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I. INVERSE PROBLEMS

INVERSE PROBLEMS OF FREQUENCY SOUNDING IN LAYERED MEDIA

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We consider the inverse problem of frequency sounding in a layered medium by a vertical magnetic dipole field. Uniqueness of the inverse problem solution is proved. We apply the method of minimum number of layers to obtain a stable solution of the sounding inverse problem for gradient media.

Keywords: inverse problems, frequency sounding, layered media, solution method for inverse problems

Introduction

In practice, the structure of layered media is often investigated by frequency soundings. This method measures the electromagnetic or acoustic field on the surface of a layered medium as a function of the frequency of the field excited by a given source. The frequency characteristics of the observed field uniquely determine the parameters of the layered medium.

Methods for the calculation of fields in layered media are well developed [1–4]. The fields are represented as a Bessel transform and the integrands are found by solving a differential equation. The solution of the inverse problem reduces to minimizing the residual functional between the observed field and the calculated field for various frequencies. This approach involves two difficulties: instability of the inverse problem solution and large computer time requirements to calculate the functional gradient.

In this article we consider approaches that overcome these difficulties for the case of layered medium soundings by the field of a vertical magnetic dipole.

Statement of the Problem

Consider a conducting layered medium with the following conductivity distribution:

$$\sigma = \begin{cases} \sigma_0 \approx 0 & \text{for} \quad z > 0, \\ \sigma(z) & \text{for} \quad z \in [0, -H], \\ \sigma_H & \text{for} \quad z < -H. \end{cases} \tag{1}$$

The field source (a vertical magnetic dipole with magnetic moment m_z) is located at the point $M_0 = (x_0 = 0, y_0 = 0, z_0)$, $z_0 \geq 0$.

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The electromagnetic field is defined in terms of the vertical component of the magnetic vector potential $\mathbf{A} = (0, 0, A_z)$ in the form

$$\mathbf{E} = \frac{1}{\sigma(z)} \operatorname{rot} \mathbf{A}; \qquad \mathbf{H} = \mathbf{A} + \frac{1}{i\omega\mu} \operatorname{grad} \operatorname{div} \frac{\mathbf{A}}{\sigma}. \tag{2}$$

The function $A_z(x, y, z)$ is the solution of the differential equation

$$\Delta\left(\frac{\mathbf{A}_z}{\sigma}\right) + i\omega\mu A_z = -i\omega\mu m_z \delta(x)\delta(y)\delta(z - z_0), \qquad z \in (-\infty, \infty) \tag{3}$$

with continuity condition for $\frac{A_z}{\sigma}$ and $\frac{\partial}{\partial z}\left(\frac{A_z}{\sigma}\right)$ on the discontinuities of $\sigma(z)$ and conditions of decrease at infinity $(\sigma \neq 0)$.

The solution of problem (3) is represented as the Bessel transform

$$A_z = \sigma u(\rho, z) = \sigma \int_0^\infty J_0(\lambda\rho) U(z, \lambda) \lambda \, d\lambda, \tag{4}$$

where the spectral function $U(z, \lambda)$ is the solution of the problem

$$\frac{\partial^2 U}{\partial z^2} - \eta^2 U = -2\delta(z - z_0), \qquad \eta = \sqrt{\lambda^2 - k^2}, \qquad \operatorname{Re}(\eta) > 0, \qquad k^2 = i\omega\mu\sigma \tag{5}$$

with continuity conditions for U and $\frac{\partial U}{\partial z}$, and also the condition $U \to 0$ as $|z| \to \infty$.

