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VARIABLE-METRIC CONTINUOUS PROXIMAL METHODS

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The article considers a continuous extra-proximal method for equilibrium-programming problems and proves the convergence of its trajectory to one of the solutions. A regularized analogue is constructed under classical assumptions regarding errors in input data. Its convergence to the normal solution is proved.

Keywords: equilibrium programming, extra-proximal method.

Equilibrium programming problems can be applied to tackle important tasks in various branches of mathematics, such as operations research, computational mathematics, and mathematical economics. The methods considered in this article solve such problems in the case of a nonsmooth objective function.

Antipin [1–3] has previously developed a number of proximal methods for solving equilibrium-programming problems. However, in some cases Antipin's algorithms displayed sluggish convergence due to specific features of the objective function.

One way to accelerate convergence is by an appropriately chosen change of variables so that the level surfaces in the space of new variables are close to spheres. The method thus introduces a new parameter: a symmetrical positive-definite matrix or, if the change of variables is applied at the current time instant, a family of such matrices. A symmetrical positive definite matrix can be applied to define a new scalar product and the corresponding metric in the given space. In the literature, methods of this type are called variable-metric methods or space-stretching methods. The fact that this approach yields substantial improvement of convergence is confirmed, for instance, by the well-known Newton's method for optimization problems. However, Newton's method is applied in cases when the matrix of second derivatives of the objective function can be evaluated without any difficulties. As a result, there is a whole range of so-called quasi-Newton methods in which the matrix — a parameter of the computational method — is chosen close to the second-derivative matrix. There are also variable-metric optimization methods that do not require evaluating the second-derivative matrix or its approximations.

Since the equilibrium-programming problem is closely linked with optimization problems, we naturally assumed that space stretching would be a fruitful idea also for the solution of equilibrium problems.

1. Variable-Metric Predictive Continuous Proximal Method of First Order

We consider the equilibrium-programming problem: find the point v_* from the conditions

$$v_* \in W, \quad \Phi(v_*, v_*) \leqslant \Phi(v_*, w) \quad \forall w \in W, \tag{1}$$

where $\Phi(v, w)$ is defined on the product of Euclidean spaces $E^n \times E^n$, $W \subseteq E^n$ is a given convex closed set.

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Problem (1) can be solved by the following predictive continuous proximal method:

$$\begin{cases} \dot{v}(t) + v(t) = \operatorname{Arg}\min_{w\in W}\left\{\frac{1}{2}\|w - v(t)\|^2_{G((v(t))} + \gamma(t)\Phi(u(t), w)\right\}, \\ u(t) = \operatorname{Arg}\min_{w\in W}\left\{\frac{1}{2}\|w - v(t)\|^2_{G(v(t))} + \gamma(t)\Phi(v(t), w)\right\}, \\ v(0) = v_0, \quad t \geqslant 0, \end{cases} \tag{2}$$

where v_0 is any fixed point from E^n, $\gamma(t) \geqslant 0$ a parameter of the method; $G(v)$ for each v is a given symmetrical positive definite matrix,

$$\|x\|^2_{G(v(t))} = \langle G(v(t))x, x\rangle.$$

Note that if the function $\Phi(v, w)$ is differentiable with respect to w, method (2) may be rewritten as an implicit variable-metric predictive gradient-projection method, specifically:

$$\begin{cases} \dot{v}(t) + v(t) = \pi_W^{G(v(t))}\big[v(t) - \gamma(t)G^{-1}(v(t))\Phi(u(t), \dot{v}(t) + v(t))\big], \\ u(t) = \pi_W^{G(v(t))}\big[v(t) - \gamma(t)G^{-1}(v(t))\Phi(v(t), u(t))\big], \\ v(0) = v_0, \quad t \geqslant 0. \end{cases}$$

We now give sufficient conditions for the convergence of method (2). We start with one of the key conditions, specifically, the condition of skew-symmetry of the function $\Phi(v, w)$ on the set W:

$$\Phi(v, v) - \Phi(v, w) - \Phi(w, v) + \Phi(w, w) \geqslant 0 \quad \forall v, w \in W. \tag{3}$$

Theorem 1.1. *Let the following conditions hold:*

1. *$W \subseteq E^n$ is a convex close set; the solution set of problem (1) W^* is nonempty;*
2. *The function $\Phi(v, w)$ is jointly continuous in the variables (v, w) on E^n, convex in w on W for every v from E^n, satisfies the skew-symmetry condition (3) on W; satisfies the Lipschitz condition*

$$|\Phi(v + h, w + k) - \Phi(v + h, w) - \Phi(v, w + k) + \Phi(v, w)| \leqslant L\|h\|\|k\|, \tag{4}$$

$$\forall v, v + h \in E^n, \quad w, w + k \in W;$$

3. *$G(v)$ is a symmetrical positive definite matrix for every v from E^n; there exists a strongly convex twice differentiable function $\Psi(v)$ and positive constants m, M, $m \leqslant M$, such that*

$$G(v) \equiv \Psi''(v); \qquad m\|w\|^2 \leqslant \langle G(v)w, w\rangle \leqslant M\|w\|^2 \quad \forall v, w \in E^n; \tag{5}$$

4. *The parameter $\gamma(t)$ satisfies the conditions*

$$\gamma(t) > 0; \qquad \lim_{t\to\infty}\gamma(t) = \gamma_0; \qquad 0 < \gamma_0 < \frac{m}{3L}; \tag{6}$$

5. *The solution $v(t)$ of system (2) exists and is unique for all $t \geqslant 0$.*

Then there exists a point v' from the solution set W^* of problem (1) such that

$$\lim_{t\to\infty} \|v(t) - v'\| = 0; \qquad \lim_{t\to\infty} \|\dot{v}(t)\| = 0.$$

Proof. Consider two functions:

$$\phi_{v,t}(w) = \frac{1}{2}\|w - v(t)\|^2_{G(v(t))} + \gamma(t)\Phi(v(t), w)$$

and

$$\psi_{u,v,t}(w) = \frac{1}{2}\|w - v(t)\|^2_{G(v(t))} + \gamma(t)\Phi(u(t), w).$$

By continuity and strong convexity in w of the function $\frac{1}{2}\|w - v(t)\|^2_{G(v(t))}$ we conclude that the functions $\psi_{u,v,t}(w)$ and $\phi_{v,t}(w)$ are strongly convex and lower-semicontinuous in w. Thus, using (2), we have

$$\frac{1}{2}\|w - \dot{v}(t) - v(t)\|^2_{G(v(t))} \leqslant \psi_{u,v,t}(w) - \psi_{u,v,t}(\dot{v}(t) + v(t)), \quad \forall w \in W,$$

$$\frac{1}{2}\|w - u(t)\|^2_{G(v(t))} \leqslant \phi_{v,t}(w) - \phi_{v,t}(u(t)), \quad \forall w \in W,$$

or

$$\frac{1}{2}\langle G(v(t))(w - \dot{v}(t) - v(t)), w - \dot{v}(t) - v(t)\rangle \leqslant \frac{1}{2}\langle G(v(t))(w - v(t)), w - v(t)\rangle - \frac{1}{2}\langle G(v(t))\dot{v}(t), \dot{v}(t)\rangle$$
$$+ \gamma(t)(\Phi(u(t), w) - \Phi(u(t), \dot{v}(t) + v(t))) \quad \forall w \in W, \tag{7}$$

$$\frac{1}{2}\langle G(v(t))(w - u(t)), w - u(t)\rangle \leqslant \gamma(t)(\Phi(v(t), w) - \Phi(v(t), u(t)))$$
$$+ \frac{1}{2}\langle G(v(t))(w - v(t)), w - v(t)\rangle$$
$$- \frac{1}{2}\langle G(v(t))(u(t) - v(t)), u(t) - v(t)\rangle \quad \forall w \in W. \tag{8}$$

