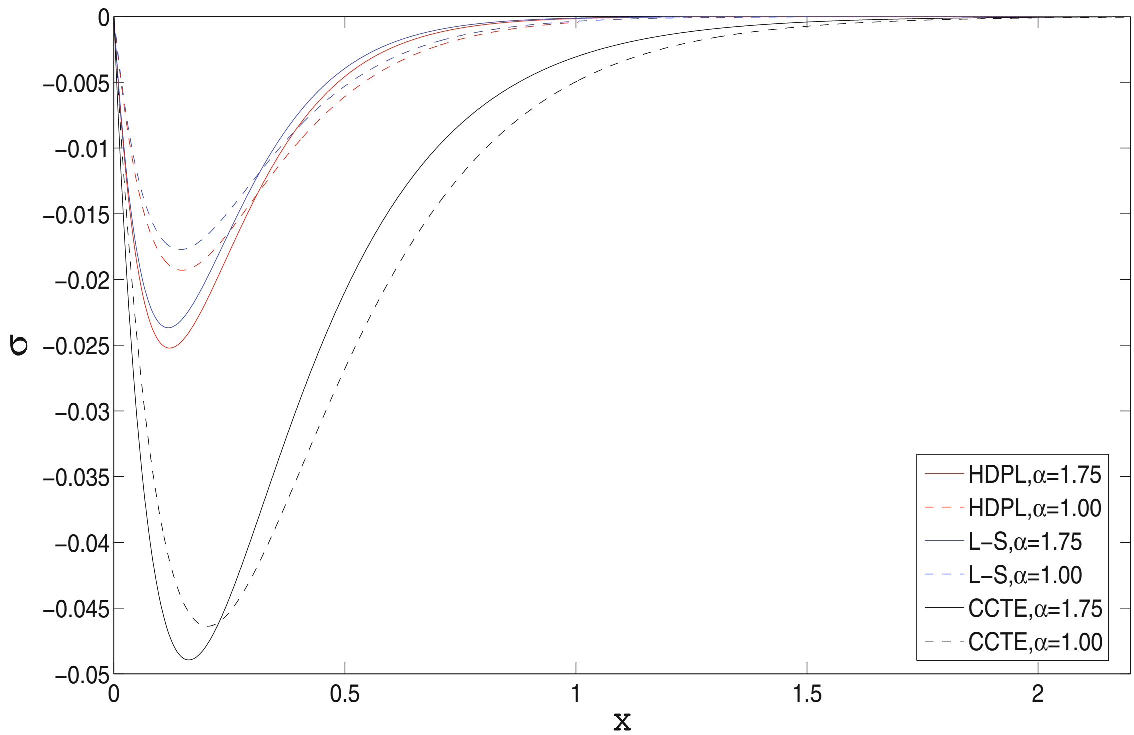

Fig. 2. The mean stress (σ) distribution vs. distance x for different α at $t = 0.3$.

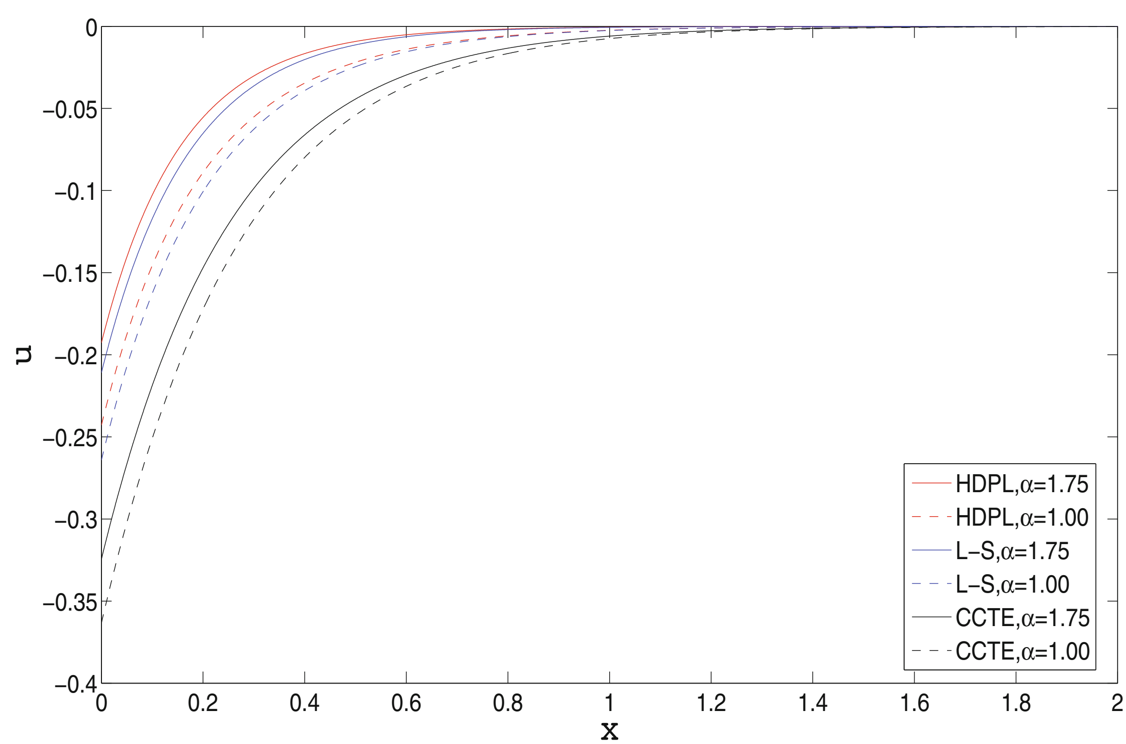

Fig. 3. The displacement distribution u vs. distance x for different α at $t = 0.3$.

