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OPTIMIZATION OF OPEN-PIT MINING BY THE GRADIENT METHOD

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A model of a two-dimensional open-pit mine is proposed and an optimal control problem is formulated with mixed constraints on the control parameters and an integral objective functional. The problem is discretized in one of the phase variables and solved by the gradient projection method with penalty functions. Numerical results illustrating the the method are also represented.

The article considers the applied economic problem of open-pit mining. A model of a two-dimensional open-pit mine is proposed and an optimal control problem is formulated with mixed constraints on the control parameters and an integral objective functional. The problem is discretized in one of the phase variables and solved by the gradient projection method with penalty functions. Numerical results illustrating the adequacy of the method are reported.

1. A Continuous Plane Model

An open-pit mine is a three-dimensional underground ore body bounded by a parallelepiped. In this section, we describe the process of open-pit mining in terms of partial differential equations and an integral objective functional, defined in our problem as a discounted flow of earnings produced by the excavation of the mine. To simplify the analysis, we drop one of the horizontal dimensions and examine a two-dimensional open-pit mine of rectangular shape. This model retains all the features of the full-dimensional model and is easily extended to 3D.

We introduce a coordinate system attached to the mine. The Ox axis is directed horizontally, the Oy axis points vertically down. The ore body is bounded by a rectangular region $0 \le x \le x_{\max}$ and $0 \le y \le y_{\max}$. The open-pit surface can be described by the curve $y(t,x)$ that varies in time in accordance with the known mining rate (see Fig. 1).

In the mathematical model of open-pit mining of an ore body, we use the following variables and parameters:

$y(t,x)$ — the depth of the open-pit mine at point x at time t;

$u(t,x)$ — the vertical rate of mining at point x at time t;

$\rho_0(t)$ — the minimum useful ore concentration (MC);

$m(x,y)$ — the density of the ore at the point (x,y);

$\rho(x,y)$ — ore concentration at the point (x,y);

q — the cost of extracting a unit of ore;

p — the cost of processing a unit of ore;

s — the market price of the extracted mineral;

$Q_{\max}$ — the maximum efficiency of the extracting equipment;

$P_{\max}$ — the maximum efficiency of the processing equipment;

T — the fixed duration of the mining process.

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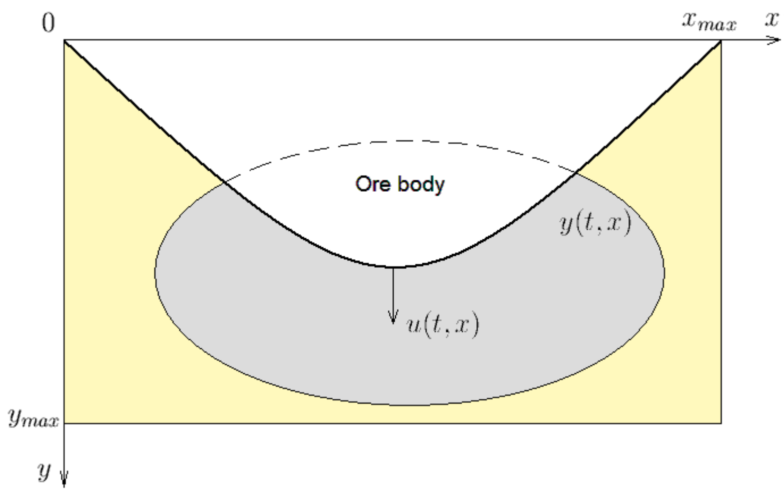

Fig. 1. A plane model of an open-pit mine.

The dynamics of the open-pit surface $y(t,x)$ is described by the partial differential equations

$$\frac{\partial y(t,x)}{\partial t} = u(t,x), \qquad t \in [0,T], \quad x \in [0, x_{\max}],$$

with the boundary condition

$$y(0,x) = 0, \quad x \in [0, x_{\max}].$$

Remark. In a more general case, the initial surface (the landscape) is not necessarily plane, and the boundary condition takes the form

$$y(0,x) = y_0(x), \quad x \in [0, x_{\max}],$$

where $y_0(x)$ is a continuous function on $[0, x_{\max}]$.

Geoengineering stability requirements impose a constraint on the maximum slope angle of the open-pit wall. We denote this angle by α_0. This constraint is equivalent to a Lipschitz condition in the variable x on the function $y(t,x)$ with a constant tan $\tan\alpha_0$, i.e., for every $x_1, x_2 \in [0, x_{\max}]$ and every $t \in [0,T]$,

$$|y(t,x_1) - y(t,x_2)| \le \tan\alpha_0 |x_1 - x_2|.$$

The geoengineering stability condition imposes an additional boundary condition

$$y(t,0) = y(t, x_{\max}) = 0, \quad t \in [0,T],$$

as it is impossible to extend the open-pit mine beyond the limits of the given rectangular region.

