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VARIABLE-METRIC DISCRETE EXTRAGRADIENT METHOD FOR SADDLE-POINT PROBLEMS

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The article considers a variable-metric discrete extragradient method to find a saddle point. The method converges in the argument to the set of saddle points.

Keywords: saddl-point problems, extragradient methods.

1. Introduction

Problems that require finding saddle points are common in various areas of mathematics and economics. There are a number of numerical methods solving this problem, but they all impose strict requirements on the objective function and are intended for a narrow class of problems. Attempts to eliminate the requirement of strong convexity–concavity lead to extrapolation method [1].

A similar approach is used to solve the equilibrium programming problem. The equilibrium formulation is essentially universal, since many problems from various areas of optimization are reducible to the equilibrium form. Various numerical methods, including extragradient methods [2], have been developed for equilibrium problems.

Of special interest is the possibility of speeding up convergence without significant computational costs. For equilibrium problems, we have an analogue of the Newton's method that suffers from the same weaknesses as the classical Newton's optimization method — local convergence and computational complexity. The second difficulty is avoided in variable-metric methods [3]. Combined with the extrapolation approach, these methods ensure sufficiently fast convergence combined with acceptable computational costs. They are furthermore appropriate for a wide class of objective functions.

A number of variable-metric extragradient methods are available at present, but these are continuous methods; no discrete analogues exist. Equilibrium programming problems are closely linked with saddle point problems — search for equilibrium points can be reduced by a change of variables to search for saddle points [4]. In this article, we consider a discrete extragradient method with variable metric for saddle point problems as a particular case of equilibrium problems.

2. Variable-Metric Discrete Extragradient Method for Saddle-Point Problems

We consider the problem of finding a saddle point (x_*, y^*) of the function $f(x, y)$:

$$f(x_*, y) \leqslant f(x_*, y^*) \leqslant f(x, y^*) \quad \forall x \in X \subseteq \mathbb{R}^n, \quad \forall y \in Y \subseteq \mathbb{R}^n. \tag{1}$$

Assume that some initial approximations $x_0 \in X,\ y_0 \in Y$ are given. We then construct the sequence $\{x_k, y_k\}$ by the rule

$$\bar{x}_k = P_X^{G(x_k)}(x_k - \alpha_k G(x_k)^{-1} f'_x(x_k, y_k)), \quad k = 0, 1, 2, \ldots, \tag{2}$$

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$$\bar{y}_k = P_Y^{G(y_k)}(y_k + \alpha_k G(y_k)^{-1} f'_y(x_k, y_k)), \quad k = 0, 1, 2, \dots, \tag{3}$$

$$x_{k+1} = P_X^{G(x_k)}(x_k - \alpha_k G(x_k)^{-1} f'_x(\bar{x}_k, \bar{y}_k)), \quad k = 0, 1, 2, \dots, \tag{4}$$

$$y_{k+1} = P_Y^{G(y_k)}(y_k + \alpha_k G(y_k)^{-1} f'_y(\bar{x}_k, \bar{y}_k)), \quad k = 0, 1, 2, \dots, \tag{5}$$

where $G(x_k)$ and $G(y_k)$ are $n \times n$ symmetrical positive definite matrices; $\alpha_k,\ k = 0, 1, \dots,$ are positive numbers; $P_X^{G(x_k)}$ and $P_Y^{G(y_k)}$ are G-projectors on the sets X and Y. The operation of G-projection on some set X is defined as follows:

$$p = P_X^G(x_0), \qquad \text{if} \quad \langle G(p - x_0), p - x_0 \rangle = \inf_{x \in X} \langle G(x - x_0), x - x_0 \rangle,$$

i.e., p actually minimizes the function

$$\rho_G(x; x_0) = \langle G(x - x_0), x - x_0 \rangle$$

on the set X. The following proposition describes the properties of G-projection.