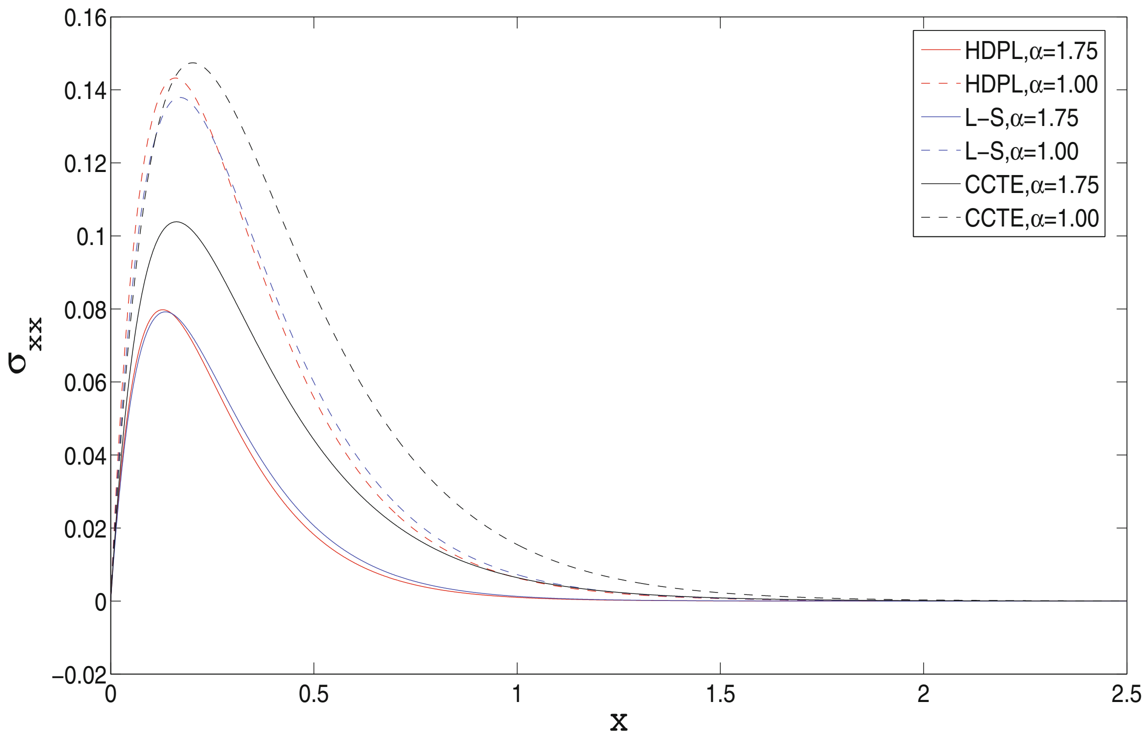

Fig. 4. The stress distribution σ_{xx} vs. distance x for different α at $t = 0.3$.

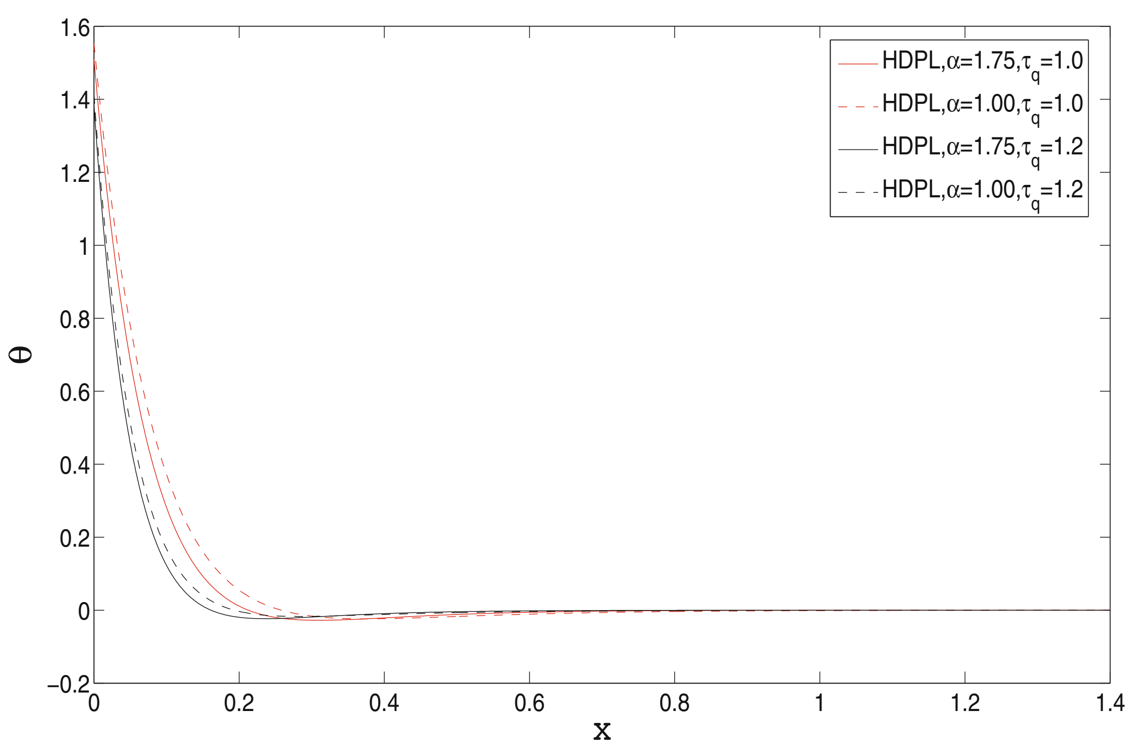

Fig. 5. The temperature distribution vs. distance x for different τ_q at $t = 0.3$.