In (7) we take for w the point $v^* \in W$ — the solution of problem (1); in (8) we take the point $\dot{v}(t) + v(t) \in W$. From now on, for simplicity we omit the argument t of the functions $\dot{v}(t),\ v(t),\ u(t)$:

$$\frac{1}{2}\langle G(v)(v^* - \dot{v} - v), v^* - \dot{v} - v\rangle$$
$$\leqslant \gamma(t)(\Phi(u, v^*) - \Phi(u, \dot{v} + v)) + \frac{1}{2}\langle G(v)(v^* - v), v^* - v\rangle - \frac{1}{2}\langle G(v)\dot{v}, \dot{v}\rangle,$$

$$\frac{1}{2}\langle G(v)(\dot{v} + v - u), \dot{v} + v - u\rangle$$
$$\leqslant \gamma(t)(\Phi(v, \dot{v} + v) - \Phi(v, u)) + \frac{1}{2}\langle G(v)\dot{v}, \dot{v}\rangle - \frac{1}{2}\langle G(v)(u - v), u - v\rangle.$$

Adding up these inequalities, we obtain

$$\langle G(v)\dot{v},\dot{v}\rangle - \langle G(v)\dot{v},u-v\rangle + \langle G(v)\dot{v},v-v^*\rangle$$
$$\leqslant \gamma(t)(\Phi(v,\dot{v}+v)-\Phi(v,u)) + \gamma(t)(\Phi(u,v^*)-\Phi(u,\dot{v}+v)) \quad \forall t \geqslant 0. \tag{9}$$

Since the function $\Phi(v,w)$ satisfies condition (3) and v^* is the solution of problem (1), we have

$$\Phi(u,v^*) \leqslant \Phi(u,u).$$

We accordingly rewrite (9) as

$$\langle G(v)\dot{v},\dot{v}\rangle - \langle G(v)\dot{v},u-v\rangle + \langle G(v)\dot{v},v-v^*\rangle + \langle G(v)(u-v),u-v\rangle$$
$$\leqslant \gamma(t)(\Phi(v,\dot{v}+v)-\Phi(v,u)) + \gamma(t)(\Phi(u,u)-\Phi(u,\dot{v}+v)). \tag{10}$$

Using (5), we obtain for the terms in the left-hand side of (10)

$$\langle G(v)\dot{v},\dot{v}\rangle + \langle G(v)(u-v),u-v\rangle - \langle G(v)\dot{v},u-v\rangle$$
$$= \frac{1}{2}\langle G(v)\dot{v},\dot{v}\rangle + \frac{1}{2}\langle G(v)(u-v),u-v\rangle$$
$$+ \frac{1}{2}\langle G(v)(\dot{v}+v-u),\dot{v}+v-u\rangle \geqslant \frac{m}{2}\|\dot{v}\|^2 + \frac{m}{2}\|u-v\|^2.$$

Then from (10)

$$\frac{m}{2}\|\dot{v}\|^2 + \langle G(v)\dot{v},v-v^*\rangle + \frac{m}{2}\|u-v\|^2$$
$$\leqslant \gamma(t)(\Phi(v,\dot{v}+v)-\Phi(v,u)+\Phi(u,u)-\Phi(u,\dot{v}+v)) \quad \forall t \geqslant 0.$$

We apply condition (4) to bound the right-hand side of the last inequality:

$$\gamma(t)(\Phi(v,\dot{v}+v)-\Phi(v,u)+\Phi(u,u)-\Phi(u,\dot{v}+v))$$
$$\leqslant \gamma(t)L\|v-u\|\|\dot{v}+v-u\|$$
$$\leqslant \gamma(t)L\left(\|u-v\|^2 + \|\dot{v}\|\|u-v\|\right) \leqslant \gamma(t)L\left(\frac{1}{2}\|\dot{v}\|^2 + \frac{3}{2}\|u-v\|^2\right),$$

and so

$$\frac{m}{2}\|\dot{v}\|^2 + \langle G(v)\dot{v},v-v^*\rangle + \frac{m}{2}\|u-v\|^2 \leqslant L\gamma(t)\left(\frac{1}{2}\|\dot{v}\|^2 + \frac{3}{2}\|v-u\|^2\right) \quad \forall t \geqslant 0. \tag{11}$$

Define the function

$$\Theta(t, v^*) = \Psi(v^*) - \Psi(v(t)) + \langle \Psi'(v(t)), v(t) - v^* \rangle,$$

where $\Psi(v)$ is the function from (5). From the properties of strongly convex function and the definition of $\Theta(t, v^*)$ it follows that

$$\frac{m}{2}\|v(t) - v^*\|^2 \leqslant \Theta(t, v^*) \leqslant \frac{M}{2}\|v(t) - v^*\|^2;$$

$$\Theta_t(t, v^*) = \langle \Psi''(v(t))\dot{v}(t), v(t) - v^* \rangle = \langle G(v(t))\dot{v}(t), v(t) - v^* \rangle \quad \forall t \geqslant 0.$$

Rewrite (11) as

$$\left(\frac{m}{2} - \frac{1}{2}L\gamma(t)\right)\|\dot{v}\|^2 + \Theta_t(t) + \left(\frac{m}{2} - \frac{3}{2}L\gamma(t)\right)\|u - v\|^2 \leqslant 0 \quad \forall t \geqslant 0. \tag{12}$$

From conditions (6) we have

$$m - 3L\gamma(t) > 2\epsilon > 0 \quad \forall t \geqslant t_0.$$

Integrating (12) on an arbitrary time interval $[\xi, t]$, where $t_0 \leqslant \xi < t$, we obtain

$$\epsilon \int_\xi^t \|\dot{v}(s)\|^2 ds + \frac{m}{2}\|v(t) - v^*\|^2 + \epsilon \int_\xi^t \|u(s) - v(s)\|^2 ds \leqslant \Theta(\xi, v^*). \tag{13}$$

From this inequality we obtain

$$\|v(t) - v^*\|^2 \leqslant \frac{M}{m}\|v(\xi) - v^*\|^2, \tag{14}$$

i.e., $\|v(t) - v^*\|^2$ is upper bounded, and all the integrals from the left-hand side of (13) converge, which yields

$$\liminf_{t\to\infty} \|\dot{v}(t)\| = 0, \qquad \liminf_{t\to\infty} \|u(t) - v(t)\| = 0.$$

Boundedness of $\|v(t) - v^*\|^2$ implies boundedness of $\|v(t)\|$, and we can thus select from the set $\{v(t)\}_{t>0}$ a convergent subsequence $\{v(t_i)\}$. Let $\lim_{t_i\to\infty}\{v(t_i)\} = v'$. Due to the convergence of the integrals, we may take

$$\lim_{t_i\to\infty} \{\dot{v}(t_i)\} = 0, \qquad \lim_{t_i\to\infty} \{u(t_i) - v(t_i)\} = 0.$$

Passing to the limit of this sequence in (7) and (8), we obtain

$$\gamma_0(\Phi(v', w) - \Phi(v', v')) \geqslant 0 \quad \forall w \in W.$$

Reducing by $\gamma_0 > 0$, we obtain that the point v' is the solution of problem (1). Setting in (14) $\xi = t_i$, $v^* = v'$, we obtain

$$\|v(t) - v'\|^2 \leqslant \frac{M}{m}\|v(t_i) - v'\|^2 \quad \forall t \geqslant t_i;$$

since the right-hand side of this inequality goes to zero as $t_i \to \infty$, we also have $\lim_{t\to\infty} \|v(t) - v'\| = 0$. Now, using the relationship

$$\langle G(v)\dot{v}, v - v'\rangle \leqslant -\frac{\epsilon}{2M^2}\|G(v)\dot{v}\|^2 - \frac{M^2}{2\epsilon}\|v - v'\|^2 \leqslant -\frac{\epsilon}{2}\|\dot{v}\|^2 - \frac{M^2}{2\epsilon}\|v - v'\|^2 \quad \forall t \geqslant 0$$

we obtain from (12) that for $v^* = v',\ t \geqslant t_0$,

$$\frac{\epsilon}{2}\|\dot{v}(t)\|^2 \leqslant \frac{M^2}{2\epsilon}\|v(t) - v'\|^2 \quad \forall t \geqslant t_0,$$

hence $\lim_{t\to\infty} \|\dot{v}(t)\| = 0$. Q.E.D.

