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INVERSE PROBLEMS IN THE OPTICS OF LAYERED MEDIA

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Inverse problems of recognition and synthesis of optical coatings are considered. Methods are proposed for fast computation of the gradient of reflection and transmission coefficients from changes in permittivity distribution. This essentially improves the efficiency of inverse problem solution.

Keywords: inverse problems, layered medium, integral equations.

Introduction

Inverse problems of optics fall into three classes:

- Synthesis of layered media in which the dielectric parameters to be determined ensure the specified reflection coefficient;
- Recognition of a layered dielectric medium whose parameters are to be determined from the frequency dependence of the reflection coefficient;
- Monitoring of a dielectric medium whose state is determined from measurements of the reflection coefficient as a function of frequency.

The reflection coefficient of a layered medium $R(\omega)$ is calculated in terms of permittivity distribution $\varepsilon(z)$ in the form

$$R(\omega) = A[\varepsilon(z), \omega], \tag{1}$$

where A is a nonlinear operator that depends on the field frequency ω. Expression (1) may be treated as the equation for the determination of $\varepsilon(z)$ given $R(\omega)$.

This problem is unstable, i.e., for every $c > 0$ there exists $\delta > 0$ such that two widely differing distributions $\varepsilon_1(z)$ and $\varepsilon_2(z)$ exist, $||\varepsilon_1(z) - \varepsilon_2(z)|| \geq c$, although the reflection coefficients are quite close:

$$||R_1(\omega) - R_2(\omega)||_{L_2} \leq \delta.$$

In synthesis problems, problem (1) is not always solvable because $R(\omega)$ is not necessarily inside the value domain of the operator A.

Tikhonov's regularization method [1, 2] is the most effective technique for solving the inverse problems of optics. Tikhonravov's work [3] has made a considerable contribution to the development of the theory of inverse problems of optics.

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The inverse problem of the optics of layered media produces a stable result [4] with the integral permittivity

$$e(z) = \int_0^z \varepsilon(z)\,dz, \quad z \in [0, H]. \tag{2}$$

The use of this integral characteristic yields efficient solutions for inverse problems of optics [5].

The regularization method reduces the inverse problem to a variational problem to minimize the smoothing functional [1]. The solution of this problem requires multiple evaluations of the functional gradient. In this study, we propose methods for fast evaluation of the functional gradient in inverse problems of optics.

1. Determining the Reflection Coefficient of a Layered Medium

Consider a dielectric layer with variable permittivity $\varepsilon(z)$, $z \in [0, H]$. For $z < 0$ we have a homogeneous subspace $\varepsilon = \varepsilon_0$, $z < 0$, and for $z > H$, we have $\varepsilon = \varepsilon_H$. An electromagnetic wave is normally incident on this layer along the Oz axis. Then the field has the following components:

– the electric field $\bar{E} = (E_x, 0, 0)$;

– the magnetic field $\overline{H} = (0, H_y, 0)$.

By Maxwell's equations, the fields are coupled by the relationships

$$\frac{dE_x}{dz} = i\omega\mu H_y; \qquad \frac{dH_y}{dz} = i\omega\varepsilon(z)E_x. \tag{3}$$

From (3) we obtain the equation for the electric field

$$\frac{d^2E_x}{dz^2} + \omega^2\mu\varepsilon(z)E_x = 0, \qquad z \in (-\infty, \infty). \tag{4}$$

For $z > 0$ we have an incident and a reflected wave

$$E_x(z) = A_0\left(e^{ik_0z} + R(\omega)e^{-ik_0z}\right), \qquad z \in (-\infty, 0), \quad k_0 = \omega\sqrt{\varepsilon_0\mu},$$

where $R(\omega)$ is the coefficient of reflection from the layer; for $z > H$ there is only a transmitted wave:

$$E_x(z) = A_0D(\omega)e^{ik_H(z-H)}, \qquad z \in (H, \infty),$$

where $D(\omega)$ is the transmission coefficient. Noting the discontinuity of $E_x(z)$ and $E_x'(z)$ at $z = 0$ and $z = H$, we obtain from these equations the boundary conditions

$$E_x'(0) + ik_0E_x(0) = 2ik_0A_0; \qquad E_x'(H) - ik_HE_x(H) = 0. \tag{5}$$

Thus, (4) and (5) produce a boundary-value problem for the determination of $E_x(z)$.