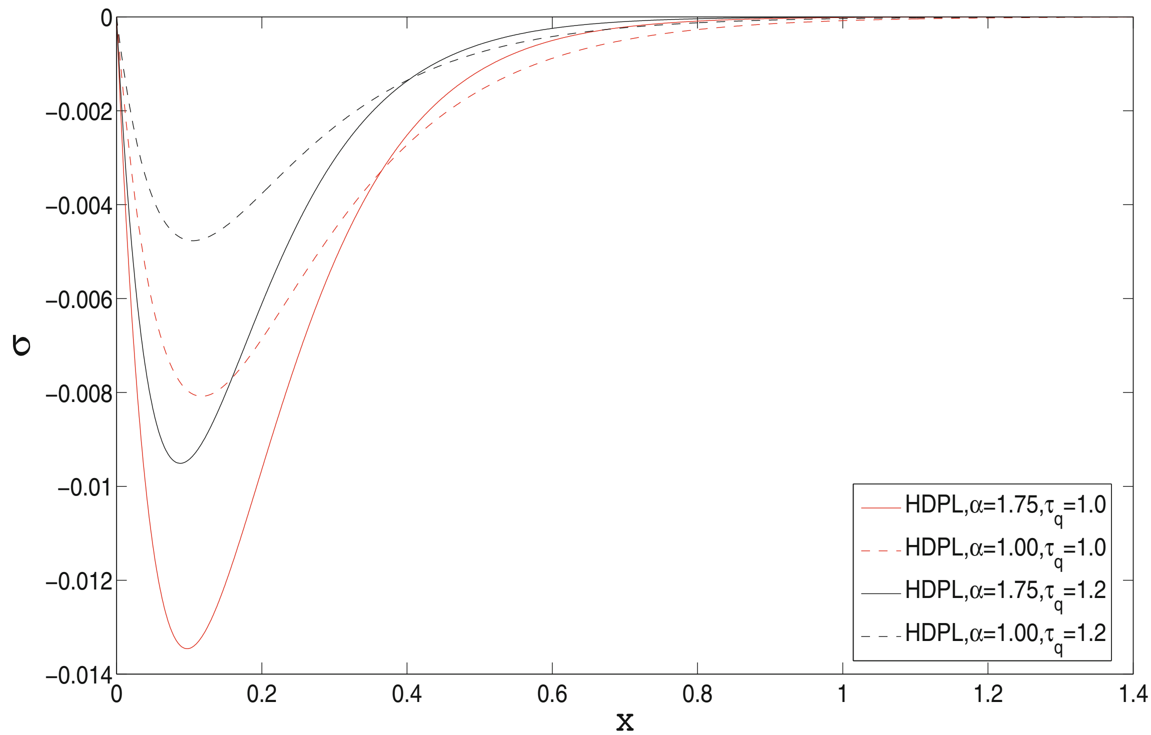

Fig. 6. The mean stress (σ) distribution vs. distance x for different τ_q at $t = 0.3$.

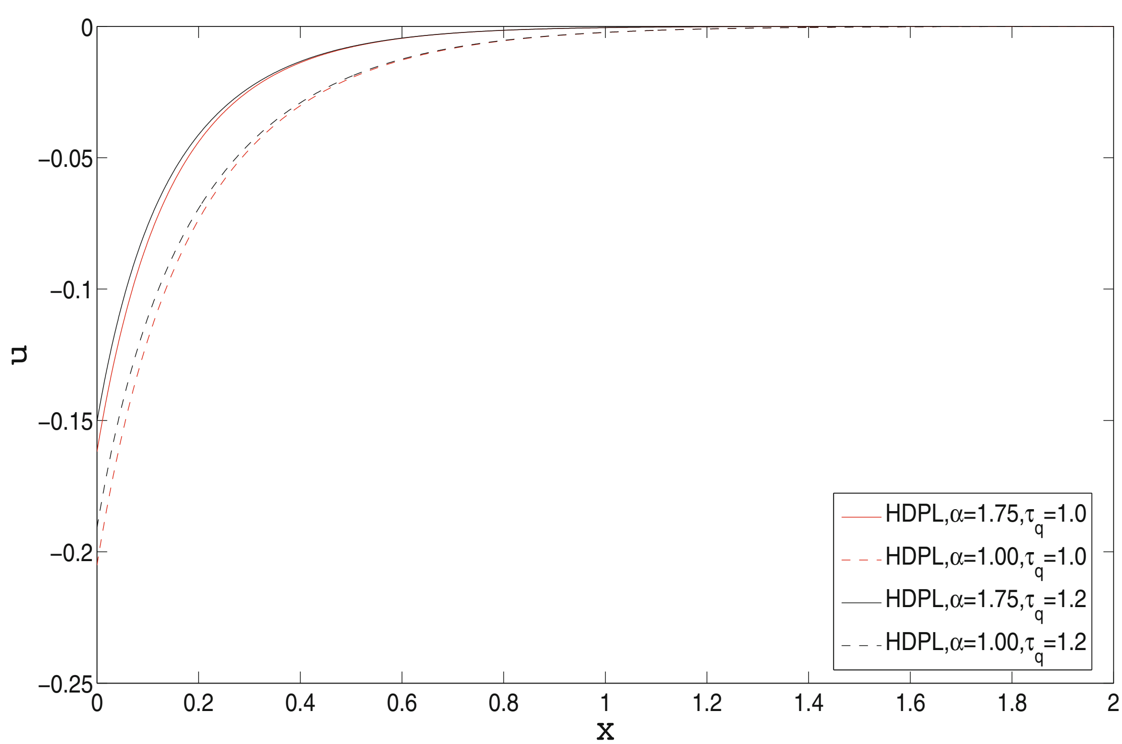

Fig. 7. The displacement distribution u vs. distance x for different τ_q at $t = 0.3$.