2. Regularized Variable-Metric Predictive Continuous Proximal Method of First Order

1. Consider the following equilibrium-programming problem: find the point v_* from the conditions

$$v_* \in W = \left\{ w \in W_0 \;\middle|\; \begin{array}{ll} g_i(w) \leqslant 0, & i = 1, \dots, l; \\ g_i(w) = 0, & i = l+1, \dots, s \end{array} \right\}; \tag{15}$$
$$\Phi(v_*, v_*) \leqslant \Phi(v_*, w) \quad \forall w \in W,$$

$W_0 \subseteq E^n$ is a given convex closed set, the function $\Phi(v, w)$ is defined on the product of Euclidean spaces $E^n \times E^n$, the functions $g_i(w),\ i = 1, \dots, s$, are defined on E^n. We assume that the solution set W^* of the original problem (15) is nonempty.

To allow for equality and inequality constraints in (15), we use the simplest penalty function

$$P(w) = \sum_{i=1}^{s} (g_i^+(w))^p, \qquad p > 1, \quad w \in W_0, \tag{16}$$

where $g_i^+(w) = \max\{0; g_i(w)\}$ for $i = 1, \dots, l;\ g_i^+(w) = |g_i(w)|$ for $i = l+1, \dots, s$. Define the Tikhonov function

$$T(v, w) = \Phi(v, w) + A(t)P(w) + \alpha(t)\langle v, w\rangle, \quad v, w \in W_0, \tag{17}$$

where $A(t) > 0,\ \alpha(t) > 0$ are given functions. We know [5, 6] that if the conditions of the theorem on the convergence of the proposed method for every $t > 0$ (see below) are satisfied, then there exists a unique equilibrium point v_t of the function $T(v, w)$ defined by the condition

$$v_t \in W_0, \qquad T(v_t, v_t) \leqslant T(v_t, w) \quad \forall w \in W_0, \tag{18}$$

where $\lim_{t\to\infty} \|v_t - v_*\| = 0$, and furthermore there is a positive number R such that $\|v_t\| \leqslant R\ \forall t > 0$.

We assume that instead of the exact values of the functions $\Phi(v, w)$ and $P(w)$ we know their approximations $\Phi(v, w, t),\ P(w, t)$, such that

$$|\Phi(v, w, t) - \Phi(v, w)| \leqslant \delta(t)(1 + \|v\| + \|w\|), \tag{19}$$
$$|P(w, t) - P(w)| \leqslant \delta(t)(1 + \|w\|) \quad \forall v, w \in E^n,\ \forall t > 0,$$

where $\delta(t) \geqslant 0$ is some given function.

Consider the following continuous method for solving problem (15):

$$\begin{cases} \dot{v}(t)+v(t) = \operatorname{Arg}\min\limits_{w\in W_0} \left\{ \dfrac{1}{2}\|w-v(t)\|^2_{G((v(t))} + \gamma(t)\Big(\Phi(u(t),w,t)+A(t)P(w,t)+\alpha(t)\langle u(t),w\rangle\Big)\right\}, \\ u(t) = \operatorname{Arg}\min\limits_{w\in W_0} \left\{ \dfrac{1}{2}\|w-v(t)\|^2_{G(v(t))} + \gamma(t)\Big(\Phi(v(t),w,t)+A(t)P(w,t)+\alpha(t)\langle v(t),w\rangle\Big)\right\}, \\ v(0)=v_0, \quad t\geqslant 0, \end{cases} \tag{20}$$

where v_0 is any fixed point from E^n, $\alpha(t)$, $\gamma(t)$, $A(t)$ are the method parameters; $G(v)$ for every v is a given symmetrical positive definite matrix,

$$\|x\|^2_{G(v(t))} = \langle G(v(t))x, x\rangle.$$

We now state sufficient conditions for the convergence of method (20).

Theorem 2.1. *Let the following conditions hold:*

1. *W_0 is a convex closed set from E^n; the solution set W_* of problem (15) is nonempty; the function $\Phi(v,w)$ is continuous in v on E^n for every w from E^n, convex and continuously differentiable with respect to w on E^n for every v from E^n, satisfies the skew-symmetry condition (3) on W_0, the functions $g_i(w)$, $i=1,\dots,l$, are convex and continuously differentiable on E^n, the functions $g_i(w)$, $i=l+1,\dots,s$, are affine, i.e., $g_i(w)=\langle a_i,w\rangle - b_i$, $a_i\in E^n$, b_i are real numbers.*

2. *There exist positive constants η, c_i, $i=1,\dots,s$, such that*

$$\Phi(v_*,v_*) \leqslant \Phi(v_*,w) + \sum_{i=1}^{s} c_i(g_i^+(w))^\eta \quad \forall w\in W_0,$$

where v_* are normal solutions of problem (15), the parameter p of the penalty function (16) satisfies the conditions $p>1$, $p\geqslant\eta$.

3. *The functions $\Phi(v,w)$ and $P(w)$ satisfy the Lipschitz condition with respect to w on the set W_0:*

$$\begin{gathered} |\Phi(v,w_1)-\Phi(v,w_2)| \leqslant L\|w_1-w_2\| \quad \forall v\in E^n, \quad w_1,w_2\in W_0, \\ |P(w_1)-P(w_2)| \leqslant L\|w_1-w_2\| \quad \forall w_1,w_2\in W_0; \end{gathered} \tag{21}$$

4. *Instead of the exact values of the functions $\Phi(v,w)$ and $P(w)$ we have their convex, lower-semicontinuous approximations $\Phi(v,w,t)$, $P(w,t)$ that satisfy conditions (19).*

5. *$G(v)$ is a symmetrical positive definite matrix for every v from E^n; there exist a strongly convex twice differentiable function $\Psi(v)$ and positive constants m, M, $m\leqslant M$, such that*

$$G(v)\equiv \Psi''(v); \qquad m\|w\|^2 \leqslant \langle G(v)w,w\rangle \leqslant M\|w\|^2 \quad \forall v,w\in E^n; \tag{22}$$

6. *The parameters $\alpha(t)$, $\gamma(t)$, $\delta(t)$, $A(t)$ satisfy the conditions*

$$\alpha(t),\gamma(t),A(t)\in C^1[0;+\infty);$$

$$\delta(t) \in C[0;+\infty); \qquad \alpha(t), \gamma(t), A(t) > 0; \qquad \delta(t) \geqslant 0;$$

$\alpha(t)$ is a convex function, $\alpha'(t) \leqslant 0$; $A(t)$ is a concave function, $A'(t) \geqslant 0$;

$$\lim_{t\to+\infty} \left(\alpha(t) + \gamma(t) + \delta(t) + \frac{\delta^2(t)A^2(t)}{\alpha(t)\gamma^{1/2}(t)}\right) = 0;$$

$$\lim_{t\to+\infty} A(t) = +\infty; \quad \lim_{t\to+\infty} \frac{\gamma^{1/2}(t)}{\alpha(t)} = 0;$$

$$\lim_{t\to+\infty} \left(\frac{|\alpha'(t)| + |A'(t)|}{\alpha^2(t)\gamma(t)} + \frac{|\gamma'(t)|}{\alpha(t)\gamma^2(t)}\right) = 0; \tag{23}$$

$$\lim_{t\to+\infty} \alpha(t)(A(t))^{\frac{\eta}{p-\eta}} = +\infty.$$

(for $p = \eta$ the last equality can be dropped).

7. *The solution of system (20) exists and is unique for every $t \geqslant 0$.*

Then

$$\lim_{t\to+\infty} \|v(t) - v_*\| = 0, \qquad \lim_{t\to+\infty} \|\dot v(t)\| = 0, \tag{24}$$

where v_* is the normal solution of problem (15), and convergence in (24) is uniform with respect to the choice of $\Phi(v,w,t)$ and $P(w,t)$ from (19).