Lemma 1. *Let X be a closed convex set from $\mathbb{R}^n$, G a symmetrical positive-definite matrix. There also exist positive numbers m and M such that $\forall x \in \mathbb{R}^n$ we have the double equality $m\|x\|^2 \leqslant \langle Gx, x \rangle \leqslant M\|x\|^2$. Then:*

1. *For every $x_0 \in \mathbb{R}^n$ there exists a unique G-projection $P_X^G(x_0)$ on the set X;*
2. *$p = P_X^G(x_0)$ if and only if $\langle G(p - x_0), x - p \rangle \geqslant 0\ \forall x \in X$;*
3. *If $p_1 = P_X^G(x_1),\ p_2 = P_X^G(x_2)$, then $\|p_1 - p_2\| \leqslant \dfrac{M}{m}\|x_1 - x_2\|$.*

In the interest of more compact notation, we introduce the G-norm $\|x\|_G = \langle Gx, x \rangle$. The matrices in Eqs. (2)–(5) are either constant (i.e., $G(x_k) \equiv G_x,\ G(y_k) \equiv G_y\ \ \forall k = 0, 1, \dots$) or such that in each step we have

$$\|x\|^2_{G(x_k)} \geqslant \|x\|^2_{G(x_{k+1})} \quad \forall x \in \mathbb{R}^n; \qquad \|y\|^2_{G(y_k)} \geqslant \|y\|^2_{G(y_{k+1})} \quad \forall y \in \mathbb{R}^n.$$

These inequalities are equivalent to nonnegative definiteness of the difference of matrices on successive iterations, which imposes strict constraints on the method of choosing these matrices. In this article we do not propose explicit formulas for $G(x_k)$ and $G(y_k)$. There are some recommendations regarding the construction of such matrices (see [5], sec. 9.2). Augmenting a symmetrical positive definite matrix in a certain way, we can ensure that the outcome is also symmetrical and positive definite. Based on this fact, we can propose for our method the following technique of choosing the matrices $G(x_k)$ and $G(y_k)$:

1. Decide on the number of iterations;
2. Choose the matrix to be applied on the last iteration;
3. Construct a sequence of matrices by the method proposed in [5].

This approach may produce an insufficiently accurate solution. In this case, we should increase the number of iterations and repeat the construction of the sequence of matrices.

Let us consider the convergence of sequences generated by the method (2)–(5).

Theorem 1. *Let the following conditions hold:*

1. $X,\ Y$ *are convex closed subsets in the space* $\mathbb{R}^n$*;*
2. *The function* $f(x,y)$ *is convex in* x *on the set* X *for every fixed* $y \in Y$ *and concave in* y *on the set* Y *for every fixed* $x \in X$*;*
3. *The partial derivatives of* $f(x,y)$ *satisfy the Lipschitz conditions*

$$\|f'_x(x_1,y_1) - f'_x(x_2,y_2)\| \leqslant L_1(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

$$\|f'_y(x_1,y_1) - f'_y(x_2,y_2)\| \leqslant L_2(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for all $x_1, x_2 \in X,\ y_1, y_2 \in Y$*;*

4. *The matrices* $G(x_k)$ *and* $G(y_k)$ *are symmetrical, there exist positive numbers* $m_1,\ m_2,\ M_1,\ M_2$ *such that for all* $k = 0, 1, \dots,$

$$\begin{aligned} m_1\|x\|^2 \leqslant \langle G(x_k)x, x\rangle \leqslant M_1\|x\|^2 \quad \forall x \in \mathbb{R}^n, \\ m_2\|y\|^2 \leqslant \langle G(y_k)y, y\rangle \leqslant M_2\|y\|^2 \quad \forall y \in \mathbb{R}^n; \end{aligned} \tag{6}$$

and moreover

$$\|x\|^2_{G(x_k)} \geqslant \|x\|^2_{G(x_{k+1})} \quad \forall x \in \mathbb{R}^n,$$

$$\|y\|^2_{G(y_k)} \geqslant \|y\|^2_{G(y_{k+1})} \quad \forall y \in \mathbb{R}^n;$$

5. *There exists a number* $\varepsilon > 0$ *such that for every* k *the step length* α_k *satisfies the conditions*

$$m_1 - 4\alpha_k L \geqslant \varepsilon, \qquad m_2 - 4\alpha_k L \geqslant \varepsilon, \tag{7}$$

where $L = \max(L_1, L_2)$.

