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NUMERICAL SOLUTION OF THE LOCALIZED INVERSE PROBLEM OF ELECTROCARDIOGRAPHY

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The localized inverse problem of electrocardiography is formulated and a solution method is proposed. The method allows determining the potential of the cardiac electric field on one of the heart sections.

Keywords: localized inverse problem of electrocardiography, boundary integral equations, regularization method.

1. Introduction

In its traditional form, the electrocardiographic inverse problem involves determining the potential on the outer surface of the heart from potential measurements on the torso surface (see, e.g., [1, 2]). One of the most important applications of inverse solution methods is diagnosis of foci of cardiac arrhythmia.

Algorithms to solve the electrocardiographic inverse problem with a model geometry of the torso and the heart have been proposed in [3]; the real geometry of the torso and the heart has been considered in [4], but a homogeneous thorax was assumed; the inverse problem of electrocardiography allowing for internal nonhomogeneity of the thorax has been considered in [5, 6, 7]; an algorithm to determine the projection of the arrhythmia focus on the outer surface of the heart has been proposed in [8]. It should be noted that in all these studies the inverse problem of determining the electric potential and localizing the arrhythmia focus was solved only on the outer cardiac surface.

The objective of the present study is to demonstrate the possibility of localization of the arrhythmia focus by measuring the electric potential not on the outer cardiac surface, but instead on the surface of a selected part of the heart where the source of arrhythmia is supposedly located.

2. The Mathematical Problem and Numerical Solution Method

Consider a three dimensional bounded region Ω_H with outer closed surface Γ_B and inner closed surface Γ_H. The surface Γ_B is the union of two surfaces Γ_T and Γ_E. These surfaces are interpreted as follows: Γ_H is the outer surface of the heart, Γ_E is the part of the human thorax surface on which the cardiac electric potential is measured, Γ_T is the union of the upper and lower cuts of the torso (Fig. 1).

The cardiac electric field is determined by sources embedded in the cardiac muscle. We assume that there is only a single source. We define three surfaces Γ_{LV}, Γ_{SP} and Γ_{RV} bounding three cardiac regions (Figure 2). The electric excitation source is located inside one of the surfaces Γ_{LV}, Γ_{SP}, Γ_{RV}. Denote by Ω_{LV}, Ω_{SP} and Ω_{RV} the regions bounded from the outside by the surface Γ_B, and from the inside by the surfaces Γ_{LV}, Γ_{SP} and Γ_{RV}, respectively.

In each of the regions Ω_{LV}, Ω_{SP} and Ω_{RV}, consider the following problem. Find the region Ω with the boundaries Γ_1 and Γ_2 and the function $u(x)$ such that

$$\Delta u(x) = 0, \quad x \in \Omega, \tag{1}$$

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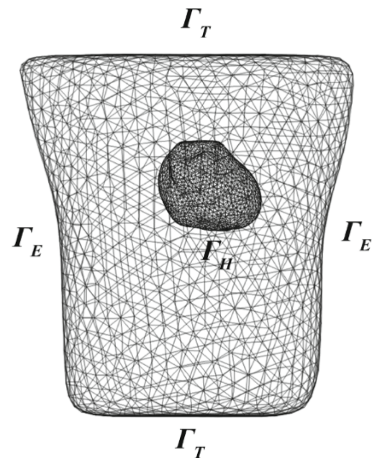

Fig. 1. The torso and the heart: a schematic.

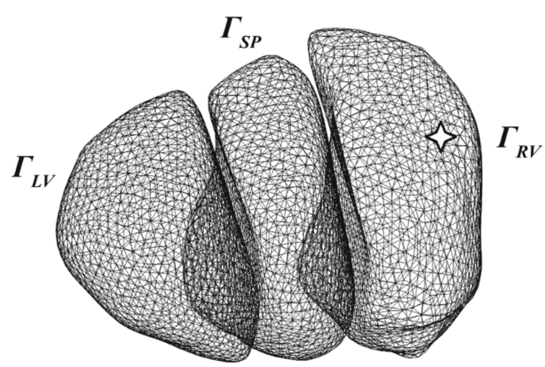

Fig. 2. Decomposition of the heart into subregions.

$$u(x) = \varphi(x), \quad x \in \Gamma_1, \tag{2}$$

$$\frac{\partial u(x)}{\partial n} = 0, \quad x \in \Gamma_1, \tag{3}$$

where $\varphi(x)$ is a known function and Ω, Γ_1 and Γ_2 are chosen for each problem as follows:

