III. NUMERICAL METHODS

SOLUTIONS OF NONLINEAR CHEMISTRY PROBLEMS BY HOMOTOPY ANALYSIS

M. Matinfar[1], M. Saeidy[2], B. Gharahsuflu[3], and M. Eslami[4]

In this paper, the application of the homotopy analysis method (HAM) is presented to obtain analytic solutions of nonlinear systems that often appear in chemical problems. Previously, D. D. Ganji et al. in ["Application of He's methods to nonlinear chemistry problems," *Comput. Math. Appl.*, **54** (2007) 1122–1132] used the variational iteration method (VIM) and the homotopy perturbation method (HPM) to obtain a solution of the above problem, but the paper contained some evident mistakes that we could easily identify. The results show that the HAM is very effective and convenient and the solutions obtained using this method have high accuracy with respect to VIM and HPM.

Keywords: Homotopy analysis method; Homotopy perturbation method; Variational iteration method; Chemistry problem.

1. Introduction

In the last two decades with the rapid development of nonlinear science, there has appeared an ever-increasing interest in analytical techniques for nonlinear problems among scientists and engineers [1–3]. The widely applied techniques are perturbation methods, but, like other nonlinear analytical techniques, perturbation methods have their own limitations. First, almost all perturbation methods are based on the assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Second, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to ideal results. However, an unsuitable choice of small parameters results in bad effects, sometimes seriously. Most scientific problems and phenomena occur nonlinearly. Except for a limited number of these problems, most of them do not have precise analytical solution; thus we have to use various approximate analytical methods. In 1992, Liao [1] employed the basic ideas of homotopy in topology to propose a general analytic method for nonlinear problems, namely the homotopy analysis method (HAM) [3–5]. This method has been successfully applied to solve many types of nonlinear problems [6–7]. The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization, which gives rise to rounding-off errors causing loss of accuracy and requires large computer power and time. The HAM is better since it does not involve discretization of the variables and hence is free from rounding-off errors and does not require large computer memory or time. This method has recently been applied widely to real-life and various engineering problems. The purpose of this paper is to use the homotopy analysis

[1] Department of Mathematics and Computer Science, University of Mazandaran, Babolsar, Iran; e-mail: m.matinfar@umz.ac.ir; corresponding author.

[2] Department of Mathematics and Computer Science, University of Mazandaran, Babolsar, Iran; e-mail: m.saidy@umz.ac.ir.

[3] Department of Mathematical, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran; e-mail: b.gharahsuflu@hotmail.com.

[4] Department of Mathematics and Computer Science, University of Mazandaran, Babolsar, Iran; e-mail: mostafa.eslami@umz.ac.ir.

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method to solve a system of differential equations, namely chemical problems, and to illustrate the advantages and simplicity of HAM as compared to VIM and HPM:

$$\frac{dy_i}{dt}(t) = f_i\left(t, y_1(t), y_2(t), \ldots, y_p(t)\right), \qquad \left(y_i(t)\right)_{t=0} = \alpha_i,$$

where α_i is a specified constant vector, $y_i(t)$ is the solution vector, and $i = 1, 2, \ldots, p$.

2. Basic Idea of HAM

To illustrate of basic idea of HAM, we consider the following differential equation:

$$\mathcal{N}\left[u(\tau)\right] = 0, \tag{1}$$

where $\mathcal{N}$ is a nonlinear operator, τ denotes an independent variable, and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. Generalizing the traditional homotopy method, Liao [4] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}\left[\phi(\tau; p) - u_0(\tau)\right] = p\hbar\mathcal{H}(\tau)\mathcal{N}\left[\phi(\tau; p)\right], \tag{2}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $\mathcal{H}(\tau) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, and $u(\tau; p)$ is an unknown function. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, we have

$$\phi(\tau; 0) = u_0(\tau), \qquad \phi(\tau; 1) = u(\tau). \tag{3}$$

Thus, as p increases from 0 to 1, the solution $u(\tau; p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $u(\tau; p)$ in a Taylor series with respect to p, we have