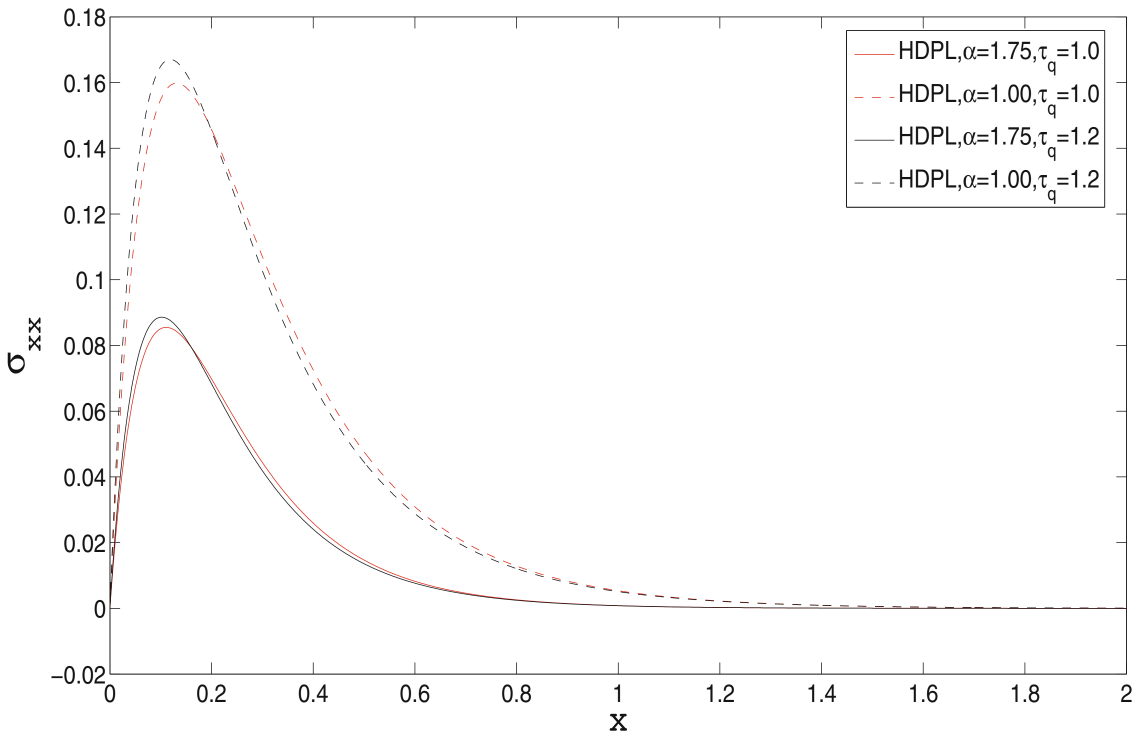

Fig. 8. The stress distribution σ_{xx} vs. distance x for different τ_q at $t = 0.3$.

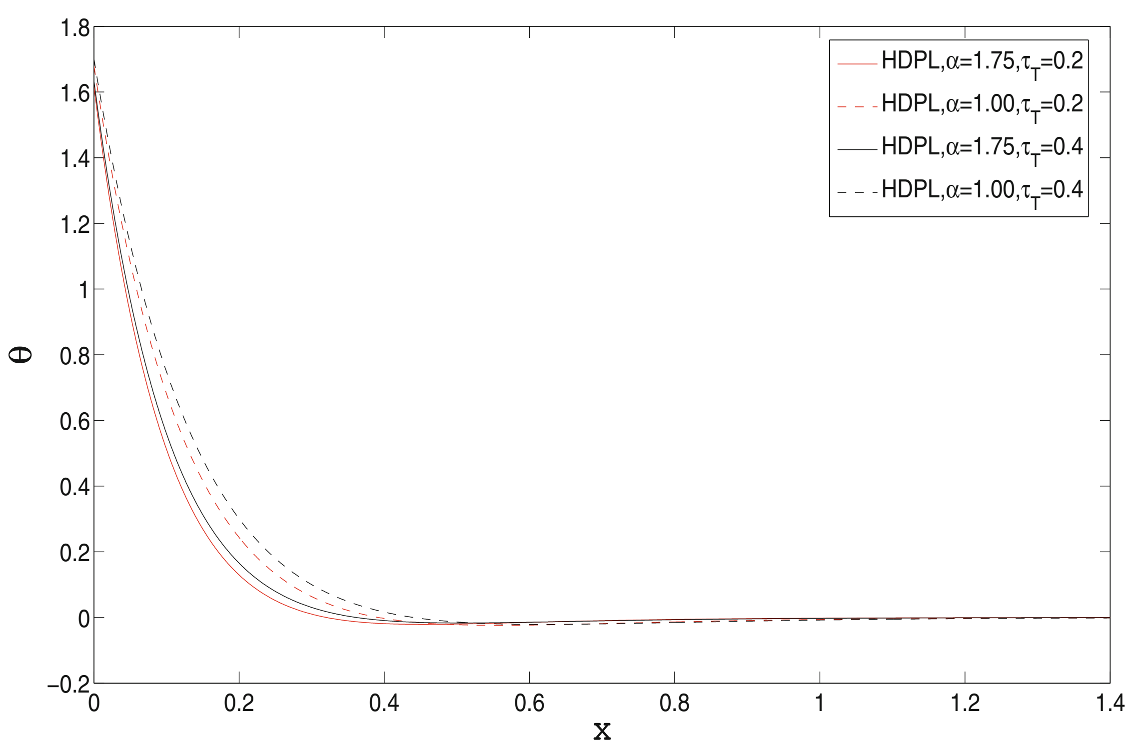

Fig. 9. The temperature distribution vs. distance x for different τ_T at $t = 0.3$.