Then for every initial approximation $(x_0, y_0) \in X \times Y$*, all the limit points* (x_*, y^*) *of the sequence* $\{(x_k, y_k)\}$ *generated by the method (2)–(5) are in the solution set* $X_* \times Y^*$ *of the saddle-point problem (1).*

Proof. Applying the properties of G-projection from Lemma 1, we rewrite the equations of the proposed method in the form of variational inequalities:

$$\big\langle G(x_k)(\bar{x}_k - x_k + \alpha_k G(x_k)^{-1} f'_x(x_k, y_k)), x - \bar{x}_k \big\rangle \geqslant 0, \tag{8}$$

$$\big\langle G(x_k)(x_{k+1} - x_k + \alpha_k G(x_k)^{-1} f'_x(\bar{x}_k, \bar{y}_k)), x - x_{k+1} \big\rangle \geqslant 0, \tag{9}$$

$$\big\langle G(y_k)(\bar{y}_k - y_k - \alpha_k G(y_k)^{-1} f'_y(x_k, y_k)), y - \bar{y}_k \big\rangle \geqslant 0, \tag{10}$$

$$\big\langle G(y_k)(y_{k+1} - y_k - \alpha_k G(y_k)^{-1} f'_y(\bar{x}_k, \bar{y}_k)), y - y_{k+1} \big\rangle \geqslant 0. \tag{11}$$

Substituting $x = x_*$ in (9) we obtain after elementary manipulations

$$\langle G(x_k)(x_{k+1} - x_k), x_* - x_{k+1}\rangle + \alpha_k \langle f'_x(\bar{x}_k, \bar{y}_k), x_* - x_{k+1}\rangle \geqslant 0$$

$$\Leftrightarrow \langle G(x_k)(x_{k+1} - x_k), x_* - x_{k+1}\rangle + \alpha_k \langle f'_x(\bar{x}_k, \bar{y}_k), x_* - \bar{x}_k\rangle$$

$$- \alpha_k \langle f'_x(x_k, y_k) - f'_x(\bar{x}_k, \bar{y}_k), \bar{x}_k - x_{k+1}\rangle + \alpha_k \langle f'_x(x_k, y_k), \bar{x}_k - x_{k+1}\rangle \geqslant 0. \tag{12}$$

The function $f(x, y)$ is convex in x on the set X, and therefore the second term in (12) can be bounded by using the convexity criterion for differentiable functions [3, p. 185], Theorem 2:

$$\alpha_k \langle f'_x(\bar{x}_k, \bar{y}_k), x_* - \bar{x}_k\rangle \leqslant \alpha_k (f(x_*, \bar{y}_k) - f(\bar{x}_k, \bar{y}_k)).$$

By the Cauchy–Bunyakovskii inequality and the Lipschitz condition, the third term in (12) is upper bounded as follows:

$$-\alpha_k \langle f'_x(x_k, y_k) - f'_x(\bar{x}_k, \bar{y}_k), \bar{x}_k - x_{k+1}\rangle$$

$$\leqslant \alpha_k \|f'_x(x_k, y_k) - f'_x(\bar{x}_k, \bar{y}_k)\| \cdot \|\bar{x}_k - x_{k+1}\|$$

$$\leqslant \alpha_k L(\|x_k - \bar{x}_k\| + \|y_k - \bar{y}_k\|) \cdot \|x_{k+1} - \bar{x}_k\| \leqslant \{2ab \leqslant a^2 + b^2\}$$

$$\leqslant \alpha_k L \cdot \frac{1}{2}(\|x_{k+1} - \bar{x}_k\|^2 + (\|x_k - \bar{x}_k\| + \|y_k - \bar{y}_k\|)^2)$$

$$\leqslant \alpha_k L \left(\frac{1}{2}\|x_{k+1} - \bar{x}_k\|^2 + \|x_k - \bar{x}_k\|^2 + \|y_k - \bar{y}_k\|^2\right).$$