$$\phi(\tau; p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) p^m, \tag{4}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau; p)}{\partial p^m}\bigg|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (4) converges at $p = 1$, and we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{6}$$

which must be one of the solutions of the original nonlinear equation, as proved by [10]. As $\hbar = -1$ and $\mathcal{H}(\tau) = 1$, Eq. (2) becomes

$$(1-p)\mathcal{L}\left[\phi(\tau;p) - u_0(\tau)\right] + p\mathcal{N}\left[\phi(\tau;p)\right] = 0, \tag{7}$$

which is used mostly in the homotopy perturbation method [8], whereas the solution is obtained directly in [9–11] without using the Taylor series. According to the definition, the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}.$$

Differentiating Eq. (2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}\left[u_m(\tau) - \chi_m u_{m-1}(\tau)\right] = \hbar\,\mathcal{H}(\tau)\mathcal{R}_m\left(\vec{u}_{m-1}\right), \tag{8}$$

where

$$\mathcal{R}_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left[\phi(\tau;p)\right]}{\partial p^{m-1}}\bigg|_{p=0}, \tag{9}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m \ge 1. \end{cases} \tag{10}$$

It should be emphasized that $u_m(\tau)$ for $m \ge 1$ is governed by the linear EQUATION (8) under the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method, we refer the reader to Liao's work [4]. If Eq. (1) admits a unique solution, then this method will produce the unique solution. If Eq. (1) does not possess a unique solution, the HAM will give a solution among many other (possible) solutions.

3. He's Variational Iteration Method

For the purpose of illustrating of the methodology of the proposed method, using variational iteration method, we begin by considering a differential equation in the formal form

$$\mathcal{L}\left[u(t)\right] + \mathcal{N}\left[u(t)\right] = f(t),$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator, and $f(t)$ is the source inhomogeneous term. According to the variational iteration method, we can construct a correction functional as follow:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(\mathcal{L}\left[u_n(\xi)\right] + \mathcal{N}\left[\tilde{u}_n(\xi)\right] - g(\xi)\right)d\xi, \quad n \ge 0,$$

where λ is a general Lagrangian multiplier [10], which can be identified optimally via the variational theory, the subscript n denotes the n th-order approximation, and $\tilde{u}_n$ is considered as a restricted variation [9, 10, 13], i.e., $\delta\tilde{u}_n = 0$. So, we first determine the Lagrange multiplier λ, which will be identified optimally via integration by parts. The successive approximations $u_{n+1}(t)$, $n \geq 0$, of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(t)$. Consequently, the solution

$$u(t) = \lim_{n \to +\infty} u_n(t).$$

4. Error Estimates

It is essential to estimate the exactness of the obtained solutions. For this purpose, in the following, two error estimates are defined. Here we have to distinguish the following two situations.

(i) If the exact solution of the problem, $u_{ex}(t)$, is known, we shall calculate the differences $\Delta(t) = u_{ap}(t) - u_{ex}(t)$, where u_{ap} is the approximate solution of the m th-order $(m \geq 1,\ a \leq t \leq b)$, and we define the error estimates as $\delta = \max|\Delta(t)|$.

(ii) Mostly the exact solution is unknown. For this case, the following procedure is recommended. First we solve our equation for some level of resolution m, the result of which is denoted by $\phi_m(t)$; then we repeat these calculations for $m-1$, getting in this way the function $\phi_{m-1}(t)$. Next we define the differences

$$\Delta(t) = \phi_m(t) - \phi_{m-1}(t),$$

where $m \geq 1$ is the order of the approximate solution. We shall define the error estimates as

$$\delta = \max|\Delta(t)|, \quad a \leq t \leq b.$$

5. Numerical Example

Example 5.1. In order to illustrate the method discussed above, we take an example arising in a chemistry problem is taken from Robertson [14, 15], as follows:

$$\begin{cases} \dfrac{dy_1}{dt} = -k_1 y_1 + k_2 y_2 y_3, \\ \dfrac{dy_2}{dt} = k_3 y_1 - k_4 y_2 y_3 - k_5 y_2^2, \\ \dfrac{dy_3}{dt} = k_6 y_2^2, \end{cases} \tag{11}$$

where k_1, k_2, k_3, k_4, k_5, and k_6 are constant parameters $(k_1 = 0.04,\ k_2 = 0.01,\ k_3 = 400,\ k_4 = 100,\ k_5 = 30000,\ k_6 = 30)$. The initial conditions are given by

$$y_1(0) = 1, \qquad y_2(0) = 0, \qquad y_3(0) = 0.$$

HAM Approach:

To solve the system (11) by the HAM, we choose the linear operator $\mathcal{L}_i = \dfrac{dy_i}{dt}$, $i = 1,2,3$, with the property $\mathcal{L}_i[c] = 0$, where c is an integral constant. Now, we define the system of nonlinear operators as

$$\begin{cases} \mathcal{N}_1[\phi_1,\phi_2,\phi_3] = \dfrac{\partial\phi_1(t;p)}{\partial t} + k_1\phi_1(t;p) - k_2\phi_2(t;p)\phi_3(t;p), \\ \mathcal{N}_2[\phi_1,\phi_2,\phi_3] = \dfrac{\partial\phi_2(t;p)}{\partial t} - k_3\phi_1(t;p) + k_4\phi_2(t;p)\phi_3(t;p) + k_5\phi_2^2(t;p), \\ \mathcal{N}_3[\phi_1,\phi_2,\phi_3] = \dfrac{\partial\phi_3(t;p)}{\partial t} - k_6\phi_2^2(t;p). \end{cases}$$

Using the above definition, we construct the system of zeroth-order deformation equations

$$(1-p)\,\mathcal{L}_i\left[\phi_i(t;p) - y_{i,0}(t)\right] = p\hbar_i\mathcal{H}_i(t)\mathcal{N}_i\left[\phi_1,\phi_2,\phi_3\right], \quad i = 1,2,3.$$

For $p = 0$ and $p = 1$, we write

$$\phi_i(t;0) = y_{i,0}(t), \qquad \phi_i(t;1) = y_i(t), \quad i = 1,2,3.$$

Thus, we obtain the system of mth-order deformation equations as follows:

$$\mathcal{L}_i\left[y_{i,m}(t) - \chi_m y_{i,m-1}(t)\right] = \hbar_i\mathcal{H}_i(t)\mathcal{R}_{i,m}\left(\vec{y}_{1,m-1}, \vec{y}_{2,m-1}, \vec{y}_{3,m-1}\right), \tag{12}$$

with the initial conditions

$$y_{i,m}(0) = 0, \qquad (y_{i,m})_t(0) = 0,$$

where

$$\begin{cases} \mathcal{R}_{1,m} = \dfrac{\partial y_{1,m-1}}{\partial t} + k_1 y_{1,m-1} - k_2\sum_{k=0}^{m-1} y_{2,k}y_{3,m-1-k}, \\ \mathcal{R}_{2,m} = \dfrac{\partial y_{2,m-1}}{\partial t} - k_3 y_{1,m-1} + \sum_{k=0}^{m-1}\left[k_4 y_{2,k}y_{3,m-1-k} + k_5 y_{2,k}y_{2,m-1-k}\right], \\ \mathcal{R}_{3,m} = \dfrac{\partial y_{3,m-1}}{\partial t} - k_6\sum_{k=0}^{m-1} y_{3,k}y_{3,m-1-k}. \end{cases}$$

Now, the solution of system (12), for $(m \geq 1)$ is

$$y_{i,m}(t) = \chi_m y_{i,m-1}(t) + h_i\int_0^t\left[\mathcal{H}_i(t_1)\mathcal{R}_{i,m}\left(\vec{y}_{1,m-1}, \vec{y}_{2,m-1}, \vec{y}_{3,m-1}\right)\right]dt_1.$$