The fields are determined in the form

$$E_x = \frac{i\omega\mu m_z}{4\pi} \frac{\partial u}{\partial y}, \qquad E_y = -\frac{i\omega\mu m_z}{4\pi} \frac{\partial u}{\partial x}, \tag{6}$$

$$H_x = \frac{m_z}{4\pi} \frac{\partial^2 u}{\partial x \partial z}, \qquad H_y = \frac{m_z}{4\pi} \frac{\partial^2 u}{\partial y \partial z}, \tag{7}$$

$$H_z = \frac{m_z}{4\pi}\left(\frac{\partial^2 u}{\partial z^2} + k^2 u\right), \tag{8}$$

where

$$u = \int_0^\infty J_0(\lambda\rho) U(z, \lambda) \lambda \, d\lambda. \tag{9}$$

The inverse problem of soundings is usually posed for observations of the vertical magnetic field $H_z^{\text{obs}}(\omega)$ or observations of the horizontal electric field $H_x^{\text{obs}}(\omega)$. The fields are observed at a fixed distance l from the source. Given $\sigma(z)$, we find $U(z,\lambda)$ from problem (5) and by (6)–(9) we obtain

$$E_x[\omega,l,\sigma(z)] = -\frac{i\omega\mu m_z}{4\pi}\int_0^\infty J_1(\lambda l)U(z,\lambda)\lambda^2\,d\lambda, \tag{10}$$

$$H_z[\omega,l,\sigma(z)] = \frac{m_z}{4\pi}\int_0^\infty J_0(\lambda l)U(z,\lambda)\lambda^3\,d\lambda. \tag{11}$$

The fields are nonlinear operators acting on the conductivity $\sigma(z)$ and they depend on the frequency ω and the distance from the source to the observation point l.

To calculate the fields, we need to determine the function $U(z,\lambda)$, which is the solution of problem (5) on the infinite line. This problem is easily reduced to a boundary-value problem using the representation of the function $U(z,\lambda)$ for $z_0 \geq 0$ and $\sigma_0 \equiv 0$ in the form

$$U(z,\lambda) = \frac{e^{-\lambda|z-z_0|}}{\lambda} + C_1 e^{-\lambda z} \qquad \text{for} \quad z \geq 0, \tag{12}$$

$$U(z,\lambda) = C_2 e^{\eta_H z} \qquad \text{for} \quad z \leq -H, \tag{13}$$

Whence, eliminating C_1 and C_2, we obtain the boundary conditions

$$\frac{\partial U}{\partial z} + \lambda U = 2e^{-\lambda z_0} \qquad \text{for} \quad z = 0, \tag{14}$$

$$\frac{\partial U}{\partial z} - \eta_H U = 0, \quad \eta_H = \sqrt{\lambda^2 - k_H^2} \qquad \text{for} \quad z = -H. \tag{15}$$

To solve Eq. (5) with the boundary conditions (14)–(15), we introduce the function

$$Y(z) = \frac{1}{U}\frac{\partial U}{\partial z}. \tag{16}$$

Then

$$\frac{d^2U}{dz^2} = \frac{dY}{dz}U + Y\frac{dU}{dz} = \left(\frac{dY}{dz} + Y^2\right)U = \eta^2 U,$$

Whence we obtain a problem for $Y(z)$

$$\begin{cases} \dfrac{dY}{dz} + Y^2 = \eta^2 & \text{for} \quad z \in [-H,0], \\ Y = \eta_H & \text{for} \quad z = -H. \end{cases} \tag{17}$$

Solving the Cauchy problem (17), we find $Y(z=0)$. Then

$$Y(z=0)U(z=0,\lambda) = \frac{dU(z=0,\lambda)}{dz}.$$

This equality combined with boundary condition (14) gives

$$U(z=0,\lambda) = \frac{2e^{-\lambda z_0}}{\lambda + Y(z=0,\lambda)}. \quad (18)$$

High-Frequency Field Asymptotic

Further analysis and proof of the uniqueness theorem for the inverse problem requires the asymptotics of the fields (10) and (11) as $\omega \to \infty$ $(k \to \infty)$.