The mining rate satisfies the natural constraint

$$u(t,x) \ge 0, \qquad t \in [0,T], \quad x \in [0, x_{\max}],$$

i.e., the open pit cannot be buried.

We introduce the indicator function

$$I\big(\rho_0(t), \rho(x,y)\big) = \begin{cases} 1, & \text{if} \quad \rho \geq \rho_0(t), \\ 0, & \text{if} \quad \rho < \rho_0(t), \end{cases}$$

that depends on the minimum concentration of the mineral and its actual concentration at some point (x, y). This function enables us to determine if the ore mined at a given point is processed (value 1) or not (value 0).

The bounded efficiency of the extracting equipment imposes the constraint

$$\int_0^{x_{\max}} u(t,x) m\big(x, y(t,x)\big) dx \leq Q_{\max}, \quad t \in [0,T],$$

and the constraint on processing capacity is representable in the form

$$\int_0^{x_{\max}} u(t,x) m(x, y(t,x)) I(\rho_0(t), \rho(x, y(t,x))) dx \leq P_{\max}, \quad t \in [0,T].$$

The total discounted flow of earnings from open-pit mining is represented in integral form as

$$\int_0^T e^{-\delta t} \int_0^{x_{\max}} \big[-q + \big(s\rho(x, y(t,x)) - p\big) I\big(\rho_0(t), \rho(x, y(t,x))\big)\big] u(t,x) m\big(x, y(t,x)\big) dx dt \to \max,$$

where δ is the discounting factor. This flow of earnings is to be maximized in our problem.

Treating the functions $u(t,x)$ and $\rho_0(t)$ as the control parameters, we obtain an optimal control problem for a partial differential equation with an integral objective functional and a fixed terminal time

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = u(t,x), \quad t \in [0,T], \quad x \in [0, x_{\max}], \\ y(0,x) = 0, \quad x \in [0, x_{\max}], \\ y(t,0) = y(t, x_{\max}) = 0, \quad t \in [0,T], \\ u(t,x) \geq 0, \quad t \in [0,T], \quad x \in [0, x_{\max}], \\ \rho_0(t) \geq 0, \quad t \in [0,T], \\ \displaystyle\int_0^{x_{\max}} u(t,x) m(x, y(t,x)) dx \leq Q_{\max}, \quad t \in [0,T], \\ \displaystyle\int_0^{x_{\max}} u(t,x) m(x, y(t,x)) I(\rho_0(t), \rho(x, y(t,x))) dx \leq P_{\max}, \quad t \in [0,T], \\ |y(t,x_1) - y(t,x_2)| \leq \tan\alpha_0 |x_1 - x_2|, \quad x_1, x_2 \in [0, x_{\max}], \quad t \in [0,T], \\ \displaystyle\int_0^T e^{-\delta t} \int_0^{x_{\max}} \big[-q + (s\rho(x, y(t,x)) - p) I(\rho_0(t), \rho(x, y(t,x)))\big] \\ \qquad\qquad\qquad \times\, u(t,x) m(x, y(t,x)) dx dt \ \to\ \max. \end{cases} \tag{1}$$

2. Discrete Plane Model

For purposes of numerical solution, we discretize the optimal control problem (1) and reduce it to an optimal control problem for a system of ordinary differential equations.

We introduce a uniform grid of points on the interval $[0, x_{\max}]$,

$$\{x_i\}_{i=0}^{N} \quad \text{with the increment} \quad \Delta = \frac{x_{\max}}{N},$$

and change the variables

$$y(t, x_i) = y_i(t), \quad i \in \{0, N\},$$

$$u(t, x_i) = u_i(t), \quad i \in \{1, N-1\}.$$

We obtain a new optimal control problem, but this time without partial differential equations. We solve this problem by the gradient projection method:

$$\begin{cases} \dfrac{dy_i(t)}{dt} = u_i(t), \quad t \in [0,T], \quad i \in \{1, N-1\}, \\ y_i(0) = 0, \quad i \in \{1, N-1\}, \\ y_0(t) = y_N(t) = 0, \quad t \in [0,T], \\ u_i(t) \geq 0, \quad t \in [0,T], \quad i \in \{1, N-1\}, \\ \rho_0(t) \geq 0, \quad t \in [0,T], \\ \displaystyle\sum_{i=1}^{N-1} u_i(t) m(x_i, y_i(t))\Delta \leq Q_{\max}, \quad t \in [0,T], \\ \displaystyle\sum_{i=1}^{N-1} u_i(t) m(x_i, y_i(t)) I(\rho_0(t), \rho(x_i, y_i(t)))\Delta \leq P_{\max}, \quad t \in [0,T], \\ |y_{i+1}(t) - y_i(t)| \leq \Delta \tan \alpha_0, \quad t \in [0,T], \quad i \in \{0, N-1\}, \\ \displaystyle\int_0^T e^{-\delta t} \sum_{i=1}^{N-1} \big[-q + (s\rho(x_i, y_i(t)) - p) I(\rho_0(t), \rho(x_i, y_i(t)))\big] \\ \qquad\qquad\qquad\qquad \times u_i(t) m(x_i, y_i(t))\Delta dt \ \to \ \max. \end{cases} \tag{2}$$

The solution of problem (2) consists of the optimal minimum concentration $\rho_0(t)$, the digging rate $u_i(t)$, $i \in \{1, N-1\}$, and the value of the integral functional, which gives the optimal profit from open-pit mining.

3. Gradient Projection Method

In this section, we solve problem (2) by the gradient method [1, 2]. To this end, we transform the problem to an equivalent form subject to some additional assumptions. We assume that the ore density is constant

$m(x,y) = m$ at all the points of the ore body. Furthermore, the phase constraints on the slopes of the open-pit walls and the processing constraints are allowed for by introducing penalty functions. The optimization problem takes the form

$$\begin{cases} \dfrac{dy_i(t)}{dt} = u_i(t), \quad t \in [0,T], \quad i \in \{1, N-1\}, \\ y_i(0) = 0, \quad i \in \{1, N-1\}, \\ u_i(t) \geq 0, \quad t \in [0,T], \quad i \in \{1, N-1\}, \\ \rho_0(t) \geq 0, \quad t \in [0,T], \\ \displaystyle\sum_{i=1}^{N-1} u_i(t) \leq \frac{Q_{\max}}{m\Delta}, \quad t \in [0,T], \\ \displaystyle\int_0^T \Big[e^{-\delta t} \sum_{i=1}^{N-1} \big[-q + \big(s\rho(x_i, y_i(t)) - p\big) I\big(\rho_0(t), \rho(x_i, y_i(t))\big)\big] u_i(t) m\Delta \\ \quad -A_1 \Phi_1\big(y_1(t), y_2(t), \dots, y_{N-1}(t)\big) \\ \quad -A_2 \Phi_2\big(y_1(t), y_2(t), \dots, y_{N-1}(t), \rho_0(t), u_1(t), u_2(t), \dots, u_{N-1}(t)\big)\Big] dt \ \to \ \max, \end{cases} \tag{3}$$

where the penalty functions are defined as

$$\Phi_1(y_1(t), y_2(t), \dots, y_{N-1}(t)) = \sum_{i=0}^{N-1} \phi_1(y_{i+1}(t) - y_i(t)),$$

$$\phi_1(z) = \begin{cases} (|z| - \Delta \tan \alpha_0)^2, & \text{if} \quad |z| > \Delta \tan \alpha_0, \\ 0, & \text{if} \quad |z| \leq \Delta \tan \alpha_0, \end{cases}$$

$$\Phi_2(y_1(t), y_2(t), \dots, y_{N-1}(t), \rho_0(t), u_1(t), u_2(t), \dots, u_{N-1}(t)) = \phi_2\left(\sum_{i=1}^{N-1} u_i(t) I(\rho_0(t), \rho(x_i, y_i(t)))\right),$$

$$\phi_2(z) = \begin{cases} \left(|z| - \dfrac{P_{\max}}{m\Delta}\right)^2, & \text{if} \quad |z| > \dfrac{P_{\max}}{m\Delta}, \\ 0, & \text{if} \quad |z| \leq \dfrac{P_{\max}}{m\Delta}, \end{cases}$$

$A_1 > 0,\ A_2 > 0$ are sufficiently large constants the values of which are fitted experimentally.