As the parameters $\alpha(t),\ \gamma(t),\ \delta(t),\ A(t)$ satisfying conditions (23) for $p = \eta$ we can take, for instance,

$$\alpha(t) = (1+t)^{-1/6}, \qquad \gamma(t) = (1+t)^{-1/2},$$

$$\delta(t) = (1+t)^{-1/2}, \qquad A(t) = (1+t)^{1/6}.$$

Proof. Consider two functions:

$$\phi_{v,t}(w) = \frac{1}{2}\|w - v(t)\|^2_{G(v(t))} + \gamma(t)\Big(\Phi(v(t),w,t) + A(t)P(w,t) + \alpha(t)\langle v(t), w\rangle\Big)$$

and

$$\psi_{u,v,t}(w) = \frac{1}{2}\|w - v(t)\|^2_{G(v(t))} + \gamma(t)\Big(\Phi(u(t),w,t) + A(t)P(w,t) + \alpha(t)\langle u(t), w\rangle\Big).$$

Continuity and strong convexity in w of the function $\frac{1}{2}\|w - v(t)\|^2_{G(v(t))}$ imply strong convexity and lower-semicontinuity in w of the functions $\psi_{u,v,t}(w)$ and $\phi_{v,t}(w)$. Therefore, from equations (20) we have

$$\frac{1}{2}\|w - \dot v(t) - v(t)\|^2_{G(v(t))} \leqslant \psi_{u,v,t}(w) - \psi_{u,v,t}(\dot v(t) + v(t)), \quad \forall w \in W_0;$$

$$\frac{1}{2}\|w - u(t)\|^2_{G(v(t))} \leqslant \phi_{v,t}(w) - \phi_{v,t}(u(t)), \quad \forall w \in W_0.$$

For simplicity we drop in what follows the argument t of the functions $\dot{v}(t),\ v(t),\ u(t)$. We rewrite the last two inequalities allowing for the definitions of the functions $\phi_{v,t}(w)$ and $\psi_{u,v,t}(w)$. We obtain

$$\frac{1}{2}\langle G(v)(w-\dot{v}-v), w-\dot{v}-v\rangle \leqslant \gamma(t)(\Phi(u,w,t)-\Phi(u,\dot{v}+v,t))$$
$$+ A(t)\gamma(t)(P(w,t)-P(\dot{v}+v,t)) + \alpha(t)\gamma(t)\langle u, w-\dot{v}-v\rangle$$
$$+ \frac{1}{2}\langle G(v)(w-v), w-v\rangle - \frac{1}{2}\langle G(v)\dot{v},\dot{v}\rangle \quad \forall w \in W_0,$$

$$\frac{1}{2}\langle G(v)(w-u), w-u\rangle \leqslant \gamma(t)(\Phi(v,w,t)-\Phi(v,u,t))$$
$$+ A(t)\gamma(t)(P(w,t)-P(u,t)) + \alpha(t)\gamma(t)\langle v, w-u\rangle$$
$$+ \frac{1}{2}\langle G(v)(w-v), w-v\rangle - \frac{1}{2}\langle G(v)(u-v), u-v\rangle \quad \forall w \in W_0.$$

Substituting for w in the first of these inequalities the point $v_\tau \in W_0$ — the solution of problem 18) for $t=\tau$, and in the second inequality the point $\dot{v}+v \in W_0$, we obtain

$$\frac{1}{2}\langle G(v)(v_\tau-\dot{v}-v), v_\tau-\dot{v}-v\rangle \leqslant \gamma(t)(\Phi(u,v_\tau,t)-\Phi(u,\dot{v}+v,t))$$
$$+ A(t)\gamma(t)(P(v_\tau,t)-P(\dot{v}+v,t)) + \alpha(t)\gamma(t)\langle u, v_\tau-\dot{v}-v\rangle$$
$$+ \frac{1}{2}\langle G(v)(v_\tau-v), v_\tau-v\rangle - \frac{1}{2}\langle G(v)\dot{v},\dot{v}\rangle \quad \forall \tau, t \geqslant 0;$$

$$\frac{1}{2}\langle G(v)(\dot{v}+v-u), \dot{v}+v-u\rangle \leqslant \gamma(t)(\Phi(v,\dot{v}+v,t)-\Phi(v,u,t))$$
$$+ A(t)\gamma(t)(P(\dot{v}+v,t)-P(u,t)) + \alpha(t)\gamma(t)\langle v, \dot{v}+v-u\rangle$$
$$+ \frac{1}{2}\langle G(v)\dot{v},\dot{v}\rangle - \frac{1}{2}\langle G(v)(u-v), u-v\rangle \quad \forall t \geqslant 0.$$

Adding up these inequalities, we obtain

$$\langle G(v)\dot{v},\dot{v}\rangle - \langle G(v)\dot{v}, u-v\rangle + \langle G(v)\dot{v}, v-v_\tau\rangle + \langle G(v)(u-v), u-v\rangle$$
$$\leqslant \gamma(t)(\Phi(v,\dot{v}+v,t)-\Phi(v,u,t)) + \gamma(t)(\Phi(u,v_\tau,t)-\Phi(u,\dot{v}+v,t))$$
$$+ A(t)(P(v_\tau,t)-P(u,t)) + \alpha(t)\gamma(t)\langle u, v_\tau-\dot{v}-v\rangle + \alpha(t)\gamma(t)\langle v, \dot{v}+v-u\rangle \quad \forall \tau, t \geqslant 0. \qquad (25)$$

Since v_τ is the solution of problem (18) for $t=\tau$, we obtain

$$0 \leqslant \Phi(w,v_\tau) - \Phi(v_\tau,v_\tau) + A(\tau)(P(w)-P(v_\tau)) + \alpha(\tau)\langle v_\tau, w-v_\tau\rangle \quad \forall w \in W_0.$$

Multiply this inequality by $\gamma(t) > 0,$ set $w = u(t) \in W_0$ $\forall t \geqslant 0,$ and add up with (25). This gives

$$\begin{aligned}
&\langle G(v)\dot v, \dot v\rangle - \langle G(v)\dot v, u - v\rangle + \langle G(v)\dot v, v - v_\tau\rangle + \langle G(v)(u - v), u - v\rangle\\
&\quad\leqslant \gamma(t)(\Phi(v, \dot v + v, t) - \Phi(v, u, t)) + \gamma(t)(\Phi(u, v_\tau, t) - \Phi(u, \dot v + v, t) + \Phi(v_\tau, u) - \Phi(v_\tau, v_\tau))\\
&\quad+ A(t)\gamma(t)(P(v_\tau, t) - P(u, t)) + A(\tau)\gamma(t)(P(u) - P(v_\tau))\\
&\quad+ \alpha(t)\gamma(t)(\langle u - v, u - \dot v - v\rangle + \langle u, v_\tau - u\rangle) + \alpha(\tau)\gamma(t)\langle v_\tau, u - v_\tau\rangle \quad \forall \tau, t \geqslant 0.
\end{aligned} \tag{26}$$

Since

$$\begin{aligned}
&\alpha(t)\gamma(t)\langle u - v, u - \dot v - v\rangle + \alpha(t)\gamma(t)\langle u, v_\tau - u\rangle + \alpha(\tau)\gamma(t)\langle v_\tau, u - v_\tau\rangle\\
&= \alpha(t)\gamma(t)\|u - v\|^2 - \alpha(t)\gamma(t)\langle u - v, \dot v\rangle - \alpha(t)\gamma(t)\|u - v_\tau\|^2\\
&\quad+ (\alpha(\tau) - \alpha(t))\gamma(t)\langle v_\tau, u - v_\tau\rangle\\
&= \alpha(t)\gamma(t)\langle v - u, \dot v\rangle - \alpha(t)\gamma(t)\|v - v_\tau\|^2 - 2\alpha(t)\gamma(t)\langle u - v, v - v_\tau\rangle\\
&\quad+ (\alpha(t) - \alpha(\tau))\gamma(t)\langle v_\tau, v_\tau - u\rangle \quad \forall \tau, t \geqslant 0,
\end{aligned}$$

and, using (22),

$$\begin{aligned}
&\langle G(v)\dot v, \dot v\rangle - \langle G(v)\dot v, u - v\rangle + \langle G(v)(u - v), u - v\rangle\\
&= \frac{1}{2}\langle G(v)(u - v), u - v\rangle + \frac{1}{2}\langle G(v)\dot v, \dot v\rangle + \frac{1}{2}\langle G(v)(\dot v + v - u), \dot v + v - u\rangle\\
&\geqslant \frac{m}{2}\|\dot v\|^2 + \frac{m}{2}\|u - v\|^2 \quad \forall t \geqslant 0,
\end{aligned}$$

we obtain from (26)

$$\begin{aligned}
&\frac{m}{2}\|\dot v\|^2 + \frac{m}{2}\|u - v\|^2 + \langle G(v)\dot v, v - v_\tau\rangle + \alpha(t)\gamma(t)\|v - v_\tau\|^2\\
&\quad\leqslant \gamma(t)(\Phi(v, \dot v + v, t) - \Phi(v, u, t) + \Phi(u, v_\tau, t) - \Phi(u, \dot v + v, t))\\
&\quad+ \gamma(t)(\Phi(v_\tau, u) - \Phi(v_\tau, v_\tau)) + \gamma(t)(A(t)(P(v_\tau, t) - P(u, t))\\
&\quad+ \gamma(t)A(\tau)(P(u) - P(v_\tau)) + \gamma(t)(\alpha(t) - \alpha(\tau))\langle v_\tau, v_\tau - u\rangle\\
&\quad+ \alpha(t)\gamma(t)(\langle v - u, \dot v\rangle - 2\langle u - v, v - v_\tau\rangle).
\end{aligned} \tag{27}$$