Normalizing the electric field as

$$u(z) = \frac{E_x(z)}{A_0} \tag{6}$$

we obtain the following boundary-value problem for $u(z)$:

$$\begin{cases} u''(z) + \omega^2 \mu\varepsilon(z) u(z) = 0, \quad z \in [0, H], \\ u'(0) + ik_0 u(0) = 2ik_0, \\ u'(H) - ik_H u(H) = 0. \end{cases} \tag{7}$$

Given $u(z)$, we can easily find $R(\omega)$ and $D(\omega)$ in the form

$$R(\omega) = u(z=0) - 1, \qquad D(\omega) = u(z=H). \tag{8}$$

Boundary-value problem (7) is usually solved by reduction to Riccati equation for the admittance function

$$Y(z) = \frac{u'(z)}{u(z)}, \tag{9}$$

which, by (7), is the solution of the Cauchy problem

$$\begin{cases} Y'(z) + Y^2(z) = -\omega^2 \mu\varepsilon(z), \quad z \in [0, H], \\ Y(H) = ik_H = i\omega\sqrt{\mu\varepsilon_H}. \end{cases} \tag{10}$$

Having determined $Y(z=0)$ from (10), we easily find the reflection coefficient. By (7), the boundary condition at $z = 0$ gives

$$u'(0) + ik_0 u(0) = Y(z=0)u(z=0) + ik_0 u(z=0) = 2ik_0.$$

Hence,

$$u'(z=0) = \frac{2ik_0}{Y(z=0) + ik_0}; \qquad R(\omega) = \frac{ik_0 - Y(z=0)}{ik_0 + Y(z=0)}. \tag{11}$$

To find the transmission coefficient $D(\omega)$, we have to solve the Cauchy problem for $u(z)$:

$$\begin{cases} u''(z) + \omega^2 \mu\varepsilon(z) u(z) = 0, \quad z \in [0, H], \\ u(z=0) = \dfrac{2ik_0}{(ik_0 + Y(0))}, \\ u'(z=0) = \dfrac{2ik_0 Y(0)}{(ik_0 + Y(0))}. \end{cases} \tag{12}$$

Having determined $u(z=H)$, we use (8) to find $D(\omega) = u(H)$.

If the inverse problem requires only $R(\omega)$, then this approach is quite efficient. If, however, we also need $D(\omega)$ or $u(z)$ at interior points $z \in (0, H)$, it is better to reduce problem (7) to an integral equation. To this end, introduce the function

$$u_0(z) = \frac{2k_0}{Q}\Big((k_c + k_H)e^{ik_c z} + (k_c - k_H)e^{ik_c(2H-z)} \Big), \tag{13}$$

where

$$Q = (k_c + k_H)(k_c - k_0) - (k_c - k_H)(k_c - k_0)e^{2ik_c H}, \tag{14}$$

which is the solution of the boundary-value problem

$$\begin{cases} u_0'' + k_c^2 u_0 = 0, \quad z \in [0, H], \quad k_c = \omega\sqrt{\mu\varepsilon_c}, \\ u'(0) + ik_0 u_0(0) = 2ik_0, \\ u'(H) - ik_H u_0(H) = 0. \end{cases} \tag{15}$$