Inequality (12) is now rewritten as

$$\langle G(x_k)(x_{k+1} - x_k), x_* - x_{k+1}\rangle + \alpha_k (f(x_*, \bar{y}_k) - f(\bar{x}_k, \bar{y}_k))$$

$$+ \alpha_k \left\langle f'_x(x_k, y_k), \bar{x}_k - x_{k+1}\right\rangle + \alpha_k L \left(\frac{1}{2}\|x_{k+1} - \bar{x}_k\|^2 + \|x_k - \bar{x}_k\|^2 + \|y_k - \bar{y}_k\|^2\right) \geqslant 0. \tag{13}$$

Setting in inequality (8) $x = x_{k+1}$, we have

$$\langle G(x_k)(\bar{x}_k - x_k), x_{k+1} - \bar{x}_k\rangle + \alpha_k \langle f'_x(x_k, y_k)), x_{k+1} - \bar{x}_k\rangle \geqslant 0,$$

and adding this inequality with (13) we obtain

$$\langle G(x_k)(x_{k+1} - x_k), x_* - x_{k+1}\rangle + \alpha_k (f(x_*, \bar{y}_k) - f(\bar{x}_k, \bar{y}_k))$$

$$+ \langle G(x_k)(\bar{x}_k - x_k), x_{k+1} - \bar{x}_k\rangle + \alpha_k L \left(\frac{1}{2}\|x_{k+1} - \bar{x}_k\|^2 + \|x_k - \bar{x}_k\|^2 + \|y_k - \bar{y}_k\|^2\right) \geqslant 0. \tag{14}$$

Repeat the same operations for the main and the predictor steps in the variable y. In inequality (11) set $y = y^*$:

$$\langle G(y_k)(y_{k+1} - y_k), y^* - y_{k+1}\rangle - \alpha_k\langle f'_y(\bar{x}_k, \bar{y}_k)), y^* - y_{k+1}\rangle \geqslant 0.$$

Represent the resulting inequality in the form

$$\langle G(y_k)(y_{k+1} - y_k), y^* - y_{k+1}\rangle - \alpha_k\langle f'_y(\bar{x}_k, \bar{y}_k), y^* - \bar{y}_k\rangle$$
$$- \alpha_k\langle f'_y(\bar{x}_k, \bar{y}_k) - f'_y(x_k, y_k), \bar{y}_k - y_{k+1}\rangle + \alpha_k\langle f'_y(x_k, y_k), \bar{y}_k - y_{k+1}\rangle \geqslant 0. \tag{15}$$

Bound the second term in (15) using concavity of the function $f(x, y)$ in the variable y on the set Y:

$$-\alpha_k\big\langle f'_y(\bar{x}_k, \bar{y}_k), y^* - \bar{y}_k\big\rangle \leqslant -\alpha_k(f(\bar{x}_k, y^*) - f(\bar{x}_k, \bar{y}_k)).$$