Now, applying the Cauchy–Bunyakovskii inequalities, the elementary inequalities

$$\|a + b\| \leqslant \|a\| + \|b\|, \qquad 2ab \leqslant \epsilon a^2 + \epsilon^{-1}b^2, \quad \epsilon > 0, \qquad (a_1 + \ldots + a_m)^2 \leqslant m(a_1^2 + \ldots + a_m^2),$$

and the fact that $\|v_\tau\| \leqslant R,$ we bound the terms in the right-hand side of (27). First note that, by conditions (19), (21), (3), we have

$$\Phi(v, \dot{v} + v, t) - \Phi(v, u, t) = \Phi(v, \dot{v} + v, t) - \Phi(v, \dot{v} + v)$$

$$+ \Phi(v, \dot{v} + v) - \Phi(v, u) + \Phi(v, u) - \Phi(v, u, t);$$

$$\Phi(u, v_\tau, t) - \Phi(u, \dot{v} + v, t) = \Phi(u, v_\tau, t) - \Phi(u, v_\tau)$$

$$+ \Phi(u, v_\tau) - \Phi(u, \dot{v} + v) + \Phi(u, \dot{v} + v) - \Phi(u, \dot{v} + v, t);$$

$$|\Phi(v, \dot{v} + v, t) - \Phi(v, \dot{v} + v)| \leqslant \delta(t)(1 + \|v\| + \|\dot{v} + v\|)$$

$$\leqslant \delta(t)(1 + 2\|v - v_\tau\| + \|\dot{v}\| + 2\|v_\tau\|);$$

$$|\Phi(v, u) - \Phi(v, u, t)| \leqslant \delta(t)(1 + \|v\| + \|u\|)$$

$$\leqslant \delta(t)(1 + 2\|v - v_\tau\| + \|u - v\| + 2\|v_\tau\|);$$

$$|\Phi(u, v_\tau, t) - \Phi(u, v_\tau)| \leqslant \delta(t)(1 + \|u\| + \|v_\tau\|)$$

$$\leqslant \delta(t)(1 + \|u - v\| + \|v - v_\tau\| + 2\|v_\tau\|);$$

$$|\Phi(u, \dot{v} + v) - \Phi(u, \dot{v} + v, t)| \leqslant \delta(t)(1 + \|u - v\| + 2\|v - v_\tau\| + \|\dot{v}\| + 2\|v_\tau\|);$$

$$\Phi(v, \dot{v} + v) - \Phi(v, u) + \Phi(u, v_\tau) - \Phi(u, \dot{v} + v) + \Phi(v_\tau, u) - \Phi(v_\tau, v_\tau)$$

$$= \Phi(v, \dot{v} + v) - \Phi(v, u) + \Phi(u, u) - \Phi(u, \dot{v} + v)$$

$$- (\Phi(u, u) - \Phi(u, v_\tau) - \Phi(v_\tau, u) + \Phi(v_\tau, v_\tau))$$

$$\leqslant \Phi(v, \dot{v} + v) - \Phi(v, u) + \Phi(u, u) - \Phi(u, \dot{v} + v) \leqslant 2L(\|\dot{v}\| + \|u - v\|).$$

Then for the first term we have

$$\gamma(t)(\Phi(v, \dot{v} + v, t) - \Phi(v, u, t) + \Phi(u, v_\tau, t) - \Phi(u, \dot{v} + v, t))$$

$$+ \gamma(t)(\Phi(v_\tau, u) - \Phi(v_\tau, v_\tau)) \leqslant \gamma(t)\delta(t)(4 + 2\|\dot{v}\| + 3\|u - v\|$$

$$+ 7\|v - v_\tau\| + 8\|v_\tau\|) + 2L\gamma(t)(\|\dot{v}\| + \|u - v\|)$$

$$\leqslant \frac{1}{4}\gamma^{1/2}(t)\delta^2(t)(4 + 2\|\dot{v}\| + 3\|u - v\| + 7\|v - v_\tau\| + 8\|v_\tau\|)^2$$

$$+ \gamma^{3/2}(t) + L^2\gamma^{1/2}(t)(\|\dot{v}\| + \|u - v\|)^2 + \gamma^{3/2}(t)$$

$$\leqslant C_1\gamma^{1/2}(t)(1+\delta^2(t))\left(\|\dot v\|^2+\|u-v\|^2\right)$$

$$+\,C_2\gamma^{1/2}(t)\delta^2(t)\left(\|v-v_\tau\|^2+(1+R)^2\right)+2\gamma^{3/2}(t). \tag{28}$$

In (28) and below we denote by C_i constants whose specific form is immaterial for our purposes. Note that these constants may depend only on $L,\ m,\ M,\ R$, but not on $u,\ v,\ v_\tau,\ \dot v,\ t$.

For the second term:

$$\gamma(t)(A(t)(P(v_\tau,t)-P(u,t))+A(\tau)(P(u)-P(v_\tau)))$$

$$=A(t)\gamma(t)(P(v_\tau,t)-P(v_\tau))$$

$$+\,A(t)\gamma(t)(P(u)-P(u,t))+\gamma(t)(A(t)-A(\tau))(P(v_\tau)-P(u))$$

$$\leqslant A(t)\gamma(t)\delta(t)(2+\|u\|+\|v_\tau\|)+\gamma(t)\left(\sqrt{\frac{\alpha(t)}{4}}\|v_\tau-u\|\right)\left(L\frac{|A(t)-A(\tau)|}{\sqrt{\frac{\alpha(t)}{4}}}\right)$$

$$\leqslant \gamma^{3/2}(t)+A^2(t)\gamma^{1/2}(t)\delta^2(t)(2+\|u-v\|+\|v-v_\tau\|+2\|v_\tau\|)^2$$

$$+\,\frac{\alpha(t)\gamma(t)}{8}(\|v_\tau-v\|+\|v-u\|)^2+2L^2\gamma(t)\frac{(A(t)-A(\tau))^2}{\alpha(t)}$$

$$\leqslant C_3A^2(t)\gamma^{1/2}(t)\delta^2(t)\left(\|u-v\|^2+\|v-v_\tau\|^2+(1+R)^2\right)^2$$

$$+\,\frac{\alpha(t)\gamma(t)}{4}\left(\|v-v_\tau\|^2+\|u-v\|^2\right)+2L^2\gamma(t)\frac{(A(t)-A(\tau))^2}{\alpha(t)}+\gamma^{3/2}(t)\qquad\forall\tau,t\geqslant 0. \tag{29}$$

The third term is bounded as

$$\alpha(t)\gamma(t)(\langle v-u,\dot v\rangle-2\langle u-v,v-v_\tau\rangle)$$

$$\leqslant\alpha(t)\gamma(t)\left(\frac14\|\dot v\|^2+5\|u-v\|^2+\frac14\|v-v_\tau\|^2\right)\quad\forall\tau,t\geqslant 0. \tag{30}$$