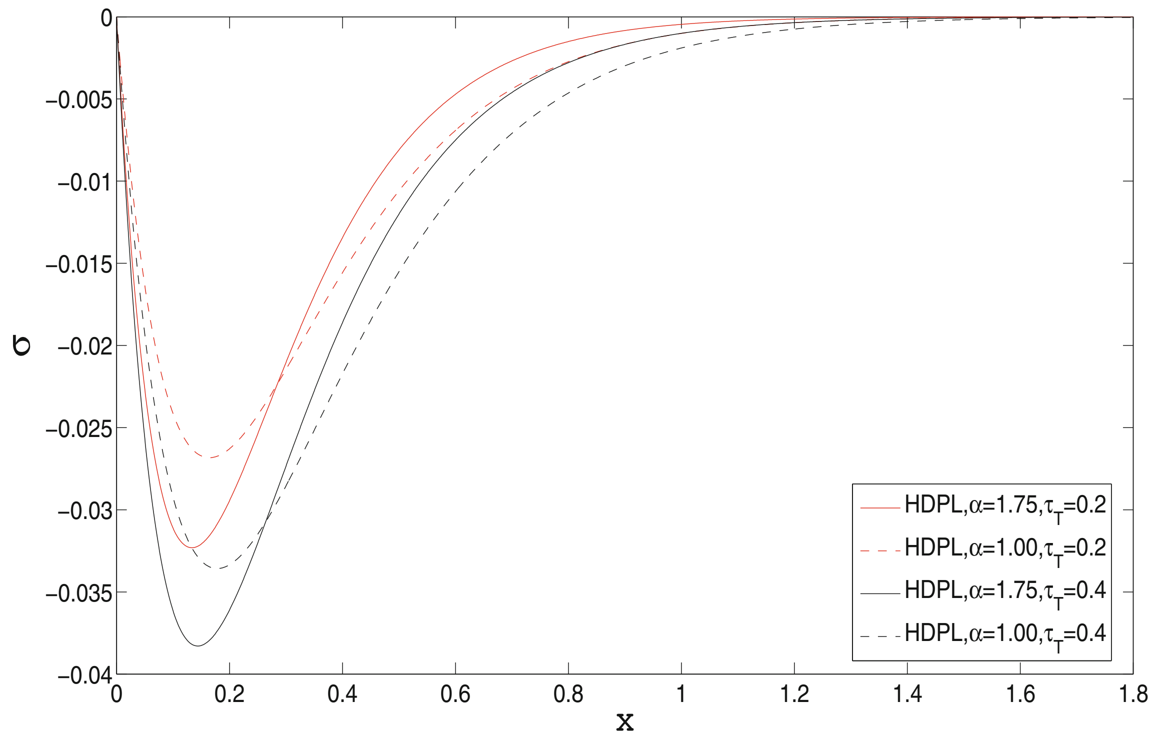

Fig. 10. The mean stress (σ) distribution vs. distance x for different τ_T at $t = 0.3$.

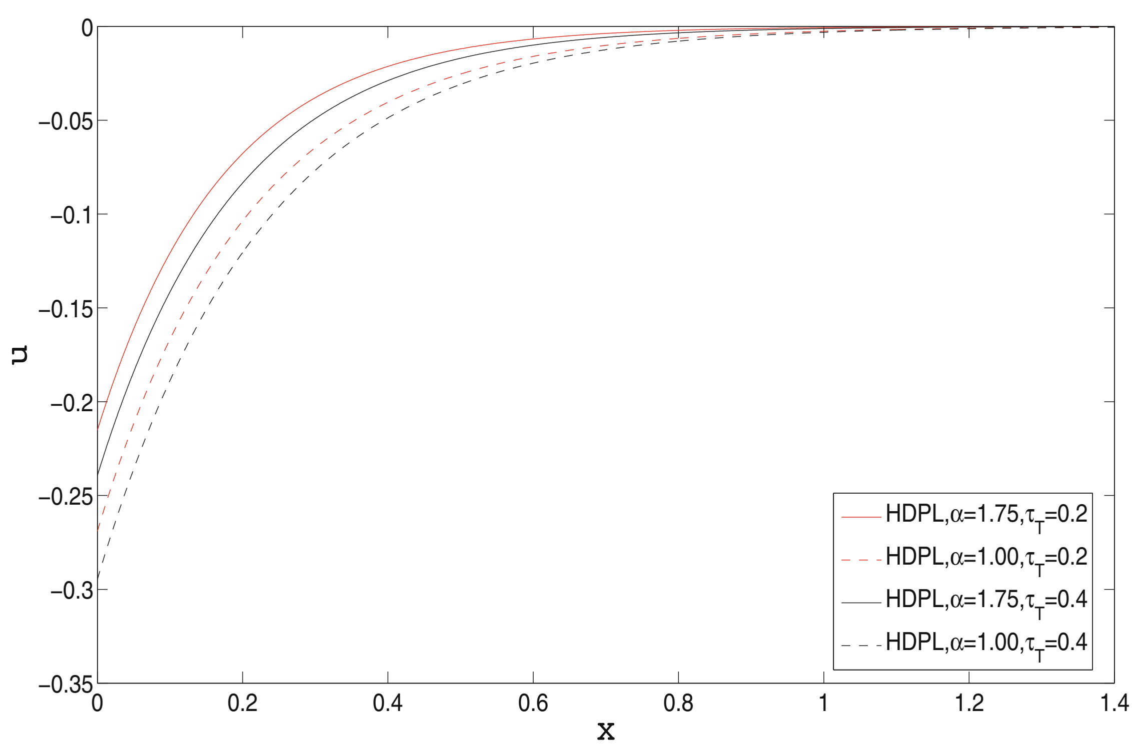

Fig. 11. The displacement u distribution for different τ_T at $t = 0.3$.