Then represent the solution of problem (7) in the form

$$u(z) = u_0(z) + v(z). \tag{16}$$

By (7) and (15), the function $v(z)$ is the solution of the boundary-value problem

$$\begin{cases} v''(z) + k_c^2 v(z) = \Big(k_c^2 - k^2(z)\Big)u(z), \quad z \in [0, H], \quad k(z) = \omega\sqrt{\mu\varepsilon(z)}, \\ v'(0) + ik_0 v(0) = 0, \\ v'(H) - ik_H v(H) = 0. \end{cases} \tag{17}$$

Introduce the Green's function $G(z, z_0)$ for problem (17), which is the solution of the problem

$$\begin{cases} \dfrac{d^2G}{dz^2} + k_c^2 G = \delta(z - z_0), \quad z \in [0, H], \quad z_0 \in [0, H], \\ \left.\dfrac{dG}{dz}\right|_{z=0} + ik_0 G\big|_{z=0} = 0, \quad z_0 \in [0, H], \\ \left.\dfrac{dG}{dz}\right|_{z=H} + ik_H G\big|_{z=H} = 0, \quad z_0 \in [0, H]. \end{cases} \tag{18}$$

Problem (18) is easy to solve and the Green's function has the form

$$G(z,z_0) = \frac{1}{2ik_c}\left(e^{ik_c|z-z_0|} + c_1 e^{ik_c(z+z_0)} + c_2 e^{ik_c(2H-z-z_0)}\right), \tag{19}$$

$$c_1 = \frac{k_c - k_0}{Q}\left((k_c + k_H) + (k_c - k_H)e^{2ik_c(H-z_0)}\right), \tag{20}$$

$$c_2 = \frac{k_c - k_H}{Q}\left((k_c + k_0) + (k_c - k_0)e^{2ik_0 z_0}\right), \tag{21}$$

where Q is determined by (14).

Applying the Green's function, we reduce problem (17) to the integral equation

$$u(z) = u_0(z) + \int_0^H G(z,z_0)\left(k_c^2 - k^2(z_0)\right)u(z_0)\,dz_0 . \tag{22}$$

This is a Fredholm integral equation of second kind. Solving (22), we find $u(z)$ on the entire interval $z \in [0, H]$ and thus, by (8), find $R(\omega)$ and $D(\omega)$.

Accordingly, there are two methods for calculating the reflection and transmission coefficients for a layered medium. One is a differential method based on the solution of the Riccati equation for the admittance function, and the other requires solution of an integral equation.

2. The Inverse Problems

We solve the inverse problem of the optics of layered media using the integral permittivity (2). The sought permittivity is represented as a piecewise-constant function on a grid with a constant increment h,

$$\varepsilon(z) = \varepsilon_n \quad \text{for} \quad (n-1)h < z < nh, \quad n \in [1, N], \quad h = \frac{H}{N}. \tag{23}$$

Then the calculated reflection coefficient, by (1), is a function of the vector of variables $\overline{\varepsilon} = \{\varepsilon_n\}$, $n \in [1, N]$, and the frequency ω:

$$R^c(\overline{\varepsilon}, \omega) = A\left[\varepsilon(z), \omega\right]. \tag{24}$$

In the inverse synthesis problem, the reflection coefficient $R^0(\omega)$ is given as a function of ω, and in the recognition problem we have the observed reflection coefficient $R^0(\omega)$. In the inverse problem, we seek a quasi-solution that minimizes the discrepancy between the calculated $R^c(\varepsilon, \omega_m)$ and the observed $R^0(\omega_m)$ on some frequency grid $\{\omega_m\}$, $m \in [1, M]$, i.e.,

$$\min_{\overline{\varepsilon}} \sum_{m=1}^{N} \left| R^c(\overline{\varepsilon}, \omega_m) - R^0(\omega) \right|^2 . \tag{25}$$