For the fourth term in the right-hand side of (27) we have

$$\gamma(t)(\alpha(t)-\alpha(\tau))\langle v_\tau,v_\tau-u\rangle\leqslant\gamma(t)\sqrt{\frac{\alpha(t)}{4}}\|v_\tau-u\|\left(\frac{|\alpha(t)-\alpha(\tau)|}{\sqrt{\frac{\alpha(t)}{4}}}\|v_\tau\|\right)$$

$$\leqslant\frac{\alpha(t)\gamma(t)}{8}(\|v_\tau-v\|+\|v-u\|)^2+2R^2\gamma(t)\frac{(\alpha(t)-\alpha(\tau))^2}{\alpha(t)}\quad\forall\tau,t\geqslant 0. \tag{31}$$

Using (28)–(31), we obtain from (27)

$$\left(\frac m2-C_1\gamma^{1/2}(t)(1+\delta^2(t))-\frac{\alpha(t)\gamma(t)}{4}\right)\|\dot v\|^2$$

$$+ \langle G(v)\dot{v}, v - v_\tau \rangle + \Big(\frac{m}{2} - C_1 \gamma^{1/2}(t)(1 + \delta^2(t)) - \frac{21}{4}\alpha(t)\gamma(t)$$

$$- C_3 A^2(t)\gamma^{1/2}(t)\delta^2(t) \Big) \|u - v\|^2$$

$$+ \alpha(t)\gamma(t) \left(\frac{1}{4} - \frac{(C_2 + C_3 A^2(t))\delta^2(t)}{\alpha(t)\gamma^{1/2}(t)} \right) \|v - v_\tau\|^2$$

$$\leqslant f(t,\tau) = 3\gamma^{3/2}(t) + (C_2 + C_3 A^2(t))\gamma^{1/2}(t)\delta^2(t)(1 + R)^2$$

$$+ 2\frac{\gamma(t)}{\alpha(t)} \left(R^2(\alpha(t) - \alpha(\tau))^2 + L^2(A(t) - A(\tau))^2 \right) \quad \forall \tau, t \geqslant 0. \tag{32}$$

By conditions (23), the coefficients of $\|\dot{v}\|^2,\ \|u - v\|^2,\ \|v - v_\tau\|^2$ in (32) satisfy the following inequalities:

$$\frac{m}{2} - C_1 \gamma^{1/2}(t)(1 + \delta^2(t)) - \frac{1}{4}\alpha(t)\gamma(t) \geqslant \frac{m}{4};$$

$$\frac{m}{2} - C_1 \gamma^{1/2}(t)(1 + \delta^2(t)) - \frac{21}{4}\alpha(t)\gamma(t) - C_3 A^2(t)\gamma^{1/2}(t)\delta^2(t) \geqslant \frac{m}{4};$$

$$\frac{1}{4} - \frac{(C_2 + C_3 A^2(t))\delta^2(t)}{\alpha(t)\gamma^{1/2}(t)} \geqslant \frac{1}{8} \quad \forall t \geqslant t_0.$$

Thus from (32) we have

$$\frac{m}{4}\|\dot{v}\|^2 + \frac{m}{4}\|u - v\|^2 + \langle G(v)\dot{v}, v - v_\tau \rangle + \frac{1}{8}\alpha(t)\gamma(t)\|v - v_\tau\|^2 \leqslant f(t,\tau) \quad \forall \tau \geqslant 0, \quad t \geqslant t_0.$$

Define the function

$$\Theta(t, v_\tau) = \Psi(v_\tau) - \Psi(v(t)) + \langle \Psi'(v(t)), v(t) - v_\tau \rangle,$$

where $\Psi(v)$ is the functions from (22). From the definition of $\Theta(t, v_\tau)$ and (22) we obtain

$$\Theta_t(t, v_\tau) = \langle \Psi''(v(t))\dot{v}(t), v(t) - v_\tau \rangle = \langle G(v(t))\dot{v}(t), v(t) - v_\tau \rangle;$$

$$\frac{m}{2}\|v(t) - v_\tau\|^2 \leqslant \Theta(t, v_\tau) \leqslant \frac{M}{2}\|v(t) - v_\tau\|^2 \quad \forall \tau, t \geqslant 0.$$

Thus we have

$$\frac{m}{4}\|\dot{v}\|^2 + \frac{m}{4}\|u - v\|^2 + \Theta_t(t, v_\tau) + \frac{1}{8}\alpha(t)\gamma(t)\|v - v_\tau\|^2 \leqslant f(t,\tau) \quad \forall t \geqslant t_0. \tag{33}$$

Multiply inequality (33) by the function $h(t) = e^{b\int_0^t \alpha(s)\gamma(s)ds}$, where $b > 0$ is chosen from the condition $4Mb < 1$, and integrate it on an arbitrary closed interval $[\xi, t],\ t_0 \leqslant \xi < t$. We obtain

$$\int_\xi^t \frac{m}{4}h(s)\|\dot{v}(s)\|^2 ds + \int_\xi^t \frac{m}{4}h(s)\|u(s)-v(s)\|^2 ds$$

$$+\int_\xi^t h(s)\Theta_t(s,\tau)ds + \int_\xi^t \frac{1}{8}\alpha(s)\gamma(s)h(s)\|v(s)-v_\tau\|^2 ds$$

$$\leqslant \int_\xi^t f(s,\tau)h(s)ds \quad \forall\tau\geqslant 0, \quad t_0\leqslant\xi<t. \tag{34}$$

Noting that $h'(s)=b\alpha(s)\gamma(s)h(s)>0,$ we have

$$\int_\xi^t h(s)\Theta_t(s,\tau)ds = h(s)\Theta(s,\tau)\Big|_{s=\xi}^{s=t} - \int_\xi^t h'(s)\Theta(s,\tau)ds$$

$$\geqslant \frac{m}{2}h(t)\|v(t)-v_\tau\|^2 - \frac{M}{2}h(\xi)\|v(\xi)-v_\tau\|^2 - \int_\xi^t \frac{M}{2}b\alpha(s)\gamma(s)h(s)\|v(s)-v_\tau\|^2 ds.$$

Then from (34), noting that the first two terms in its left-hand side are nonnegative, we obtain the inequality

$$\frac{m}{2}h(t)\|v(t)-v_\tau\|^2 + \int_\xi^t \alpha(s)\gamma(s)h(s)\left(\frac{1}{8}-\frac{Mb}{2}\right)\|v(s)-v_\tau\|^2 ds$$

$$\leqslant \int_\xi^t f(s,\tau)h(s)ds + M\left(\|v(\xi)\|^2+R^2\right)h(\xi) \quad \forall\tau\geqslant 0, \quad t_0\leqslant\xi<t,$$

or, recalling that $4Mb<1,$

$$\frac{m}{2}h(t)\|v(t)-v_\tau\|^2 \leqslant \int_\xi^t f(s,\tau)h(s)ds + C_4(\xi) \quad \forall\tau\geqslant 0, \quad t_0\leqslant\xi<t. \tag{35}$$

Since $\alpha(t)$ is convex and decreasing, we have

$$0\leqslant \alpha(s)-\alpha(t)\leqslant \alpha'(s)(s-t) \quad \forall s,t,$$

since $A(t)$ is concave and increasing, we have

$$0\leqslant A(t)-A(s)\leqslant A'(s)(t-s) \quad \forall s,t.$$

Hence, recalling the definition (32) of the function $f(t,\tau)$ and (35), we obtain

$$\|v(t)-v_\tau\|^2 \leqslant \frac{2C_3}{mh(t)} \int\limits_\xi^t \Bigg[3\gamma^{3/2}(s)h(s) + (C_2 + C_3A^2(s))\gamma^{1/2}(s)\delta^2(s)(1+R)^2h(s)$$

$$+ C_5\frac{\gamma(s)}{\alpha(s)}((\alpha'(s))^2 + (A'(s))^2)(s-\tau)^2\Bigg]ds \quad \forall\tau\geqslant 0, \quad t_0\leqslant\xi<t.$$

In this inequality set $\tau = t$:

$$\|v(t)-v_t\|^2 \leqslant \frac{2C_3}{mh(t)} \int\limits_\xi^t \Bigg[3\gamma^{3/2}(s)h(s) + (C_2 + C_3A^2(s))\gamma^{1/2}(s)\delta^2(s)(1+R)^2h(s)$$

$$+ C_5\frac{\gamma(s)}{\alpha(s)}((\alpha'(s))^2 + (A'(s))^2)(t-s)^2\Bigg]ds \quad \forall t_0\leqslant\xi<t. \tag{36}$$

In what follows we will need two lemmas [4].