	Ω	Γ_1	Γ_2
problem 1	Ω_{LV}	Γ_E	$\Gamma_T \cup \Gamma_{LV}$
problem 2	Ω_{SP}	Γ_E	$\Gamma_T \cup \Gamma_{SP}$
problem 3	Ω_{RV}	Γ_E	$\Gamma_T \cup \Gamma_{RV}$

Problem (1)–(3) is a Cauchy problem for the Laplace equation and it is ill-posed. One of the essential manifestations of this property is instability of the potential $u(x)$ in Ω under small changes in the initial values $\varphi(x)$. Uniqueness and conditional stability of the Cauchy problem for the Laplace equation as well as development of numerical solution methods have been studied by many authors (see, e.g., [9, 13] and the references therein).

Solving problem (1)–(3) in different regions Ω_{LV}, Ω_{SP}, Ω_{LV} we can test the conjecture concerning the region containing the cardiac electric source by using electric field observations on the surface Γ_1.

The Cauchy problem (1)–(3) can be restated as the problem to find the function $u(x)$ on the surface Γ_2 given that $u(x)$ satisfies (1)–(3). Denote the unknown values of $u(x)$ on Γ_2 by $v(x)$ and consider the boundary-value problem

$$\Delta u(x) = 0, \quad x \in \Omega, \tag{4}$$

$$u(x) = v(x), \quad x \in \Gamma_2, \tag{5}$$

$$\frac{\partial u(x)}{\partial n} = 0, \quad x \in \Gamma_1. \tag{6}$$

The boundary-value problem (4)–(6) determines the operator A that maps the values of the potential $v(x)$ on the surface Γ_2 into its values $\varphi(x)$ on the surface Γ_1. Our inverse problem thus becomes a problem for an operator equation of the first kind

$$Av = \varphi, \tag{7}$$

where v is unknown and φ is given.

We apply the boundary integral equation method to construct a discrete analogue of Eq. (7). The surface $\Gamma_1 \cup \Gamma_2$ is approximated by the polygonal surface $\Sigma = \hat{\Gamma}_1 \cup \hat{\Gamma}_2$ formed as the union of N plane triangles (we call them boundary elements), $\Sigma = \zeta_1 \cup \zeta_2 \cup \ldots \cup \zeta_N$. The set of boundary elements forms the boundary-element grid. The nodes of the boundary-element grid are the points $x_i \in \Sigma$, $i = 1, 2, \ldots, N$ at the centers of gravity of the corresponding boundary elements ζ_i.

On the surface Σ we define a system of linearly independent compact-support basis functions $\phi_j(x)$, $x \in \Sigma$, $j = 1, 2, \ldots, N$

$$\begin{cases} \phi_j(x) = 1, & x \in \zeta_j, \\ \phi_j(x) = 0, & x \notin \zeta_j. \end{cases} \tag{8}$$

Consider the approximate representation of the functions $u(x)$ and $q(x) \equiv \frac{\partial u(x)}{\partial n}$ as an expansion in the system of basis functions $\phi_j(x)$

$$\tilde{u}(x) = \sum_{j=1}^{N} \alpha_j \cdot \phi_j(x), \tag{9}$$

$$\tilde{q}(x) = \sum_{j=1}^{N} \beta_j \cdot \phi_j(x), \tag{10}$$

where the expansion coefficients α_j and β_j are the values of the functions $\tilde{u}(x)$ and $\tilde{q}(x)$ at the nodes of the boundary-element grid.

For each node x_i we can write out a discrete analogue of Green's third formula

$$2\pi \tilde{u}(x_i) = \int\limits_{\Sigma} \tilde{q}(y) \frac{1}{|x_i - y|} d\Sigma - \int\limits_{\Sigma} \tilde{u}(y) \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|} d\Sigma, \tag{11}$$

where $i = 1, 2, \ldots, N$, $x_i \in \zeta_i$, $y \in \Sigma$, $|x_i - y|$ is the distance between the points x_i and y. Substituting (9) and (10) in (11), we obtain

$$2\pi\alpha_i = \int\limits_{\Sigma} \left(\sum_{j=1}^{N} \beta_j \cdot \phi_j(y) \right) \frac{1}{|x_i - y|} d\Sigma - \int\limits_{\Sigma} \left(\sum_{j=1}^{N} \alpha_j \cdot \phi_j(y) \right) \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|} d\Sigma. \tag{12}$$