By the Cauchy–Bunyakovskii inequality and the Lipschitz condition for the derivative, we obtain the following bound for the third term in (15):

$$- \alpha_k\big\langle f'_y(\bar{x}_k, \bar{y}_k) - f'_y(x_k, y_k), \bar{y}_k - y_{k+1}\big\rangle$$
$$\leqslant \alpha_k \left\| f'_y(\bar{x}_k, \bar{y}_k) - f'_y(x_k, y_k)\right\| \cdot \|\bar{y}_k - y_{k+1}\|$$
$$\leqslant \alpha_k L(\|x_k - \bar{x}_k\| + \|y_k - \bar{y}_k\|) \cdot \|y_{k+1} - \bar{y}_k\| \leqslant \{2ab \leqslant a^2 + b^2\}$$
$$\leqslant \alpha_k L \cdot \frac{1}{2}(\|y_{k+1} - \bar{y}_k\|^2 + (\|x_k - \bar{x}_k\| + \|y_k - \bar{y}_k\|)^2)$$
$$\leqslant \alpha_k L\left(\frac{1}{2}\|y_{k+1} - \bar{y}_k\|^2 + \|x_k - \bar{x}_k\|^2 + \|y_k - \bar{y}_k\|^2\right).$$

Collecting the various bounds, we rewrite inequality (15) in the form

$$\langle G(y_k)(y_{k+1} - y_k), y^* - y_{k+1}\rangle - \alpha_k(f(\bar{x}_k, y^*) - f(\bar{x}_k, \bar{y}_k))$$
$$- \alpha_k\langle f'_y(x_k, y_k), \bar{y}_k - y_{k+1}\rangle + \alpha_k L\left(\frac{1}{2}\|y_{k+1} - \bar{y}_k\|^2 + \|x_k - \bar{x}_k\|^2 + \|y_k - \bar{y}_k\|^2\right) \geqslant 0. \tag{16}$$

Setting in inequality (10) $y = y_{k+1}$, we have

$$\langle G(y_k)(\bar{y}_k - y_k), y_{k+1} - \bar{y}_k\rangle - \alpha_k\big\langle f'_y(x_k, y_k)), y_{k+1} - \bar{y}_k\big\rangle \geqslant 0,$$

and adding this inequality with (16) we obtain

$$\langle G(y_k)(y_{k+1} - y_k), y^* - y_{k+1}\rangle - \alpha_k(f(\bar{x}_k, y^*) - f(\bar{x}_k, \bar{y}_k))$$

$$- \alpha_k \langle f'_y(x_k, y_k), \bar{y}_k - y_{k+1} \rangle + \alpha_k L \biggl(\frac{1}{2} \|y_{k+1} - \bar{y}_k\|^2 + \|x_k - \bar{x}_k\|^2 + \|y_k - \bar{y}_k\|^2 \biggr)$$

$$+ \langle G(y_k)(\bar{y}_k - y_k, y_{k+1} - \bar{y}_k \rangle - \alpha_k \langle f'_y(x_k, y_k)), y_{k+1} - \bar{y}_k \rangle \geqslant 0. \tag{17}$$

To obtain a joint bound for the errors $\|x_k - x_*\|$ and $\|y_k - y^*\|$, we add (14) and (17):

$$\langle G(x_k)(x_{k+1} - x_k), x_* - x_{k+1} \rangle + \langle G(x_k)(\bar{x}_k - x_k), x_{k+1} - \bar{x}_k \rangle$$

$$+ \langle G(y_k)(y_{k+1} - y_k), y^* - y_{k+1} \rangle + \langle G(y_k)(\bar{y}_k - y_k), y_{k+1} - \bar{y}_k \rangle$$

$$+ \alpha_k (f(x_*, \bar{y}_k) - f(\bar{x}_k, y^*))$$

$$+ \alpha_k L \biggl(\frac{1}{2} \|x_{k+1} - \bar{x}_k\|^2 + \frac{1}{2} \|y_{k+1} - \bar{y}_k\|^2 + 2\|x_k - \bar{x}_k\|^2 + 2\|y_k - \bar{y}_k\|^2 \biggr) \geqslant 0. \tag{18}$$