Fields in a homogeneous space with wavenumber k have two asymptotics: the high-frequency asymptotic $kl \gg 1$ and the low-frequency asymptotic $kl \ll 1$, where l is the distance between the source and the field observation point. In a layered medium an additional intermediate asymptotic arises, when $kl \gg 1$ in some layers and $kl \ll 1$ in other layers. This asymptotic is known as the far-zone field. It often arises at the interface between air and the conducting medium, because while $k_0 l \ll 1$ in air, we have $k(z)l \gg 1$ in the conducting medium. We thus have to find the asymptotics of fields represented by Bessel transforms (10) and (11) for $l \to \infty$.

The asymptotics of Bessel integrals are calculated in [5], where it is proved that the Bessel transform

$$I(\rho) = \int_0^\infty J_0(\lambda\rho)F(\lambda)d\lambda$$

has the following asymptotic as $\rho \to \infty$:

$$I(\rho) = \sum_{k=0}^{n} \frac{c_k}{\rho^{k+1}} \frac{d^k F(\lambda)}{d\lambda^k}\bigg|_{\lambda=0} + \frac{\varepsilon(\rho)}{\rho^{n+1}}, \quad \varepsilon(\rho) \to 0 \quad \text{as} \quad \rho \to 0, \quad (19)$$

where

$$c_{2k-1} = 0, \quad c_{2k} = (-1)^k \frac{(2k-1)!!}{(2k)!!}. \quad (20)$$

We accordingly obtain

$$I(\rho) = \frac{F(\lambda=0)}{\rho} - \frac{F''(\lambda=0)}{2\rho^3} + \frac{F^{(IV)}(\lambda=0)}{8\rho^5} + \frac{\varepsilon(\rho)}{\rho^5}. \quad (21)$$

From (21) we obtain the asymptotic of the first-order Bessel transform

$$I_1(\rho) = \int_0^\infty J_1(\lambda\rho) F(\lambda)\lambda\, d\lambda = -\frac{dI(\rho)}{d\rho}$$

$$= \frac{F(\lambda=0)}{\rho^2} - \frac{3F''(\lambda=0)}{2\rho^4} + \frac{15\ F^{(IV)}(\lambda=0)}{8\rho^6} + \frac{\varepsilon(\rho)}{\rho^6}. \quad (22)$$

Using (21) and (22), we easily obtain the high-frequency field asymptotic. Note that making the change of variable $\lambda = |k_1| t$ in the integrals (10) and (11), we obtain a large parameter $\rho = |k| l \to \infty$ as $\omega \to \infty$. This implies that the high-frequency field asymptotic is obtained as $l \to \infty$.

First consider the asymptotic of the electric field E_x with the representation (10) $F(\lambda) = \lambda U(z,\lambda)$. Then, by (22), we have as $l \to \infty$ and $z = 0$

$$E_x[x=0, y=l, z=0] \cong -\frac{3i\omega\mu m_z}{4\pi l^4} \frac{\partial U(z=0,\lambda)}{d\lambda}\bigg|_{\lambda=0}, \quad (23)$$

since $F(0) = 0$, $F''(0) = 2U'(z=0,\lambda)|_{\lambda=0}$. For the magnetic field H_z by (1) we have

$$F(\lambda) = \lambda^3 U(z,\lambda).$$

Then $F(0) = 0$, $F''(0) = 0$ and $F^{(IV)}(\lambda=0) = 24U'(z,\lambda=0)$.

Finally, by (21), we obtain the asymptotic of the magnetic field as $l \to \infty$, $z \geq 0$:

$$H_z[x=0, y=l, z\geq 0] \cong \frac{9m_z}{4\pi l^5} \frac{\partial U(z,\lambda)}{d\lambda}\bigg|_{\lambda=0}. \quad (24)$$

Thus, to find the final asymptotic for the fields we have to evaluate $\dfrac{\partial U(z,\lambda=0)}{d\lambda}$ for $z \geq 0$.

