DOI 10.1007/s10598-015-9295-9

Computational Mathematics and Modeling, Vol. 26, No. 4, October, 2015

A NOTE ON “SOLVING FULLY FUZZY LINEAR SYSTEMS BY USING IMPLICIT GAUSS–CHOLESKY ALGORITHM”

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This paper shows that the exact solutions for fully fuzzy linear systems for all examples in [1] are non-fuzzy solutions, and that the proposed solutions in [1] do not correspond to these systems. In addition, approximate fuzzy solutions are provided for all the examples. Finally, this paper shows the efficiency of the provided solutions by using the distance metric function introduced in [13].

Keywords: fully fuzzy linear system; fuzzy number; triangular fuzzy number; fuzzy number vector solution.

1. Preliminaries

The following section reviews the basic definitions and remarks in fuzzy theory, which will be needed in the succeeding sections.

Definition 1.1. *Let X be a universal set. We define the fuzzy subset $\tilde{A}$ of X by its membership function $\mu_{\tilde{A}}(x) : \mathbb{R} \to [0;1]$, which assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$, where the value $\mu_{\tilde{A}}(x)$ represents the membership grade of x in $\tilde{A}$.*

A fuzzy set A is written as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X, \mu_{\tilde{A}}(x) \in [0,1]\}$.

Definition 1.2. *A fuzzy set $\tilde{A}$ in $X = \mathbb{R}^n$ is a convex fuzzy set if*

$$\forall x_1, x_2 \in X, \quad \forall \lambda \in [0,1],$$

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda) x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)).$$

Definition 1.3. *Let $\tilde{A}$ be a fuzzy set defined by the set of real numbers $\mathbb{R}$. $\tilde{A}$ is called a normal fuzzy set if there exists $x \in \mathbb{R}$, such that $\mu_{\tilde{A}}(x) = 1$.*

Definition 1.4. *A fuzzy number is a normal and convex fuzzy set when its membership function $\mu_{\tilde{A}}(x)$ is defined in the real line $\mathbb{R}$ and is piecewise continuous.*

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Definition 1.5. (LR Fuzzy number). *A fuzzy number* $\tilde{m}$ *is called an LR fuzzy number if its membership function is defined as follows:*

$$\mu_{\tilde{m}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & \text{for} \quad x \le m, \quad \alpha > 0, \\ R\left(\frac{x-m}{\beta}\right) & \text{for} \quad m \le x, \quad \beta > 0, \end{cases}$$

where $m, \alpha, \beta \in \mathbb{R}$.

The function $L(.)$ is also called a left shape function if the following hold:

1. $L(x) = L(-x)$,
2. $L(0) = 1$, $L(1) = 0$,
3. L is non-increasing on $[0, \infty]$.

The definition of function $R(.)$, called right shape, is similar to that of $L(.)$. An LR fuzzy number is symbolically written as $\tilde{m} = (m, \alpha, \beta)_{LR}$, where m represents the mean value, while α and β are the left and right spreads respectively, i.e., α and β are the coefficients of "fuzziness." As the spreads increase, $\tilde{m}$ becomes fuzzier and fuzzier.

Definition 1.6. *An LR fuzzy number* $\tilde{m} = (m, \alpha, \beta)_{LR}$ *is called a triangular fuzzy number, where* $L = R = \max(0, 1-x)$.

The set of all triangular fuzzy numbers is denoted by $F(\mathfrak{R})$.

Definition 1.7. (Arithmetic operations on LR fuzzy numbers). *The arithmetic operations for two LR fuzzy numbers* $\tilde{m} = (m, \alpha, \beta)_{LR}$ *and* $\tilde{n} = (n, \gamma, \delta)_{LR}$ *are as follows:*

Addition:

$$(m, \alpha, \beta)_{LR} \oplus (n, \gamma, \delta)_{LR} = (m+n, \alpha+\gamma, \beta+\delta)_{LR}.$$

Opposite:

$$-(m, \alpha, \beta)_{LR} = (-m, \beta, \alpha)_{RL}.$$

Subtraction:

$$(m, \alpha, \beta)_{LR} \ominus (n, \gamma, \delta)_{RL} = (m-n, \alpha+\delta, \beta+\gamma)_{LR}.$$

Approximated multiplication operation of two fuzzy numbers:

(i) If $\tilde{m} > 0$ and $\tilde{n} > 0$, then

$$(m, \alpha, \beta)_{LR} \otimes (n, \gamma, \delta)_{LR} \cong (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR} .$$

(ii) If $\tilde{m} < 0$ and $\tilde{n} > 0$, then

$$(m, \alpha, \beta)_{RL} \otimes (n, \gamma, \delta)_{LR} \cong (mn, n\alpha - m\delta, n\beta - m\gamma)_{RL} .$$