Consider more closely the fifth term in (18):

$$\alpha_k (f(x_*, \bar{y}_k) - f(\bar{x}_k, y^*)) = \alpha_k (f(x_*, \bar{y}_k) - f(\bar{x}_k, y^*)) = \{\pm f(x_*, y^*)\}$$

$$= \alpha_k \bigl(f(x_*, \bar{y}_k) - f(x_*, y^*) + f(x_*, y^*) - f(\bar{x}_k, y^*) \bigr).$$

Since (x_*, y^*) is a saddle point of the function $f(x, y)$, we have

$$f(x_*, y) \leqslant f(x_*, y^*) \leqslant f(x, y^*) \quad \forall x \in X, \quad \forall y \in Y.$$

Setting in this relationship $x = \bar{x}_k,\ y = \bar{y}_k$, we obtain

$$f(x_*, \bar{y}_k) \leqslant f(x_*, y^*) \leqslant f(\bar{x}_k, y^*),$$

and then $f(x_*, \bar{y}_k) - f(x_*, y^*) \leqslant 0$ and $f(x_*, y^*) - f(\bar{x}_k, y^*) \leqslant 0$. Thus, their sum $f(x_*, \bar{y}_k) - f(x_*, y^*) + f(x_*, y^*) - f(\bar{x}_k, y^*)$ is also nonpositive. Since $\alpha_k > 0$, the fifth term in inequality (18) is not greater than zero and we obtain

$$\langle G(x_k)(x_{k+1} - x_k), x_* - x_{k+1} \rangle + \langle G(x_k)(\bar{x}_k - x_k), x_{k+1} - \bar{x}_k \rangle$$

$$+ \langle G(y_k)(y_{k+1} - y_k), y^* - y_{k+1} \rangle + \langle G(y_k)(\bar{y}_k - y_k), y_{k+1} - \bar{y}_k \rangle$$

$$+ \alpha_k L \left(\frac{1}{2} \|x_{k+1} - \bar{x}_k\|^2 + \frac{1}{2} \|y_{k+1} - \bar{y}_k\|^2 + 2\|x_k - \bar{x}_k\|^2 + 2\|y_k - \bar{y}_k\|^2 \right) \geqslant 0. \tag{19}$$

Applying the identity $2\langle Ga, b \rangle = \langle G(a+b), a+b \rangle - \langle Ga, a \rangle - \langle Gb, b \rangle$, we transform inequality (19) as follows:

$$\langle G(x_k)(x_* - x_k), x_* - x_k \rangle - \langle G(x_k)(x_{k+1} - x_k), x_{k+1} - x_k \rangle$$

$$- \langle G(x_k)(x_* - x_{k+1}), x_* - x_{k+1}\rangle + \langle G(x_k)(x_{k+1} - x_k), x_{k+1} - x_k\rangle$$

$$- \langle G(x_k)(\bar{x}_k - x_k), \bar{x}_k - x_k\rangle - \langle G(x_k)(x_{k+1} - \bar{x}_k), x_{k+1} - \bar{x}_k\rangle$$

$$+ \langle G(y_k)(y^* - y_k), y^* - y_k\rangle - \langle G(y_k)(y_{k+1} - y_k), y_{k+1} - y_k\rangle$$

$$- \langle G(y_k)(y^* - y_{k+1}), y^* - y_{k+1}\rangle + \langle G(y_k)(y_{k+1} - y_k), y_{k+1} - y_k\rangle$$

$$- \langle G(y_k)(\bar{y}_k - y_k), \bar{y}_k - y_k\rangle - \langle G(y_k)(y_{k+1} - \bar{y}_k), y_{k+1} - \bar{y}_k\rangle$$

$$+ 2\alpha_k L\left(\frac{1}{2}\|x_{k+1} - \bar{x}_k\|^2 + \frac{1}{2}\|y_{k+1} - \bar{y}_k\|^2 + 2\|x_k - \bar{x}_k\|^2 + 2\|y_k - \bar{y}_k\|^2\right) \geqslant 0.$$