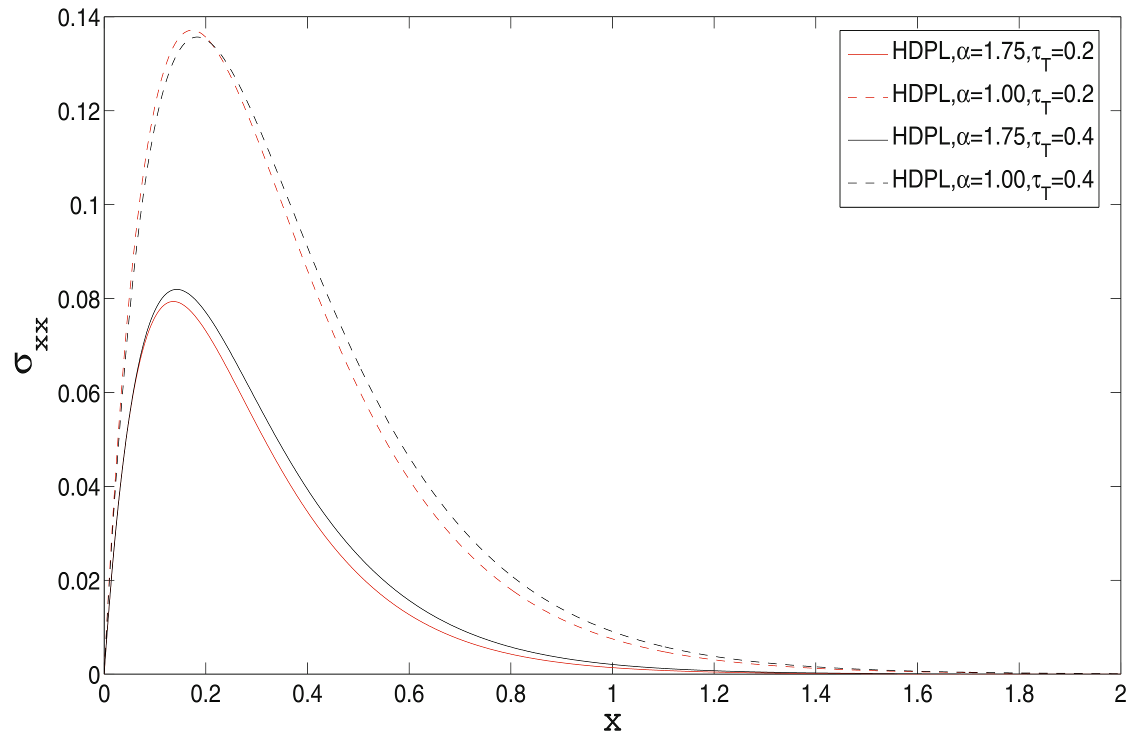

Fig. 12. The stress distribution σ_{xx} vs. distance x for different τ_T at $t = 0.3$.

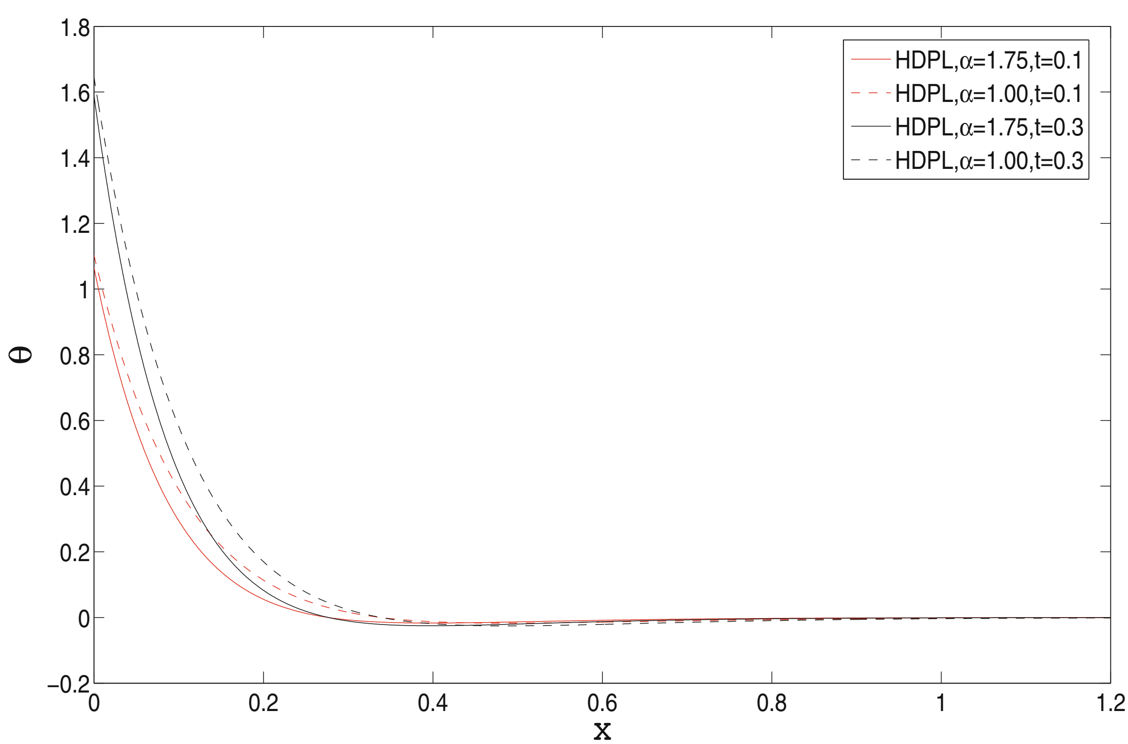

Fig. 13. The temperature distribution vs. distance x for $t = 0.1, 0.3$.