Interchanging integration and summation, we obtain

$$2\pi\alpha_i = \sum_{j=1}^{N} \beta_j \int\limits_{\Sigma} \phi_j(y) \frac{1}{|x_i - y|} d\Sigma - \sum_{j=1}^{N} \alpha_j \int\limits_{\Sigma} \phi_j(y) \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|} d\Sigma. \tag{13}$$

From (8) we obtain a system of equations for α_j and β_j $(i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, N)$

$$2\pi\alpha_i + \sum_{j=1}^{N} \alpha_j \int\limits_{\zeta_j} \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|} d\zeta_j = \sum_{j=1}^{N} \beta_j \int\limits_{\zeta_j} \frac{1}{|x_i - y|} d\zeta_j. \tag{14}$$

This system can be rewritten in matrix form

$$\mathrm{H}\mathbf{u} = \mathrm{G}\mathbf{q}, \tag{15}$$

where the matrices H and G are evaluated as follows:

$$\mathrm{H} \equiv [h_{ij}] = \begin{cases} \displaystyle\int\limits_{\zeta_j} \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|}\, d\zeta_j, & i \neq j, \\ \displaystyle\int\limits_{\zeta_j} \frac{\partial}{\partial n_y} \frac{1}{|x_i - y|}\, d\zeta_j + 2\pi, & i = j, \end{cases} \tag{16}$$

$$\mathrm{G} \equiv [g_{ij}] = \int\limits_{\zeta_j} \frac{1}{|x_i - y|}\, d\zeta_j, \tag{17}$$

$\mathbf{u} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ and $\mathbf{q} = [\beta_1, \beta_2, \ldots, \beta_N]^T$.

Regrouping the elements h_{ij} and g_{ij} of the matrices H and G, we rewrite system (15) in the form

$$\begin{bmatrix} \mathrm{H}_{11} & \mathrm{H}_{12} \\ \mathrm{H}_{21} & \mathrm{H}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{G}_{11} & \mathrm{G}_{12} \\ \mathrm{G}_{21} & \mathrm{G}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \tag{18}$$

or

$$\begin{aligned} \mathrm{H}_{11}\mathbf{u}_1 + \mathrm{H}_{12}\mathbf{u}_2 &= \mathrm{G}_{11}\mathbf{q}_1 + \mathrm{G}_{12}\mathbf{q}_2 \\ \mathrm{H}_{21}\mathbf{u}_1 + \mathrm{H}_{22}\mathbf{u}_2 &= \mathrm{G}_{21}\mathbf{q}_1 + \mathrm{G}_{22}\mathbf{q}_2, \end{aligned} \tag{19}$$

where the matrices H_{kl} and G_{kl} are formed from the elements h_{ij} and g_{ij} such that $x_i \in \hat{\Gamma}_k$, $\zeta_j \in \hat{\Gamma}_l$, where $k = 1, 2$, $l = 1, 2$, and the vectors $\mathbf{u}_l$ and $\mathbf{q}_l$ are formed from the values α_j and β_j at the nodes x_j such that $x_j \in \hat{\Gamma}_l$.

The vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{q}_1$, $\mathbf{q}_2$ are discrete approximations of the function $u(x)$ and its normal derivative $\frac{\partial u(x)}{\partial n}$ on the surfaces $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$. By condition (3) $\mathbf{q}_1 = 0$ and the system takes the form

$$\begin{aligned} \mathrm{H}_{11}\mathbf{u}_1 + \mathrm{H}_{12}\mathbf{u}_2 &= \mathrm{G}_{12}\mathbf{q}_2, \\ \mathrm{H}_{21}\mathbf{u}_1 + \mathrm{H}_{22}\mathbf{u}_2 &= \mathrm{G}_{22}\mathbf{q}_2, \end{aligned} \tag{20}$$

Solving (20) for $\mathbf{u}_2$, we obtain a linear algebraic system

$$\hat{\mathrm{A}}\mathbf{u}_2 = \mathbf{u}_1 \tag{21}$$

and an expression linking the discrete approximations of the function and its normal derivative

$$\mathbf{q}_2 = \hat{\mathrm{R}}\mathbf{u}_2, \tag{22}$$

where

$$\hat{\mathrm{A}} = \left(\mathrm{H}_{11} - \mathrm{G}_{12}\mathrm{G}_{22}^{-1}\mathrm{H}_{21}\right)^{-1}\left(\mathrm{G}_{12}\mathrm{G}_{22}^{-1}\mathrm{H}_{22} - \mathrm{H}_{12}\right), \tag{23}$$

$$\hat{\mathrm{R}} = \left(\mathrm{G}_{22} - \mathrm{H}_{21}\mathrm{H}_{11}^{-1}\mathrm{G}_{12}\right)^{-1}\left(\mathrm{H}_{22} - \mathrm{H}_{21}\mathrm{H}_{11}^{-1}\mathrm{H}_{12}\right). \tag{24}$$

System (21) is the discrete analogue of the operator equation (7). Its solution is found by applying Tikhonov's regularization method [14].