We write the Hamilton–Pontryagin function

$$\begin{aligned}
&H(t, y_1, \ldots, y_{N-1}, \rho_0, u_1, \ldots, u_{N-1}, \psi_1, \ldots, \psi_{N-1}) \\
&\qquad = \sum_{i=1}^{N-1} u_i \psi_i + e^{-\delta t} \sum_{i=1}^{N-1} [-q + (s\rho(x_i, y_i) - p) I(\rho_0(t), \rho(x_i, y_i))] u_i m \Delta \\
&\qquad\quad - A_1 \Phi_1(y_1, y_2, \ldots, y_{N-1}) - A_2 \Phi_2(y_1, y_2, \ldots, y_{N-1}, \rho_0, u_1, u_2, \ldots, u_{N-1}).
\end{aligned}$$

Here the conjugate variables $\psi_i(t)$ are the solution of the conjugate system

$$\begin{cases}
\dot{\psi}_i(t) = -\dfrac{\partial H(t, y_1(t), \ldots, y_{N-1}(t), \rho_0(t), u_1(t), \ldots, u_{N-1}(t), \psi_1(t), \ldots, \psi_{N-1}(t))}{\partial y_i}, \\
\psi_i(T) = 0, \quad i \in \{1, N-1\}.
\end{cases}$$

The functional gradient in problem (3) equals the gradient of the Hamilton–Pontryagin function.

The Algorithm. The gradient projection method is an iterative procedure. Initially we chose some crude approximation of the control parameters. This approximation is usually in the form of constants. Then in each iteration, we calculate a new approximation using the preceding approximation and the value of the functional gradient. The process stops when the functional no longer increases from iteration to iteration. The current approximation of the control parameters is taken as optimal.

The algorithm is described below.

Step 0. First choose an approximation for the control parameters

$$\rho_0(t), u_1(t), \ldots, u_{N-1}(t).$$

Denote the initial approximation by

$$\rho_0^0(t), u_1^0(t), \ldots, u_{N-1}^0(t).$$

Here the superscript is the number of the iteration in the gradient projection method.

Step *k*. We now describe step k of the method with $(k \geq 1)$. From the preceding iteration $(k-1)$ we have the approximation of the control parameters

$$\rho_0^{(k-1)}(t), u_1^{(k-1)}(t), \ldots, u_{N-1}^{(k-1)}(t).$$

Substituting these values in the dynamical system

$$\begin{cases}
\dfrac{dy_i(t)}{dt} = u_i^{(k-1)}(t), \qquad t \in [0, T], \quad i \in \{1, N-1\}, \\
y_i(0) = 0, \qquad\qquad\qquad i \in \{1, N-1\}.
\end{cases}$$

we obtain new values of the variables $y_1(t), \ldots, y_{N-1}(t)$.

We then solve the conjugate system in reverse time

$$\begin{cases} \dot{\psi}_i(t) = -\dfrac{\partial H(t, y_1(t), \ldots, y_{N-1}(t), \rho_0^{(k-1)}(t), u_1^{(k-1)}(t), \ldots, u_{N-1}^{(k-1)}(t), \psi_1(t), \ldots, \psi_{N-1}(t))}{\partial y_i}, \\ \psi_i(T) = 0, \quad i \in \{1, N-1\}. \end{cases}$$

The new values of the control parameters are calculated by shifting the previous value along the functional gradient:

$$\rho_0^k(t) = pr_{[0,\rho_{\max}]}\Bigg(\rho_0^{(k-1)}(t) + \alpha \frac{\partial H(t, y_1(t), \ldots, y_{N-1}(t), \rho_0^{(k-1)}(t), u_1^{(k-1)}(t), \ldots, u_{N-1}^{(k-1)}(t), \psi_1(t), \ldots, \psi_{N-1}(t))}{\partial \rho_0^{(k-1)}(t)}\Bigg),$$

where $pr_{[0,\rho_{\max}]}$ is the projector onto the set $[0, \rho_{\max}]$, $\alpha > 0$ is the parameter of the gradient method, $t \in [0, T]$.

We similarly evaluate the digging rate

$$u_i^k(t) = pr_U\Bigg(u_i^{(k-1)}(t) + \alpha \frac{\partial H(t, y_1(t), \ldots, y_{N-1}(t), \rho_0^{(k-1)}(t), u_1^{(k-1)}(t), \ldots, u_{N-1}^{(k-1)}(t), \psi_1(t), \ldots, \psi_{N-1}(t))}{\partial u_i^{(k-1)}(t)}\Bigg),$$

where $i \in \{1, \ldots, N-1\}$, and the set

$$U = \left\{(u_1, \ldots, u_{N-1}) \in R^{N-1}\colon\ u_i \geq 0,\ i \in \{1, \ldots, N-1\},\ \sum_{i=1}^{N-1} u_i \leq \frac{Q_{\max}}{m\Delta}\right\}.$$

The parameter α is chosen so that the functional value in the current step is greater than the preceding value. If this cannot be accomplished, the iterative process stops and the approximation

$$\rho_0^k(t), u_1^k(t), \ldots, u_{N-1}^k(t)$$

is taken as the (approximate) solution of problem (3). Otherwise, we proceed to the next iteration.