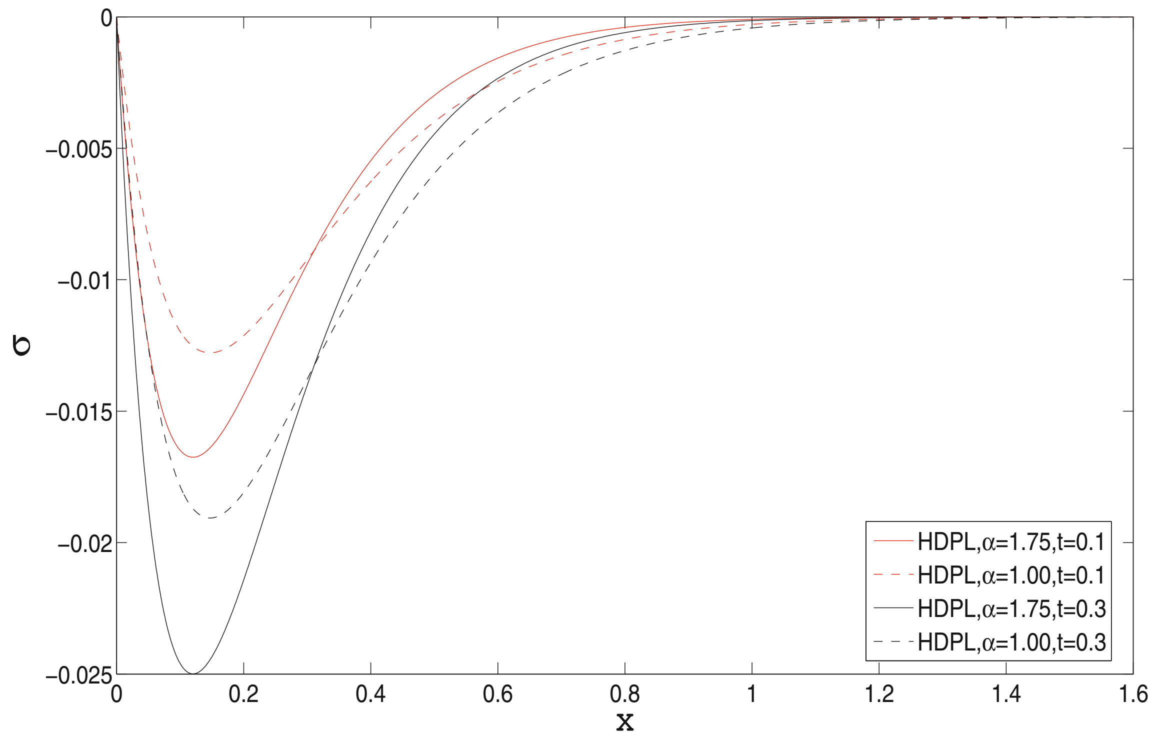

Fig. 14. The mean stress (σ) distribution vs. distance x for $t = 0.1, 0.3$.

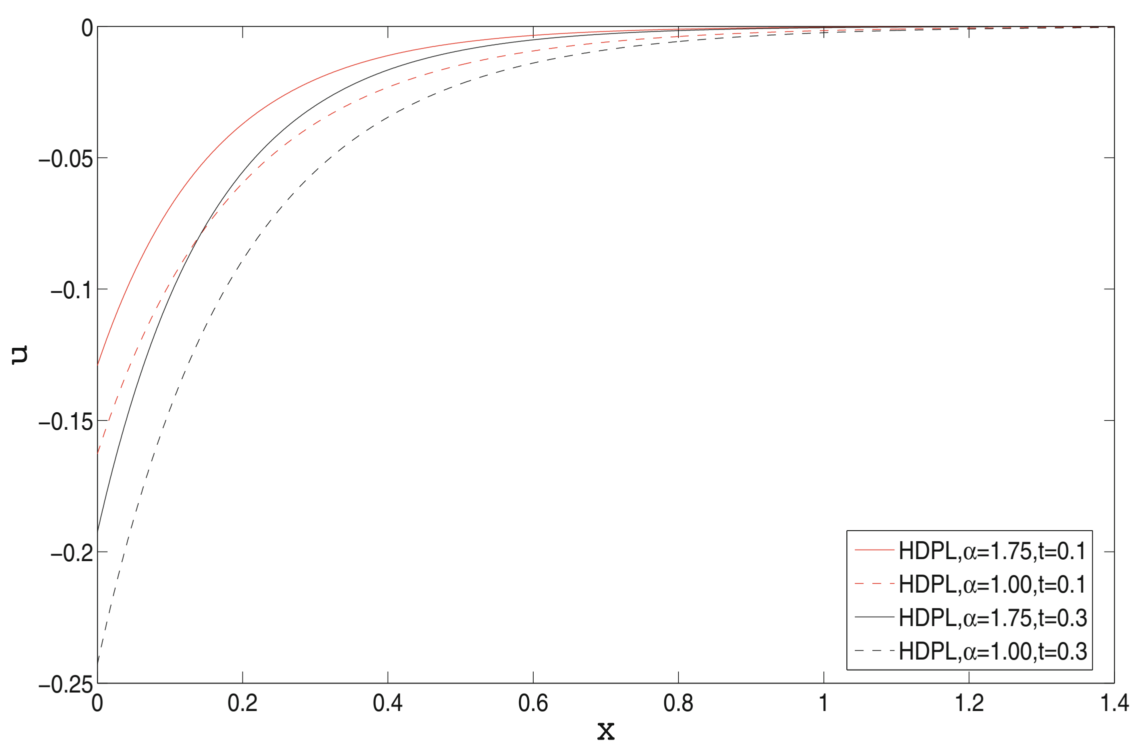

Fig. 15. The displacement u distribution for $t = 0.1, 0.3$.