Applying expression (18) for $U(z=0,\lambda)$, we obtain for $z_0 = 0$

$$\frac{dU}{d\lambda}\bigg|_{\lambda=0} = -\frac{2}{Y^2(z=0,\lambda=0)}\left(1 + \frac{dY}{d\lambda}\bigg|_{\lambda=0}\right).$$

Note that by problem (17) $Y(z,\lambda)$ depends on λ^2, because $\eta = \sqrt{\lambda^2 - k^2(z)}$. Therefore $\dfrac{dY}{d\lambda}\bigg|_{\lambda=0} = 0$ and finally

$$\frac{dU}{d\lambda}\bigg|_{\lambda=0} = -\frac{2}{Y^2(z=0,\lambda=0)}. \quad (25)$$

Given $\dfrac{dU}{d\lambda}\bigg|_{\lambda=0}$, from (23) and (24) we find the high-frequency asymptotics of E_x and H_z

$$E_x \to \frac{3i\omega\mu m_z}{2\pi l^4 Y^2(z=0,\lambda=0)} \quad \text{as} \quad \omega \to \infty, \quad (26)$$

$$H_z \to \frac{9m_z}{2\pi l^5 Y^2(z=0,\lambda=0)} \quad \text{as} \quad \omega \to \infty. \tag{27}$$

Consider the function $Y_0(z) = Y(z,\lambda=0)$. By (17) $Y_0(z)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{dY_0}{dz} + Y_0^2 = -k^2(z), \quad z \in [-H,0], \quad k^2(z) = i\omega\mu\sigma(z), \\ Y_0(z=-H) = -ik_H, \quad k_H = \sqrt{i\omega\mu\sigma_H}, \quad \mathrm{Re}(k_H) > 0. \end{cases} \tag{28}$$

This problem is identical with the problem of determining the admittance of a layered medium for the magnetotelluric field [6]. Hence, $Y(z=0,\lambda=0) = Y_0(z=0)$ is the admittance of the layered medium that determines the high-frequency asymptotic of the electromagnetic field excited by a vertical magnetic dipole on the surface of the layered medium.

To finally determine the high-frequency field asymptotic, we need to find the asymptotic of the admittance $Y_0(z)$. It is easily shown that if $\sigma(z)$ is a continuous function with a bounded derivative, the admittance has the asymptotic

$$Y_0(z) = (1-i)\sqrt{\frac{\omega\mu\sigma(z)}{2}}\left(1 + O\left(\frac{1}{\sqrt{\omega\mu}}\right)\right). \tag{29}$$

To prove this result, we introduce the function $X(z)$ such that

$$Y_0(z) = \sqrt{\omega\mu}X(z).$$

Then by (28) we obtain the following problem for $X(z)$:

$$\begin{cases} \dfrac{1}{\sqrt{\omega\mu}}\dfrac{dX}{dz} + X^2(z) = -i\sigma(z), \quad z \in [-H,0], \\ X(z=-H) = (1-i)\sqrt{\dfrac{\sigma_H}{2}}. \end{cases} \tag{30}$$

Equation (20) is an equation with a small parameter $\varepsilon = \dfrac{1}{\sqrt{\omega\mu}}$ multiplying the highest order derivative. The derivation of the asymptotic solution of this problem has been investigated in [7]. The solution of (30) is represented in the form

$$X(z) = \bar{x}(z) + O(\varepsilon),$$

where $\bar{x}(z)$ is the solution of problem (20) with $\varepsilon = \dfrac{1}{\sqrt{\omega\mu}} = 0$, $\bar{x}^2 = -i\sigma(z)$ for $z \in [-H,0]$.

The solution of this equation consistent with the initial condition is $\bar{x}(z) = (1-i)\sqrt{\frac{\sigma(z)}{2}}$. Then

$$Y_0(z) = (1-i)\sqrt{\frac{\omega\mu\sigma(z)}{2}}\left(1+O\left(\frac{1}{\sqrt{\omega\mu}}\right)\right).$$

We have proved the asymptotic (29).