(iii) $\tilde{m} < 0$ and $\tilde{n} < 0$, then

$$(m, \alpha, \beta)_{LR} \otimes (n, \gamma, \delta)_{LR} \cong (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{RL} .$$

Scalar multiplication:
Let $\lambda \in R$, then

$$\delta \otimes (m, \alpha, \beta)_{LR} = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta)_{LR}, & \lambda \geq 0, \\ (\lambda m, -\lambda\beta, -\lambda\alpha)_{RL,} & \lambda < 0. \end{cases}$$

Definition 1.8. *A matrix* $\tilde{A} = (\tilde{a}_{ij})_{nn}$ *is called a fuzzy matrix if* $\tilde{a}_{ij} \in F(\Re)$, $\forall i$, $j = 1, \ldots, n$. *A fuzzy matrix* $\tilde{A}$ *is non-negative (non-positive) and denoted by* $\tilde{A} \geq 0$ $(\tilde{A} \leq 0)$, *if each element* $\tilde{a}_{ij}$ *is non-negative (non-positive).*

Definition 1.9. *A vector* $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ *is called a fuzzy vector if* $\tilde{x}_i \in F(\Re)$, $\forall i = 1, \ldots, n$.

Definition 1.10. *Let* $\tilde{A} = (\tilde{a}_{ij})$ *and* $\tilde{B} = (\tilde{b}_{ij})$ *be two* $m \times n$ *and* $n \times p$ *matrices respectively. We define* $\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$, *which is the* $m \times p$ *matrix where*

$$\tilde{c}_{ij} = \sum_{k=1,\ldots,n}^{\oplus} \tilde{a}_{ik} \otimes \tilde{b}_{kj} .$$

Definition 1.11. (Fully fuzzy linear system). *Consider the* $n \times n$ *linear system of equations:*

$$\begin{cases} (a_{11} \otimes \tilde{x}_1) \otimes (a_{12} \otimes \tilde{x}_2) \otimes \ldots \otimes (a_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ (a_{21} \otimes \tilde{x}_1) \otimes (a_{22} \otimes \tilde{x}_2) \otimes \ldots \otimes (a_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ \vdots \\ (a_{n1} \otimes \tilde{x}_1) \otimes (a_{n2} \otimes \tilde{x}_2) \otimes \ldots \otimes (a_{nn} \otimes \tilde{x}_n) = \tilde{b}_n. \end{cases}$$

The matrix form of the above equations is

$$\tilde{A} \otimes \tilde{x} = \tilde{b},$$

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{pmatrix},$$

This system is called a fully fuzzy linear system (FFLS), where the coefficient matrix $\tilde{A} = (\tilde{a}_{ij})$ $1 \le i, j \le n$ is a fuzzy matrix, $\tilde{x}$ and $\tilde{b}$ are fuzzy vectors, and $\tilde{x}$ is the unknown to be found.

Remark 1.1. When choosing the nearest solution, the following metric distance function proposed in [13] for triangular fuzzy number is used.

If $\tilde{a} = (a, \alpha, \beta)$ and $\tilde{b} = (b, \gamma, \eta)$ are two triangular fuzzy numbers, then the distance function introduced by Ming et al. [13] is

$$D_2^2\left(\tilde{a}, \tilde{b}\right) = \left(\frac{1}{2}\right)\left(4(a-b)^2 + (\alpha-\gamma)^2 + (\beta-\eta)^2\right) + (a-b)(\gamma+\eta-\alpha-\beta).$$

For two L-R fuzzy vectors $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$, and $\tilde{Y} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ the distance function is

$$D_n^2\left(\tilde{X}, \tilde{Y}\right) = \sum_{i=1}^{n} D_2^2(\tilde{x}_i, \tilde{y}_i).$$

2. Numerical Examples

In this section, we will discuss the proposed solutions in [1]. We also show that they do not correspond to the systems by applying arithmetic operations on LR fuzzy numbers to prove that $\tilde{A}\tilde{X}_a, \ne \tilde{B}$, where $\tilde{X}_a$ is the proposed solution in [1]. Furthermore, by using the distance metric function in [13], we show that $D_n\left(\tilde{A}\tilde{X}_a, \tilde{B}\right) \ne 0$.

The exact solutions are provided to show that the proposed FFLSs do not have exact fuzzy solutions. Therefore, approximate fuzzy solutions are presented, and by using the distance metric function in [13], we show that the solutions $\tilde{X}_g$ are better because $D_n\left(\tilde{A}\tilde{X}_g, \tilde{B}\right)$ is smaller compared with $D_n\left(\tilde{A}\tilde{X}_a, \tilde{B}\right)$. The solutions $\tilde{X}_g$ are provided using the methods in [9, 10].