Reducing identical terms, we obtain

$$\langle G(x_k)(x_* - x_k), x_* - x_k\rangle - \langle G(x_k)(x_* - x_{k+1}), x_* - x_{k+1}\rangle$$

$$- \langle G(x_k)(\bar{x}_k - x_k), \bar{x}_k - x_k\rangle - \langle G(x_k)(x_{k+1} - \bar{x}_k), x_{k+1} - \bar{x}_k\rangle$$

$$+ \langle G(y_k)(y^* - y_k), y^* - y_k\rangle - \langle G(y_k)(y^* - y_{k+1}), y^* - y_{k+1}\rangle$$

$$- \langle G(y_k)(\bar{y}_k - y_k), \bar{y}_k - y_k\rangle - \langle G(y_k)(y_{k+1} - \bar{y}_k), y_{k+1} - \bar{y}_k\rangle$$

$$+ 2\alpha_k L\left(\frac{1}{2}\|x_{k+1} - \bar{x}_k\|^2 + \frac{1}{2}\|y_{k+1} - \bar{y}_k\|^2 + 2\|x_k - \bar{x}_k\|^2 + 2\|y_k - \bar{y}_k\|^2\right) \geqslant 0. \tag{20}$$

Rewrite inequality (20) using the previously introduced notation for the G-norm:

$$\|x_* - x_k\|^2_{G(x_k)} - \|x_* - x_{k+1}\|^2_{G(x_k)} - \|\bar{x}_k - x_k\|^2_{G(x_k)} - \|\bar{x}_k - x_{k+1}\|^2_{G(x_k)}$$

$$+ \|y^* - y_k\|^2_{G(y_k)} - \|y^* - y_{k+1}\|^2_{G(y_k)} - \|\bar{y}_k - y_k\|^2_{G(y_k)} - \|\bar{y}_k - y_{k+1}\|^2_{G(y_k)}$$

$$+ \alpha_k L(\|x_{k+1} - \bar{x}_k\|^2 + \|y_{k+1} - \bar{y}_k\|^2 + 4\|x_k - \bar{x}_k\|^2 + 4\|y_k - \bar{y}_k\|^2) \geqslant 0.$$

Leave in the left-hand side of the inequality the norms of the k th step deviations from the solution and in the right-hand side bound some G-norms with the aid of point 3 in the conditions of the theorem:

$$\|x_* - x_k\|^2_{G(x_k)} + \|y^* - y_k\|^2_{G(y_k)}$$

$$\geqslant \|x_* - x_{k+1}\|^2_{G(x_k)} + \|y^* - y_{k+1}\|^2_{G(y_k)} + m_1\|\bar{x}_k - x_k\|^2$$

$$+ m_2\|\bar{y}_k - y_k\|^2 + m_1\|\bar{x}_k - x_{k+1}\|^2 + m_2\|\bar{y}_k - y_{k+1}\|^2$$

$$- \alpha_k L(\|x_{k+1} - \bar{x}_k\|^2 + \|y_{k+1} - \bar{y}_k\|^2 + 4\|x_k - \bar{x}_k\|^2 + 4\|y_k - \bar{y}_k\|^2).$$

By the choice of the matrices, we have the inequality

$$\|x_* - x_{k+1}\|^2_{G(x_k)} + \|y^* - y_{k+1}\|^2_{G(y_k)} \geqslant \|x_* - x_{k+1}\|^2_{G(x_{k+1})} + \|y^* - y_{k+1}\|^2_{G(y_{k+1})}.$$

Furthermore, apply point 5 in the conditions of the theorem to bound the constants in the right-hand side. We finally obtain

$$\begin{aligned}
&\|x_* - x_k\|^2_{G(x_k)} + \|y^* - y_k\|^2_{G(y_k)} \\
&\quad \geqslant \|x_* - x_{k+1}\|^2_{G(x_{k+1})} + \|y^* - y_{k+1}\|^2_{G(y_{k+1})} \\
&\qquad + \varepsilon\left(\|\bar{x}_k - x_k\|^2 + \|\bar{y}_k - y_k\|^2 + \|\bar{x}_k - x_{k+1}\|^2 + \|\bar{y}_k - y_{k+1}\|^2\right).
\end{aligned} \tag{21}$$