Substituting the asymptotic

$$Y_0(z=0) \to (1-i)\sqrt{\frac{\omega\mu\sigma(0)}{2}} \quad \text{as} \quad \omega \to \infty$$

in (26) and (27), we obtain the field asymptotics

$$E_x \to -\frac{3m_z}{2\pi l^4 \sigma(0)} \quad \text{as} \quad \omega \to \infty, \tag{31}$$

$$H_z \to -\frac{9im_z}{2\pi l^5 \omega\mu\sigma(0)} \quad \text{as} \quad \omega \to \infty. \tag{32}$$

Investigation of the Inverse Problem

Tikhonov [8] has shown that different conductivity distributions $\sigma^{(1)}(z) \neq \sigma^{(2)}(z)$ in the layered medium produce different admittances $Y_0{}^{(1)}(\omega) \neq Y_0{}^{(2)}(\omega)$. Since the high-frequency field asymptotics are inversely proportional to the admittance $Y_0(\omega)$, this implies that different distributions $\sigma^{(1)}(z)$ and $\sigma^{(2)}(z)$ correspond to different frequency characteristics of the fields $E_x^{(1)} \neq E_x^{(2)}$, $H_z^{(1)} \neq H_z^{(2)}$. This in turn implies uniqueness of the solution of the inverse problem of frequency sounding, because only a unique conductivity distribution $\sigma(z)$ may correspond to a given frequency characteristic of the field E_x or H_z.

To solve the inverse problem, instead of the field itself we use its normalized value, the so-called "apparent resistivity". Consider a homogeneous half-space with conductivity σ. Then by (31) and (32), we easily find the resistivity $\rho = 1/\sigma$ of this half-space from the high-frequency field asymptotics in the form:

$$\rho_E = \frac{2\pi l^4}{3m_z}|E_x|, \quad \rho_H = \frac{2\pi l^5 \omega\mu}{9m_z}|H_z|. \tag{33}$$

Given the frequency dependence of the fields, we evaluate $\rho_E(\omega)$ and $\rho_H(\omega)$ from (33), which give the apparent resistivities of the medium with respect to the electric and magnetic field. It determines the resistivity of the homogeneous half-space when the fields at a given frequency are identical with the fields on the surface of the layered medium. The apparent resistivity taken as a function of frequency qualitatively reflects the variation of conductivity with depth.

When solving the inverse problem, we have to find a conductivity distribution $\sigma(z)$ which for a given source produces a field that is identical with the observed fields at the corresponding frequencies. The apparent resistivity may vary by several orders of magnitude. The frequency also varies by several orders of magnitude. It is therefore advisable to compare the log of apparent resistivity with logged frequency, i.e., the residual functional in the inverse problem is defined as

$$|\Delta\rho| = \int_{\omega_0}^{\omega_m} \ln^2 \frac{\rho^c(\omega)}{\rho^e(\omega)} \frac{d\omega}{\omega}, \tag{34}$$

where $\rho^c(\omega)$ is the calculated apparent resistivity, $\rho^e(\omega)$ is the experimentally observed apparent resistivity, ω_0 is the minimum field frequency, and ω_m is the maximum field frequency in the series of observations.

When solving the inverse problem, we can first determine the conductivity of the lower half-space σ_H. It is obtained from the low-frequency field asymptotic. To this end, we have to find the asymptotic as $\omega \to 0$ of the admittance function $Y(z)$, which is the solution of problem (17). Represent $Y(z)$ in the form

$$Y(z) = \lambda + X(z). \tag{35}$$

Then, by (17), we have the following problem for $X(z)$:

$$\begin{cases} \dfrac{dX(z)}{dz} + 2\lambda X(z) + X^2(z) = -i\omega\mu\sigma(z), \quad z \in [-H, 0], \\ Y(z=-H) = \sqrt{\lambda^2 - i\omega\mu\sigma_L} - \lambda. \end{cases} \tag{36}$$