Example 2.1. [1] Consider the following FFLS:

$$\begin{cases} (4,3,2)\otimes(x_1,y_1,z_1)\otimes(5,2,1)\otimes(x_2,y_2,z_2)\otimes(3,0,3)\otimes(x_3,y_3,z_3) = (71,54,76), \\ (7,4,3)\otimes(x_1,y_1,z_1)\otimes(10,6,3)\otimes(x_2,y_2,z_2)\otimes(2,1,1)\otimes(x_3,y_3,z_3) = (118,115,129), \\ (6,2,2)\otimes(x_1,y_1,z_1)\otimes(7,1,2)\otimes(x_2,y_2,z_2)\otimes(15,5,4)\otimes(x_3,y_3,z_3) = (155,89,151). \end{cases}$$

According to [1], the exact fuzzy solution for this FFLS is

$$\tilde{X}_a = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (4, 2, 2) \\ (8, 3, 5) \\ (5, 1, 4) \end{pmatrix}.$$

Note that $\tilde{A}\tilde{X}_a \neq \tilde{B}$. Hence,

$$\tilde{A}\tilde{X}_a = \begin{pmatrix} (71,54,76) \\ (118,115,113) \\ (155,89,151) \end{pmatrix}$$

Moreover, using Remark 1.1, we obtain

$$D_3\left(\tilde{A}\tilde{X}_a, \tilde{B}\right) = 11.3137 .$$

In fact, the exact solution for this system is a non-fuzzy vector as follows:

$$\tilde{X}_e = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (4, 2, -16.782) \\ (8, 3, 19.608) \\ (5, 1, 4.695) \end{pmatrix}$$

which satisfies

$$D_3\left(\tilde{A}\tilde{X}_e, \tilde{B}\right) = 0 .$$

In this case, we have provided an approximate fuzzy solution $\tilde{X}_g$,

$$\tilde{X}_g = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (3.569, 3.569, 0.) \\ (8.294, 1.912, 8.171) \\ (5.034, 0.905, 3.328) \end{pmatrix}$$

then

$$\tilde{A}\tilde{X}_g = \begin{pmatrix} (70.8544, 53.8544, 81.3822) \\ (118, 115, 129) \\ (155, 89, 151) \end{pmatrix}$$

and

$$D_3\left(\tilde{A}\tilde{X}_g, \tilde{B}\right) = 3.9114 .$$

Example 2.2. [1] Consider the following FFLS:

$$\begin{cases} (4,3,2)\otimes(x_1,y_1,z_1)\otimes(5,2,1)\otimes(x_2,y_2,z_2)\otimes(3,0,3)\otimes(x_3,y_3,z_3) = (127,108,140), \\ (7,4,3)\otimes(x_1,y_1,z_1)\otimes(10,6,3)\otimes(x_2,y_2,z_2)\otimes(2,1,1)\otimes(x_3,y_3,z_3) = (201,184,206), \\ (6,2,2)\otimes(x_1,y_1,z_1)\otimes(7,1,2)\otimes(x_2,y_2,z_2)\otimes(15,5,4)\otimes(x_3,y_3,z_3) = (291,224,287). \end{cases}$$

According to [1], the exact fuzzy solution for this FFLS is

$$\tilde{X}_a = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (13, 7, 2) \\ (9, 1, 8) \\ (10, 6, 9) \end{pmatrix}.$$

Note that $\tilde{A}\tilde{X}_a \neq \tilde{B}$, where

$$\tilde{A}\tilde{X}_a = \begin{pmatrix} (127, 108, 140) \\ (201, 187, 188) \\ (291, 224, 287) \end{pmatrix}.$$

Moreover, using Remark 1.1, we get

$$D_3\left(\tilde{A}\tilde{X}_a, \tilde{B}\right) = 12.903 .$$

In fact, the exact solution for this system is a non-fuzzy vector as follows:

$$\tilde{X}_e = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (13, 10.5217, -19.1304) \\ (9, -1.73913, 24.4348) \\ (10, 5.86957, 9.78261) \end{pmatrix}$$

which satisfies

$$D_3\left(\tilde{A}\tilde{X}_e, \tilde{B}\right) = 0 .$$

Similar to the previous example, we provide an approximate fuzzy solution $\tilde{X}_g$,

$$\tilde{X}_g = \begin{pmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{pmatrix} = \begin{pmatrix} (13, 7.978, 0) \\ (9\ , 0\ , 11.352) \\ (10\ , 6.075, 8.235) \end{pmatrix};$$

hence,

$$\tilde{A}\tilde{X}_g = \begin{pmatrix} (127.,107.14,146.471) \\ (201.,184.,206) \\ (291.,224.,287) \end{pmatrix}$$

and

$$D_3\left(\tilde{A}\tilde{X}_g, \tilde{B}\right) = 4.6156 .$$

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