Assume that for exact values of the vector $\bar{\mathbf{u}}_1$ Eq. (21) has an exact solution $\bar{\mathbf{u}}_2$, but $\bar{\mathbf{u}}_1$ is unknown and we only have its approximation $\mathbf{u}_{1_\delta}$, where the error δ is such that $\|\mathbf{u}_{1_\delta} - \bar{\mathbf{u}}_1\| \le \delta$. Given $\mathbf{u}_{1_\delta}$ and the error δ, it is requires to construct an approximate solution $\mathbf{u}_{2_\delta}$.

Consider the functional

$$M^\lambda[\mathbf{u}_2] = \|\hat{\mathrm{A}}\mathbf{u}_2 - \mathbf{u}_{1_\delta}\|^2 + \lambda\|\hat{\mathrm{R}}\mathbf{u}_2\|^2, \tag{25}$$

where λ is a positive parameter. The approximate solution $\mathbf{u}_{2_\delta}$ is defined as the element that minimizes the functional $M^\lambda[\mathbf{u}_2]$, where the regularization parameter λ depends in an appropriate manner on the error δ, i.e., $\lambda = \lambda(\delta)$ and can be obtained from the discrepancy principle

$$\|\hat{\mathrm{A}}\mathbf{u}_{2_\delta} - \mathbf{u}_{1_\delta}\|^2 = \delta \tag{26}$$

The necessary condition of minimum for the regularizing functional (25) implies that the approximate solution $\mathbf{u}_{2_\delta}$ is the solution of the operator equation

$$(\hat{\mathrm{A}}^T\hat{\mathrm{A}} + \lambda\hat{\mathrm{R}}^T\hat{\mathrm{R}})\mathbf{u}_{2_\delta} = \hat{\mathrm{A}}^T\mathbf{u}_{1_\delta}. \tag{27}$$

3. Results of Numerical Experiments

We will now consider the results of numerical experiments with a real torso and heart geometry. The torso surface Γ_B and the outer surface of the heart Γ_H were reconstructed from CT data. There were 2784 boundary elements on the surface Γ_B and 3082 on Γ_H. The cardiac surface was further partitioned into three parts corresponding to the anatomical parts of the heart:

– Γ_{LV} is the left ventricle surface, with 3120 boundary elements;

– Γ_{SP} is the surface of the interventricular septum, with 3056 boundary elements;

– Γ_{RV} is the right ventricle surface, with 3014 boundary elements.

Using the Oxford Cardiac Chaste software [15], we performed so-called virtual pacing from left and right ventricles. The virtual pacing procedure involves simulation of the cardiac electric field produced by electric stimulation of a section of the myocardium in the left and the right ventricles. Numerical simulation enabled us to reconstruct the values of the cardiac electric potential on the torso surface. The relative error was taken as $\delta = 3\%$.

Then the previous algorithm was applied to these simulation data to solve the three problems for each of the surfaces Γ_{LV}, Γ_{SP}, Γ_{RV} and the solution discrepancies on the torso surface were analyzed. The results are listed below:

	Source in Ω_{LV}	Source in Ω_{RV}
Solution for Γ_{LV} (part 1)	$7.03 \cdot 10^{-2}$	$3.45 \cdot 10^{-1}$
Solution for Γ_{SP} (part 2)	$1.21 \cdot 10^{-1}$	$1.83 \cdot 10^{-1}$
Solution for Γ_{RV} (part 3)	$2.46 \cdot 10^{-1}$	$5.69 \cdot 10^{-2}$

With the source in region Ω_{LV} the minimum error was attained with the solution for Γ_{LV}, whereas with the source in Ω_{RV} the minimum error corresponded to the solution for Γ_{RV}. Thus, given sufficiently accurate measurements, the proposed numerical algorithm reliably identifies the surface of one of the heart parts that contains the electric source.

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