Note that as $\omega \to 0$ problem (36) reduces to a Cauchy problem for a homogeneous equation and zero initial condition. Thus, $X(z) \to 0$ as $\omega \to 0$. For small $X(z)$ we can ignore $X^2(z)$ in Eq. (36), which gives the linear Cauchy problem

$$\begin{cases} \dfrac{dX(z)}{dz} + 2\lambda X(z) = -i\omega\mu\sigma(z), \quad z \in [-H, 0], \\ X(z=-H) = \eta_H - \lambda, \qquad \eta_H = \sqrt{\lambda^2 - i\omega\mu\sigma_L}. \end{cases} \tag{37}$$

The solution of this problem is

$$X(z) = (\eta_H - \lambda)e^{-2\lambda(H+z)} - i\omega\mu \int_{-H}^{H} \sigma(\xi) e^{-2\lambda(z-\xi)} d\xi,$$

whence with $z = 0$, using (35), we find the asymptotic as $\omega \to 0$

$$Y(z=0) = Y_0(\lambda) = \lambda + (\eta_H - \lambda)e^{-2\lambda H} - i\omega\mu \int_{-H}^{0} \sigma(\xi) e^{2\lambda\xi} d\xi. \tag{38}$$

Since $\eta_H - \lambda \to -\dfrac{-i\omega\mu\sigma_H}{2\lambda}$ as $\omega \to 0$, we finally obtain

$$Y(z=0,\lambda) = \lambda - i\omega\mu Q(\lambda) \quad \text{as} \quad \omega \to 0, \tag{39}$$

where

$$Q(\lambda) = \sigma_H \frac{e^{-2\lambda H}}{2\lambda} \int_{-H}^{0} \sigma(z) e^{2\lambda z} dz \,. \tag{40}$$

Substituting (39) in (18), we obtain with $z_0 = 0$

$$U(z=0,\lambda) = \frac{2}{2\lambda - i\omega Q(\lambda)} \approx \frac{1}{\lambda} + \frac{i\omega\mu Q(\lambda)}{2\lambda^2}. \tag{41}$$

Given $U(z=0,\lambda)$, we apply (10) and (11) to find the low-frequency field asymptotics:

$$E_x = \frac{i\omega\mu m_z}{4\pi l^2} + \frac{\omega^2\mu^2 m_z}{8\pi} \int_0^{\infty} J_1(\lambda l) Q(\lambda)\, d\lambda, \tag{42}$$

$$H_z = \frac{m_z}{4\pi l^3} + \frac{i\omega\mu m_z}{8\pi} \int_0^{\infty} J_0(\lambda l) Q(\lambda) \lambda \, d\lambda. \tag{43}$$

Substituting expression (40) for $Q(\lambda)$ and applying the standard integrals

$$\int_0^{\infty} J_0(\lambda l) e^{\lambda z}\, d\lambda = \frac{1}{\sqrt{l^2 + z^2}} \quad \text{for} \quad z \le 0,$$

$$\int_0^{\infty} J_1(\lambda l) e^{\lambda z}\, d\lambda = \frac{1}{l}\left(1 + \frac{z}{\sqrt{l^2 + z^2}}\right) \quad \text{for} \quad z \le 0,$$

$$\int_0^{\infty} J_1(\lambda l) \frac{e^{\lambda z}}{\lambda}\, d\lambda = \frac{1}{l}\left(z + \sqrt{l^2 + z^2}\right) \quad \text{for} \quad z \le 0,$$

we find the low-frequency field asymptotics

$$E_x = \frac{i\omega\mu m_z}{4\pi l^2} + \frac{\omega^2\mu^2 m_z}{8\pi}\left(\int_{-H}^{0} \sigma(z)\left(1 + \frac{z}{\sqrt{l^2 + 4z^2}}\right) dz - \sigma_H\left(H - \frac{\sqrt{l^2 + 4H^2}}{2}\right)\right), \tag{44}$$

$$H_z = \frac{m_z}{4\pi l^3} + \frac{i\omega\mu m_z}{16\pi}\left(\frac{\sigma_H}{\sqrt{l^2 + 4H^2}} - 4\int_{-H}^{0} \frac{\sigma(z) z}{\left(\sqrt{l^2 + 4z^2}\right)^3} dz\right). \tag{45}$$