Lemma 2.1. *Let the function* $f(t)\in C^1[0;+\infty)$ *be such that*

$$f(t)>0,\ f'(t)\leqslant 0 \quad \forall t>0; \qquad \lim_{t\to\infty}\frac{f'(t)}{f^2(t)} = 0.$$

Then

$$\int\limits_0^{+\infty} f(s)ds = +\infty; \qquad \lim_{t\to\infty} f^n(t)e^{\int_0^t f(s)ds} = +\infty \quad \forall n = 0,1,2,\dots.$$

Lemma 2.2. *Assume that the function* $f(t)$ *satisfies the conditions of Lemma 2.1,* $b>0$, *and*

$$g(t) = e^{b\int_0^t f(s)ds}.$$

Then

$$\lim_{t\to\infty}\frac{\frac{d}{dt}(f^n(t)g(t))}{f^{n+1}(t)g(t)} = b \quad \forall n=0,1,2,\dots\ .$$

In (36) go to the limit as $t\to+\infty$, applying Lemmas 2.1 and 2.2. Set $f(t)=\alpha(t)\gamma(t)$. From (23) we obtain

$$f'(t) = \alpha'(t)\gamma(t) + \alpha(t)\gamma'(t)\leqslant 0;$$

$$\lim_{t\to\infty}\frac{f'(t)}{f^2(t)} = \lim_{t\to\infty}\left(\frac{\alpha'(t)}{\alpha^2(t)\gamma(t)} + \frac{\gamma'(t)}{\alpha(t)\gamma^2(t)}\right) = 0.$$

Then by Lemma 2.1,

$$\lim_{t\to\infty} h(t) = +\infty$$

and by Lemma 2.2,

$$\lim_{t\to\infty} \frac{\alpha^{n+1}(t)\gamma^{n+1}(t)h(t)}{\frac{d}{dt}(\alpha^n(t)\gamma^n(t)h(t))} = \frac{1}{b}.$$

Applying this result and the classical L'Hopital's rule, we obtain from (33)

$$\lim_{t\to\infty} \frac{1}{h(t)} \int\limits_{\xi}^{t} \gamma^{3/2}(s)h(s)ds = \lim_{t\to\infty} \frac{\gamma^{3/2}(t)h(t)}{h'(t)} = \lim_{t\to\infty} \frac{\gamma^{1/2}(t)}{b\alpha(t)} = 0;$$

$$\lim_{t\to\infty} \frac{C_4(\xi)}{h(t)} = 0;$$

$$\lim_{t\to\infty} \frac{1}{h(t)} \int\limits_{\xi}^{t} (C_2 + C_3A^2(s))\gamma^{1/2}(s)\delta^2(s)(1+R)^2h(s)ds$$

$$= \lim_{t\to\infty} \frac{(C_2 + C_3A^2(t))\gamma^{1/2}(t)\delta^2(t)h(t)}{b\alpha(t)\gamma(t)h(t)} = \lim_{t\to\infty} \frac{\delta^2(t)(1+A^2(t))}{\alpha(t)\gamma^{1/2}(t)} = 0;$$

$$\lim_{t\to\infty} \frac{1}{h(t)} \int\limits_{\xi}^{t} \frac{\gamma(s)}{\alpha(s)}((\alpha'(s))^2 + (A'(s))^2)(t-s)^2h(s)ds$$

$$= \lim_{t\to\infty} \frac{\int \xi^t \frac{2\gamma(s)}{\alpha(s)}((\alpha'(s))^2 + (A'(s))^2)(t-s)h(s)ds}{b\alpha(t)\gamma(t)h(t)}$$

$$= \lim_{t\to\infty} \frac{\int \xi^t \frac{2\gamma(s)}{\alpha(s)}((\alpha'(s))^2 + (A'(s))^2)h(s)ds}{b(\alpha(t)\gamma(t))^2h(t)} \cdot \frac{(\alpha(t)\gamma(t))^2h(t)}{\frac{d}{dt}(\alpha(t)\gamma(t)h(t))}$$

$$= \lim_{t\to\infty} \frac{2\gamma(t)((\alpha'(t))^2 + (A'(t))^2)h(t)}{\alpha(t)b^2(\alpha(t)\gamma(t))^3h(t)} \cdot \frac{(\alpha(t)\gamma(t))^3h(t)}{\frac{d}{dt}((\alpha(t)\gamma(t))^2h(t))}$$

$$= \lim_{t\to\infty} \frac{2}{b^3}\left(\frac{\alpha'(t)}{\alpha^2(t)\gamma(t)}\right)^2 + \left(\frac{A'(t)}{\alpha^2(t)\gamma(t)}\right)^2 = 0.$$

Hence

$$\lim_{t\to\infty} \|v(t) - v_t\| = 0.$$

We have previously noted that under the conditions of this theorem $\lim_{t\to\infty} \|v_t - v_*\| = 0$. Thus,

$$\lim_{t\to\infty} \|v(t) - v_*\| = 0.$$

To prove the second equality in (24), we return to inequality (33) and consider it for $\tau = t$:

$$\frac{m}{4}\|\dot{v}(t)\|^2 \leqslant -\langle G(v(t))\dot{v}(t), v(t) - v_t\rangle + C\left(\gamma^{3/2}(t) + (1 + A^2(t))\gamma^{1/2}(t)\delta^2(t)\right) \quad \forall t \geqslant t_0.$$

Noting that

$$\begin{aligned}
-\langle G(v(t))\dot{v}(t), v(t) - v_t\rangle &\leqslant \|G(v(t))\dot{v}(t)\|\|v(t) - v_t\| \\
&\leqslant \frac{m}{8M^2}\|G(v(t))\dot{v}(t)\|^2 + \frac{2M^2}{m}\|v(t) - v_t\|^2 \\
&\leqslant \frac{m}{8}\|\dot{v}(t)\|^2 + \frac{2M^2}{m}\|v(t) - v_t\|^2,
\end{aligned}$$

we obtain

$$\frac{m}{8}\|\dot{v}(t)\|^2 \leqslant C\left(\gamma^{3/2}(t) + (1 + A^2(t))\gamma^{1/2}(t)\delta^2(t)\right) + \frac{2M^2}{m}\|v(t) - v_t\|^2.$$

Since $\lim_{t\to\infty} \|v(t) - v_t\|^2 = 0$, from the last inequality we have

$$0 \leqslant \liminf_{t\to\infty} \|\dot{v}(t)\|^2 \leqslant \limsup_{t\to\infty} \|\dot{v}(t)\| \leqslant 0 \Rightarrow \lim_{t\to\infty} \|\dot{v}(t)\| = 0.$$

Uniform convergence in (24) with respect to the choice of $\Phi(v, w, t)$ and $P(w, t)$ from (19) follows from the fact that the coefficients in (33) and in subsequent inequalities used to prove (24) are independent of the particular realizations of $\Phi(v, w, t)$ and $P(w, t)$. Q.E.D.

2. In practice, instead of the condition (19) with $\delta(t) \to 0$, it is more realistic to expect that we know the approximations $\Phi_\delta(v, w)$ and $P_\delta(w)$ of the functions $\Phi(v, w)$, $P(w)$ satisfying the following conditions:

$$\begin{aligned}
&|\Phi_\delta(v, w) - \Phi(v, w)| \leqslant \delta(1 + \|v\| + \|w\|), \quad v, w \in E^n, \\
&|P_\delta(w) - P(w)| \leqslant \delta(1 + \|w\|), \quad w \in E^n,
\end{aligned} \tag{37}$$

where $\delta > 0$ is a fixed number. Then we may consider the process

$$\begin{cases}
\dot{v}(t) + v(t) = \operatorname{Arg}\min\limits_{w\in W_0} \left\{\frac{1}{2}\|w - v(t)\|^2_{G((v(t))} + \gamma(t)\Big(\Phi_\delta(u(t), w) + A(t)P_\delta(w) + \alpha(t)\langle u(t), w\rangle\Big)\right\}, \\
u(t) = \operatorname{Arg}\min\limits_{w\in W_0} \left\{\frac{1}{2}\|w - v(t)\|^2_{G(v(t))} + \gamma(t)\Big(\Phi_\delta(v(t), w) + A(t)P_\delta(w) + \alpha(t)\langle v(t), w\rangle\Big)\right\}, \\
v(0) = v_0, \quad t \geqslant 0,
\end{cases} \tag{38}$$

which is obtained from (20) by replacing $\Phi(v, w, t)$ and $P(w, t)$ with $\Phi_\delta(v, w)$ and $P_\delta(w)$, respectively.