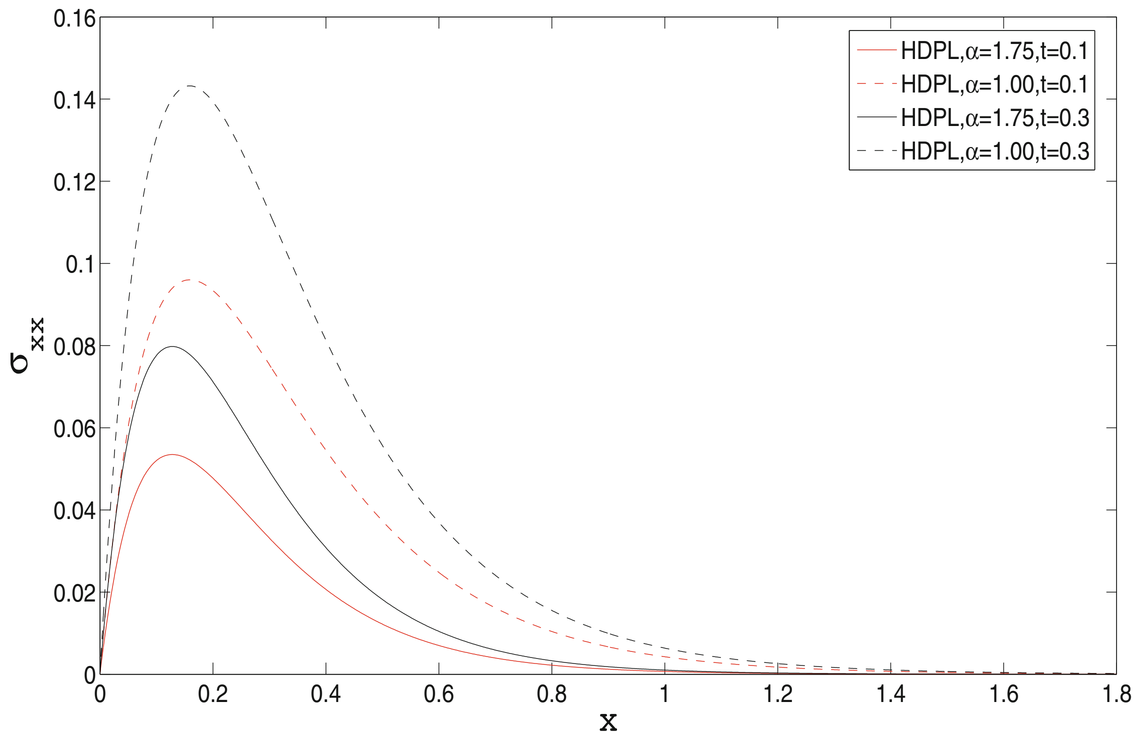

Fig. 16. The stress distribution σ_{xx} vs. distance x for $t = 0.1, 0.3$.

From Figs. 1–4, it is clear that the dependence of modulus of elasticity on the reference temperature (α^*) has decreasing effects on θ, u and σ_{xx} in the HDPL, L-S and CCTE theories while it has increasing effects on σ in the region $0 \leq x \leq 0.25$ for CCTE theory and in the region $0 \leq x \leq 0.27$ for L-S and HDPL theories and then it acts to decrease the value of σ in the region $0.25 < x \leq 2.0$ for CCTE theory and in the region $0.27 < x \leq 2.0$ for L-S and HDPL theories. The temperature θ, displacement u and the stress σ_{xx} attain their maximum values in CCTE theory for $\alpha = 1.0$ while the mean stress σ attains its maximum values in CCTE theory for $\alpha = 1.75$.

Figures 5–7 depict that the phase-lag of the heat flux (τ_q) has decreasing effects on θ, σ and u for fixed value of α^* in HDPL theory. Fig. 8 shows that τ_q has an increasing effect on σ_{xx} in the region $0 \leq x \leq 0.2$ and then it acts to decrease the value of σ in the region $0.2 < x \leq 1.4$ for $\alpha = 1.75$ and in the region $0.2 < x \leq 1.8$ for $\alpha = 1.0$.

Figures 9–11 depict that the phase-lag of the temperature gradient (τ_T) has an increasing effects on θ, σ and u for fixed value of α^* in HDPL theory. Fig. 12 shows that τ_T has decreasing effects on σ_{xx} in the region $0 \leq x \leq 0.1$ for $\alpha = 1.75$ and in the region $0 \leq x \leq 0.2$ for $\alpha = 1.0$ and then it acts to increase the value of σ_{xx} in the region $0.1 < x \leq 1.2$ for $\alpha = 1.75$ and in the region $0.2 < x \leq 1.8$ for $\alpha = 1.0$.

Figures 13–16 exhibit that the time parameter t has increasing effects on θ, σ_{xx}, u and σ for fixed α^* in HDPL theory. The temperature θ and the displacement u attain their maximum value for HDPL theory at $\alpha^* = 0$ for $(x, y, z, t) = (0.0, 0.5, 0.5, 0.3)$. The stress σ attains its maximum value for $(x, y, z, t, \alpha) = (0.16, 0.5, 0.5, 0.3, 1.75)$ while σ_{xx} attains its maximum value for $(x, y, z, t, \alpha) = (0.18, 0.5, 0.5, 0.3, 1.0)$.

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