With $l \gg H$, the low-frequency field asymptotics are simplified and take the form

$$E_x = \frac{i\omega\mu m_z}{4\pi l^2}\left(1 - \frac{i\omega\mu\sigma_H l^2}{4}\right), \qquad H_z = \frac{m_z}{4\pi l^3}\left(1 + \frac{i\omega\mu\sigma_H l^2}{4}\right).$$

It is easy to see that in this case the low-frequency asymptotics produce the lower half-space conductivity σ_H.

Solution Methods for the Inverse Problem

The main difficulty with inverse problems is the instability of their solution. In accordance with the regularization theory for unstable problems [9], the solution is stable only if it belongs to a compact class of functions. Solution of the inverse problems of frequency electromagnetic soundings is considered on conductivity functions $\sigma(z)$ from the class of piecewise-analytical functions with finitely many discontinuities. It is for this class of functions $\sigma(z)$ that the uniqueness theorem for the inverse problem has been proved. In this case, the inverse operator of the inverse problem is bounded, and as the error of field observations approaches zero, the approximate solution of the inverse problem approaches the exact solution. However, if the compact set of solutions is large, the inverse operator may have a fairly large norm.

This implies that the error in the solution of the inverse problem may be much greater than the error of field observations, which is unacceptable. The resolution of the sounding method has to match the detail with which the solution is obtained [10]. It is necessary to impose stricter constraints on the set of solutions of the inverse problem, i.e., reduce the solution detail and correspondingly reduce the solution error due to errors in the data. This is the essence of the method of minimum number of layers for solving the inverse problem.

In this method, the compact solution set is the layered medium with a piecewise-constant conductivity distribution

$$\sigma(z) = \sigma_n \quad \text{for} \quad z \in [z_{n-1}, z_n], \quad n \in [1, N], \quad z_0 = 0, \quad z_N = H. \tag{46}$$

The conductivity of the underlying half-space σ_H with $z > H$ is assumed known from the low-frequency field asymptotic. Thus, the compact space is described by $2N$ parameters (σ_n, z_n), $n \in [1, N]$, that minimize the apparent resistivity residual (34).

The solution is first found for a two-layered medium, then for a three-layered medium, and so on, until the apparent resistivity residual $|\Delta\rho|$ becomes comparable with the error δ in the determination of ρ_k from field observations. As a result, we obtain an approximation of $\sigma(z)$ in the form of a piecewise-constant function with a minimum number of discontinuities, such that the corresponding field deviates from the observed field by less than the observation error. This method fairly quickly produces the inverse-problem solution, because usually there are $n \le 6$ layers.

This method, however, is inappropriate for gradient media, where the number of conductivity discontinuities is small but between discontinuities the conductivity varies strongly with depth. In this case it is advisable to apply a compact set of solutions in the form a piecewise-linear conductivity distribution

$$\sigma(z) = \sigma_n + \sigma_n'(z - z_{n-1}) \quad \text{for} \quad z \in [z_{n-1}, z_n], \quad n \in [1, N], \quad z_0 = 0, \quad z_N = H, \tag{47}$$

where z_n is the depth of the discontinuity in $\sigma(z)$, $\sigma_n = \sigma(z_n + 0)$, and σ_n' is the mean conductivity gradient in layer $z \in [z_{n-1}, z_n]$. Thus, the solution set of the inverse problem is determined by $3N$ parameters

$(\sigma_n, \sigma'_n, z_n)$, $n \in [1, N]$. It is on this compact solution set that the method of minimum number of layers is implemented.

The proposed method quickly solves the inverse problem of frequency sounding for layered medium. It is easily extended to various inverse problems with one-dimensional distribution of the sought parameter.

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