Assume that $\alpha(t), \gamma(t), \delta(t)$, satisfying the conditions of Theorem 2.1 are fixed, $\delta(0) > \delta$. For every fixed $\delta\colon 0 < \delta < \delta(0)$ we continue the process (38) to the time instant $t = t(\delta)$ determined by the condition

$$t(\delta) = \sup\big\{t\colon \delta(s) > \delta;\ 0 \leqslant s \leqslant t\big\} \tag{39}$$

Since $\delta(t) \to \infty$ as $t \to \infty$, $\delta(0) > \delta$, we conclude that $t(\delta)$ is finite $\forall\delta > 0$.

Theorem 2.2. *Assume that all the conditions of Theorem 2.1 hold, with the exception of point 4; the approximations $\Phi_\delta(v, w)$, $P_\delta(w)$ satisfy (37); $v(t)\colon 0 \leqslant t \leqslant t(\delta)$ is the trajectory of the process (38), where the time instant $t(\delta)$ is determined from (39). Then*

$$\lim_{\delta\to 0} \|v(t(\delta)) - v_*\| = 0.$$

Proof. From (37) and (39) we obtain

$$|\Phi_\delta(v, w) - \Phi(v, w)| \leqslant \delta(t)(1 + \|v\| + \|w\|), \quad 0 \leqslant t \leqslant t(\delta);$$

$$|P_\delta(w) - P(w)| \leqslant \delta(t)(1 + \|w\|), \quad 0 \leqslant t \leqslant t(\delta).$$

i.e., the approximations $\Phi_\delta(v, w)$ and $P_\delta(w)$ satisfy conditions (19) for every $t\colon 0 \leqslant t \leqslant t(\delta)$. By the stopping rule (39), $\lim_{t\to+\infty} \delta(t) = 0$ implies that $\lim_{\delta\to 0} t(\delta) = +\infty$. Hence, for small $\delta > 0$, the time instant $t(\delta)$ may be arbitrarily large.

By the preceding theorem, when all its conditions are satisfied, the trajectory $v(t)$ generated by method (38) converges in norm to the point v_*, i.e., $\forall\epsilon > 0$ there exists $T = T(\epsilon)\colon \|v(t) - v_*\| < \epsilon$, $\forall t \geqslant T(\epsilon)$, where $T(\epsilon)$ is independent of the choice of the realizations $\Phi(v, w, t)$ and $P(w, t)$. Since $\lim_{\delta\to 0} t(\delta) = +\infty$, there exists $\delta(\epsilon) > 0\colon t(\delta) \geqslant T(\epsilon)\ \forall\delta\colon 0 < \delta < \delta(\epsilon)$.

Thus, $\forall\delta\colon 0 < \delta < \delta(\epsilon)$ method (38) for $0 \leqslant t \leqslant t(\delta)$, where $t(\delta)$ is taken from (39), generates the trajectory $v(t)$ for $0 \leqslant t \leqslant t(\delta)$, which may be obtained also by method (20) with $\Phi(v, w, t) = \Phi_\delta(v, w)$, $P(w, t) = P_\delta(w)$.

Since $t(\delta) > T(\epsilon)$, we obtain for $t = t(\delta)$

$$\|v(t(\delta)) - v_*\| \leqslant \epsilon \quad \forall\delta\colon 0 < \delta < \delta(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, this proves the assertion of the theorem. Q.E.D.

It follows from this theorem that the operator R_δ associating the point $\big(\Phi_\delta(v, w), P_\delta(w), \delta, \alpha(t), \gamma(t), \delta(t)\big)$ defined by (38) to the tuple of values $v(\delta) = v(t(\delta))$ is a regularizing operator.

3. In cases when the set W is known exactly, we can apply the following continuous method to solve problem (15):

$$\begin{cases} \dot v(t) + v(t) = \operatorname{Arg}\min\limits_{w\in W} \left\{\dfrac{1}{2}\|w - v(t)\|^2_{G((v(t))} + \gamma(t)\Big(\Phi(u(t), w, t) + \alpha(t)\langle u(t), w\rangle\Big)\right\}, \\ u(t) = \operatorname{Arg}\min\limits_{w\in W} \left\{\dfrac{1}{2}\|w - v(t)\|^2_{G(v(t))} + \gamma(t)\Big(\Phi(v(t), w, t) + \alpha(t)\langle v(t), w\rangle\Big)\right\}, \\ v(0) = v_0, \quad t \geqslant 0, \end{cases} \tag{40}$$

We now give the sufficient conditions of convergence, as $t \to +\infty$, of the trajectory $v(t)$ of the process (40) to the solution of problem (15).

Theorem 2.3. *Let the following conditions hold:*

1. *W is a convex closed set from E^n; the solution set W_* of problem (15) is nonempty.*
2. *The function $\Phi(v, w)$ is continuous in v on E^n for every w from E^n, convex and continuously differentiable with respect to w on E^n for every v from E^n, satisfies the skew-symmetry condition (3) on W; satisfies the Lipschitz condition with respect to w on the set W:*

$$|\Phi(v, w_1) - \Phi(v, w_2)| \leqslant L\|w_1 - w_2\| \quad \forall v \in E^n, \quad w_1, w_2 \in W_0;$$

3. *Instead of the exact value of the function $\Phi(v, w)$ we know its convex lower-semicontinuous approximation $\Phi(v, w, t)$ that satisfies condition (19).*
4. *$G(v)$ is a symmetrical positive definite matrix for every v from E^n; there exist a strongly convex twice continuously differentiable function $\Psi(v)$ and positive constants $m,\ M,\ m \leqslant M$, such that*

$$G(v) \equiv \Psi''(v); \qquad m\|w\|^2 \leqslant \langle G(v)w, w\rangle \leqslant M\|w\|^2 \quad \forall v, w \in E^n;$$

5. *The parameters $\alpha(t),\ \gamma(t),\ \delta(t)$ satisfy the conditions*

$$\alpha(t), \gamma(t) \in C^1[0; +\infty); \qquad \delta(t) \in C[0; +\infty); \qquad \alpha(t), \gamma(t) > 0; \qquad \delta(t) \geqslant 0;$$

$$\alpha(t) \quad \textit{is a convex function,} \qquad \alpha'(t) \leqslant 0;$$

$$\lim_{t\to+\infty} \left(\alpha(t) + \gamma(t) + \delta(t) + \frac{\delta^2(t)}{\alpha(t)\gamma^{1/2}(t)}\right) = 0;$$

$$\lim_{t\to+\infty} \frac{\gamma^{1/2}(t)}{\alpha(t)} = 0;$$

$$\lim_{t\to+\infty} \left(\frac{|\alpha'(t)|}{\alpha^2(t)\gamma(t)} + \frac{|\gamma'(t)|}{\alpha(t)\gamma^2(t)}\right) = 0;$$

6. *The solution of system (40) exists and is unique for every $t \geqslant 0$.*

Then

$$\lim_{t\to+\infty} \|v(t) - v_*\| = 0; \qquad \lim_{t\to+\infty} \|\dot{v}(t)\| = 0,$$

where v_* is the normal solution of problem (15), and convergence in (24) is uniform with respect to the choice of $\Phi(v, w, t)$ from (19).

As the parameters $\alpha(t),\ \gamma(t),\ \delta(t)$ we may take, for instance,

$$\alpha(t) = (1 + t)^{-1/6}, \qquad \gamma(t) = (1 + t)^{-1/2}, \qquad \delta(t) = (1 + t)^{-1/2}.$$

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