Since the inverse problem is unstable, the solution $\overline{\varepsilon}$ obtained from the minimization problem (25) may strongly deviate from the true solution. However, as noted above, the integral permittivity is stable. Therefore, $\tilde{l}(z)$ calculated from (2) produces an approximation of the true $l(z)$. As a result, we obtain a Volterra integral equation of the first kind for $\varepsilon(z)$:

$$\int_0^z \varepsilon(\zeta)\,d\zeta = \tilde{l}(z), \qquad z \in [0,H]. \tag{26}$$

We have thus transformed the nonlinear unstable inverse problem to a linear integral equation of the first kind. Its solution is unstable, but this problem can be solved by a regularization method. To solve the minimization problem (25), we have to compute the gradient of the reflection coefficient $R^c(\overline{\varepsilon},\omega)$ with respect to the variables $\overline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$.

3. Fast Gradient Calculation

First consider the calculation of the gradient of $R(\overline{\varepsilon},\omega)$ using the admittance function $Y(z)$. From (1),

$$q = \frac{\partial R(\overline{\varepsilon},\omega)}{\partial \varepsilon_n} = \frac{2ik_0}{\left(ik_0 + Y(z=0)\right)^2} \cdot \left.\frac{\partial Y}{\partial \varepsilon_n}\right|_{z=0} \tag{27}$$

The function $\varphi_n(z) = \dfrac{\partial Y(z)}{\partial \varepsilon_n}$, by (10), is the solution of the Cauchy problem

$$\begin{cases} \varphi_n'(z) + 2Y(z)\varphi_n(z) = -\omega^2\mu \cdot f_n(z), & z \in [0,H], \\ \varphi_n(z=H) = 0, \end{cases} \tag{28}$$

where

$$f_n(z) = \begin{cases} 1 & \text{for} \quad z \in [z_{n-1}, z_n], \\ 0 & \text{for} \quad z \notin [z_{n-1}, z_n]. \end{cases} \tag{29}$$

Since $f_n(z) = 0$ for $z \in [z_n, H]$, by (28) we obtain that $\varphi_n(z_n) = 0$. Then consider the problem

$$\begin{cases} \varphi_n'(z) + 2Y(z)\varphi_n(z) = -\omega^2\mu, & z \in [z_{n-1}, z_n], \\ \varphi_n(z_n) = 0. \end{cases} \tag{30}$$

Hence we find

$$\varphi_n(z) = \omega^2\mu e^{-2P(z)} \int_z^{z_n} e^{+2P(\xi)}\,d\xi, \tag{31}$$

where

$$P(z) = \omega^2 \int_{z_n}^{z} Y(\zeta)\, d\zeta\,. \tag{32}$$

Thus,

$$\varphi_n(z_{n-1}) = \omega^2\mu \int_{z_{n-1}}^{z_n} e^{-2\alpha(\xi)}\, d\xi\,, \tag{33}$$

where

$$\alpha(\xi) = P(z_{n-1}) - P(\xi) = \int_{\xi}^{z_{n-1}} Y(\zeta)\, d\zeta\,. \tag{34}$$

As a result, we obtain the following problem for the determination of $\varphi_n(z=0)$:

$$\begin{cases} \varphi_n'(z) + 2Y(z)\varphi_n(z) = 0, \quad z \in [0, z_{n-1}], \\ \varphi_n(z_{n-1}) = \omega^2\mu \int_{z_{n-1}}^{z_n} e^{-2\alpha(\xi)}\, d\xi. \end{cases} \tag{35}$$

Finally,

$$\varphi_n(z=0) = \varphi_n(z_{n-1}) e^{+2\int_0^{z_{n-1}} Y(\zeta)\, d\zeta}. \tag{36}$$

Now, by (27), we find

$$q_n = \frac{\partial R(\bar{\varepsilon}, \omega)}{\partial \varepsilon_n} = \frac{2ik_0\varphi_n(z_{n-1})}{\left(ik_0 Y(z=0)\right)^2} \cdot e^{2\int_0^{z_{n-1}} Y(\zeta)\, d\zeta}. \tag{37}$$