4. Prototype Example

In this section, we apply the gradient projection method to solve a prototype example that is unrelated to any specific open-pit mine. We consider an open-pit mine bounded by the rectangle

$$\{(x, y)\colon x \in [0, 2],\ y \in [0, 1]\}.$$

The dependence of the mineral concentration $\rho(x, y)$ on the location in the open-pit plane is shown by the graph in Fig. 2. The deposit consists of two ore bodies centered at the points $\left(\dfrac{2}{3}, \dfrac{1}{2}\right)$ and $\left(\dfrac{4}{3}, \dfrac{1}{4}\right)$.

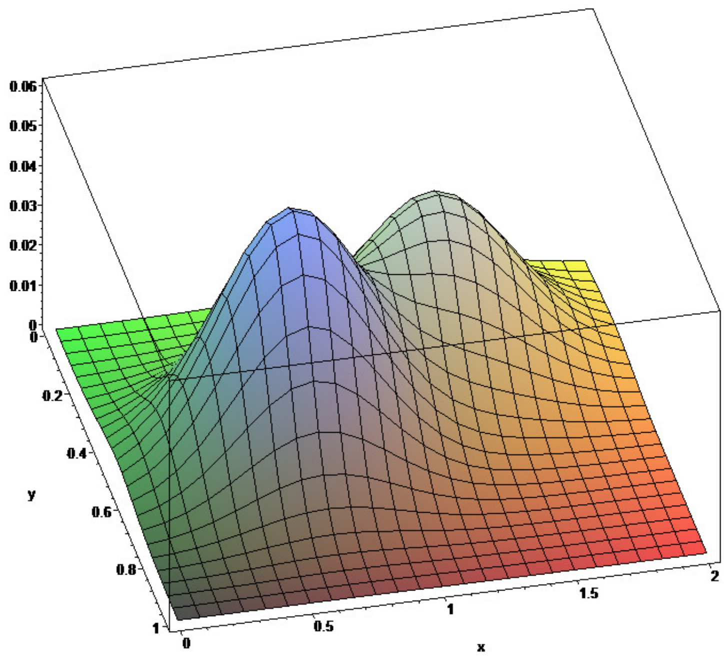

Fig. 2. Mineral concentration $\rho(x, y)$.

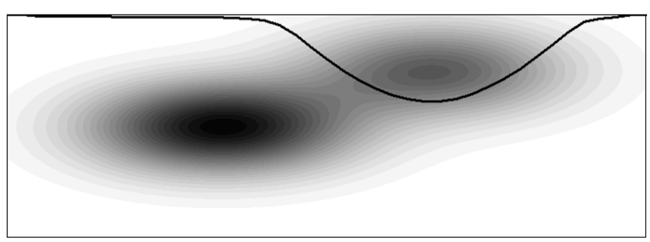

Fig. 3. Optimal shape of the open-pit mine for $T = 1$.

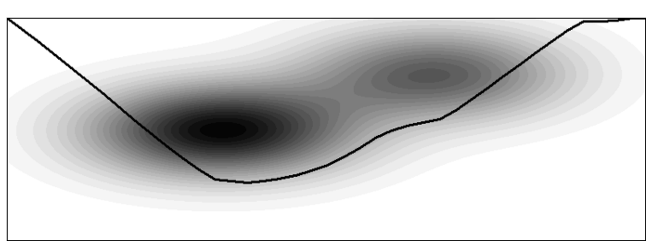

Fig. 4. Optimal shape of the open-pit mine for $T = 4$.

We solved two problems with different terminal times $T = 1$ and $T = 4$. The calculation results are presented in Figs. 3 and 4 respectively. The black lines in the figures shows the optimal shape of the open-pit mine, i.e., the value of the function $y(t, x)$ at $t = T$. The mineral concentration at each point is mapped by the intensity of the black color in the figures.

Given the economic interpretation of our optimal control problem, the solution appears to be adequate. The open-pit shape traces the level lines of the mineral concentration function subject to the slope angle constraints and digging occurs in spots with maximum concentration. If the excavation time is insufficient (the first example), the gradient method makes no attempt to "reach" a higher concentration deeper down in the mine. As the time is increased, the direction of digging is change and both ore bodies are mined.

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