Sum inequalities (21) from $k = 0$ to $k = N$:

$$\begin{aligned}
&\|x_0 - x_*\|^2_{G(x_0)} + \|y_0 - y^*\|^2_{G(y_0)} \\
&\quad \geqslant \|x_{N+1} - x_*\|^2_{G(x_{N+1})} + \|y_{N+1} - y^*\|^2_{G(y_{N+1})} \\
&\qquad + \varepsilon\left(\sum_{k=0}^{N}\|\bar{x}_k - x_{k+1}\|^2 + \sum_{k=0}^{N}\|\bar{y}_k - y_{k+1}\|^2 + \sum_{k=0}^{N}\|\bar{x}_k - x_k\|^2 + \sum_{k=0}^{N}\|\bar{y}_k - y_k\|^2\right).
\end{aligned} \tag{22}$$

From inequality (22) follows boundedness of the trajectories

$$\|x_{N+1} - x_*\|^2_{G(x_{N+1})} + \|y_{N+1} - y^*\|^2_{G(y_{N+1})} \leqslant \|x_0 - x_*\|^2_{G(x_0)} + \|y_0 - y^*\|^2_{G(y_0)},$$

and also convergence of the series

$$\sum_{k=0}^{+\infty}\|\bar{x}_k - x_{k+1}\|^2 < \infty, \ \sum_{k=0}^{+\infty}\|\bar{x}_k - x_k\|^2 < \infty,$$
$$\sum_{k=0}^{+\infty}\|\bar{y}_k - y_{k+1}\|^2 < \infty, \sum_{k=0}^{+\infty}\|\bar{y}_k - y_k\|^2 < \infty.$$

Then, by the necessary condition of convergence for series,

$$\|\bar{x}_k - x_{k+1}\|^2 \to 0, \qquad \|\bar{x}_k - x_k\|^2 \to 0, \qquad \|\bar{y}_k - y_{k+1}\|^2 \to 0, \qquad \|\bar{y}_k - y_k\|^2 \to 0$$

as $k \to +\infty$. Since the sequences $\{(x_k, y_k)\}$ and $\{\alpha_k\}$ are bounded, there exist elements $\{(x', y')\}$ and α' such that

$$x_{k_i} \to x', \qquad y_{k_i} \to y', \qquad \alpha_{k_i} \to \alpha'$$

as $k_i \to +\infty$, and

$$\|\bar{x}_{k_i} - x_{k_i+1}\|^2 \to 0, \qquad \|\bar{x}_{k_i} - x_{k_i}\|^2 \to 0, \qquad \|\bar{y}_{k_i} - y_{k_i+1}\|^2 \to 0, \qquad \|\bar{y}_{k_i} - y_{k_i}\|^2 \to 0.$$

Take the limit in equations (2)–(5). Then for all $k_i \to +\infty$ we obtain

$$x' = P_X^{G(x')}\big(x' - \alpha' G(x')^{-1} f'_x(x', y')\big),$$

$$y' = P_Y^{G(y')}\big(y' + \alpha' G(y')^{-1} f'_y(x', y')\big).$$

These relationships correspond to the projection form of the optimality criterion. Thus, $x' = x_* \in X_*,\ y' = y^* \in Y^*$, i.e., every limiting point of the sequence $\{(x_k, y_k\}$ is a solution of problem (1). The sum of norms $\|x_k - x_*\| + \|y_k - y^*\|$ decreases, which ensures uniqueness of the solution, i.e., convergence $x_k \to x_*,\ y_k \to y^*$ as $k \to +\infty$. Q.E.D.

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