Substituting (33) in (37), we finally obtain

$$\frac{\partial R}{\partial \varepsilon_n} = \frac{2ik_0\omega^2\mu}{\left(ik_0 + Y(z=0)\right)^2} \int_{z_{n-1}}^{z_n} e^{2P_0(\xi)}\, d\xi\,, \tag{38}$$

where

$$P_0(\xi) = \int_0^{\xi} Y(\zeta)\, d\zeta\,. \tag{39}$$

Expression (38) can be simplified noting that $P_0(\xi)$ does not change much for $\xi \in [z_{n-1}, z_n]$. Then we can approximately represent

$$P_n(\xi) \cong P_0(z_{n-1}) + Y(z_{n-1})(\xi - z_{n-1}), \qquad \xi \in [z_{n-1}, z_n].$$

Substituting this expression in (38) and integrating, we find

$$\frac{\partial R}{\partial \varepsilon_n} = \frac{2ik_0\omega^2\mu \cdot e^{2P_0(z_{n-1})}}{Y(z_{n-1})\left(ik_0 + Y(z=0)\right)^2} \cdot \left(e^{2Y(z_{n-1})h} - 1\right). \tag{40}$$

Thus, solving problem (10) and determining the admittance function $Y(z)$ for $z \in [0, H]$, we easily determine from (11) and (38) (or (40)) both the reflection coefficient $R(\omega)$ and its gradient $\dfrac{\partial R}{\partial \varepsilon_n}$, $n \in [1, N]$. This essentially speeds up the minimization process when solving the inverse problem (25), which is particularly significant with a large number of partitions N.

Let us now consider how the integral equation (22) is used to calculate the gradient of $R(\bar{\varepsilon}, \omega)$ and $D(\bar{\varepsilon}, \omega)$. Taking a piecewise-constant representation of permittivity (23), we represent the integral equation (22) in the form

$$u(z) = u_0(z) + \sum_{n=1}^{N} \omega^2\mu\left(\varepsilon_c - \varepsilon_n\right) \int_{(n-1)h}^{nh} G(z, z_0)u(z_0)\,dz_0. \tag{41}$$

The solution of the integral equation depends on $\bar{\varepsilon} = \{\varepsilon_n\}$, the kernel $G(z, z_0)$ of the equation does not depend on $\bar{\varepsilon}$. Introduce the function $d_n(z) = \dfrac{\partial u(z)}{\partial \varepsilon_n}$. Differentiating (41) with respect to ε_n we obtain an equation for $d_n(z)$:

$$d_n(z) = f(z) + \sum_{n=1}^{N} \omega^2\mu(\varepsilon_0 - \varepsilon_n) \int_{(n-1)h}^{nh} G(z, z_0)d_n(z_0)\,dz_0, \tag{42}$$

where

$$f(z) = -\omega^2\mu(\varepsilon_0 - \varepsilon_n) \int_{(n-1)h}^{nh} G(z, z_0)u(z_0)\,dz_0. \tag{43}$$

Note that the integral equation kernels are the same for both $u(z)$ and $d_n(z)$, $n \in [1, N]$. Therefore, the equations can be solved simultaneously by parallel computation. Solving the integral equations (41) and (42), we obtain, in accordance with (8),

$$R(\omega) = u(z=0) - 1, \qquad D(\omega) = u(z=H),$$

$$\frac{\partial R}{\partial \varepsilon_n} = \frac{\partial u(z=0)}{\partial \varepsilon_n} = d_n(z=0), \qquad \frac{\partial D}{\partial \varepsilon_n} = \frac{\partial u(z=H)}{\partial \varepsilon_n} = d_n(z=H). \tag{44}$$

We thus have a fast procedure for computing the reflection and transmission coefficients and their gradients with respect to $\bar{\varepsilon}$. This leads to efficient solution of inverse problems of the optics of layered media.

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