We start with the initial approximations $y_{1,0}(t) = 1$ and $y_{2,0}(t) = y_{3,0}(t) = 0$. If $\mathcal{H}_i(t_1) = 1$, $i = 1,2,3$, the approximate solution of Example (5.1), can be readily obtained by

$$\begin{cases} y_1(t) = \sum_{m=0}^{+\infty} y_{1,m}(t) = 1 + h_1 k_1 t + \frac{1}{2} h_1^2 k_1^2 t^2 + (h_1 + 1) h_1 k_1 t + \ldots, \\ y_2(t) = \sum_{m=0}^{+\infty} y_{2,m}(t) = -h_2 k_3 t - \frac{1}{2} h_2 h_1 k_3 k_1 t^2 - (h_2 + 1) h_2 k_3 t + \ldots, \\ y_3(t) = \sum_{m=0}^{+\infty} y_{3,m}(t) = -\frac{1}{3}\left(h_3 h_2^2 k_6 k_3^2 t^3 \right) + \ldots. \end{cases}$$

In practice, all terms of the series $y_i(t) = \sum_{j=0}^{+\infty} y_{i,j}(t)$, $i = 1, 2, 3$, cannot be determined and so we use an approximation of the solution by the following truncated series:

$$\mu_m(t) = \sum_{j=0}^{m-1} y_{i,j}(t), \quad \text{with} \quad y_i(t) = \lim_{m \to +\infty} \mu_m(t), \qquad \text{for} \quad i = 1, 2, 3.$$

VIM Approach:

Now, we apply the VIM for this example as done in [14], but there are some mistakes in their solutions. For this goal, the correction functionals for y_1, y_2, and y_3 in the VIM approach can be expressed, respectively, as follows:

$$\begin{cases} y_{1,n+1} = y_{1,n} + \int_0^t \lambda_1 \left[(y_{1,n})_\tau(\tau) + k_1 y_{1,n}(\tau) - k_2 y_{2,n}(\tau) y_{3,n}(\tau) \right] d\tau, \\ y_{2,n+1} = y_{2,n} + \int_0^t \lambda_2 \left[(y_{2,n})_\tau(\tau) - k_3 y_{1,n}(\tau) + k_4 y_{2,n}(\tau) y_{3,n}(\tau) + k_5 \left(y_{2,n}(\tau) \right)^2 \right] d\tau, \\ y_{3,n+1} = y_{3,n} + \int_0^t \lambda_3 \left[(y_{3,n})_\tau(\tau) - k_6 \left(y_{2,n}(\tau) \right)^2 \right] d\tau, \end{cases}$$

where the subscript n indicates the n th-order approximation, and λ_i, $i = 1, 2, 3$, is a general Lagrangian multiplier that can be identified optimally via the variational theory [11]. Applying Lagrange–Euler equations and natural boundary conditions, we identify the Lagrange multipliers $\lambda_1 = \lambda_2 = \lambda_3 = -1$ and obtain the following solutions (we perform the calculations until the third order approximation):

$$\begin{cases} y_{1,1}(t) = 1 - [k_1] t, \\ y_{1,2}(t) = 1 - [k_1] t + \left[\frac{1}{2} k_1^2 \right] t^2, \\ \vdots \end{cases} \qquad \begin{cases} y_{2,1}(t) = [k_3] t, \\ y_{2,2}(t) = [k_3]\, t - \left[\frac{1}{2} k_1 k_2 \right] t^2 - \left[\frac{1}{3} k_3^2 k_5 \right] t^2, \\ \vdots \end{cases}$$

$$\begin{cases} y_{3,1}(t) = 0, \\ y_{3,2}(t) = \left[\frac{1}{3} k_3^2 k_6 \right] t^3, \\ \vdots \end{cases}$$

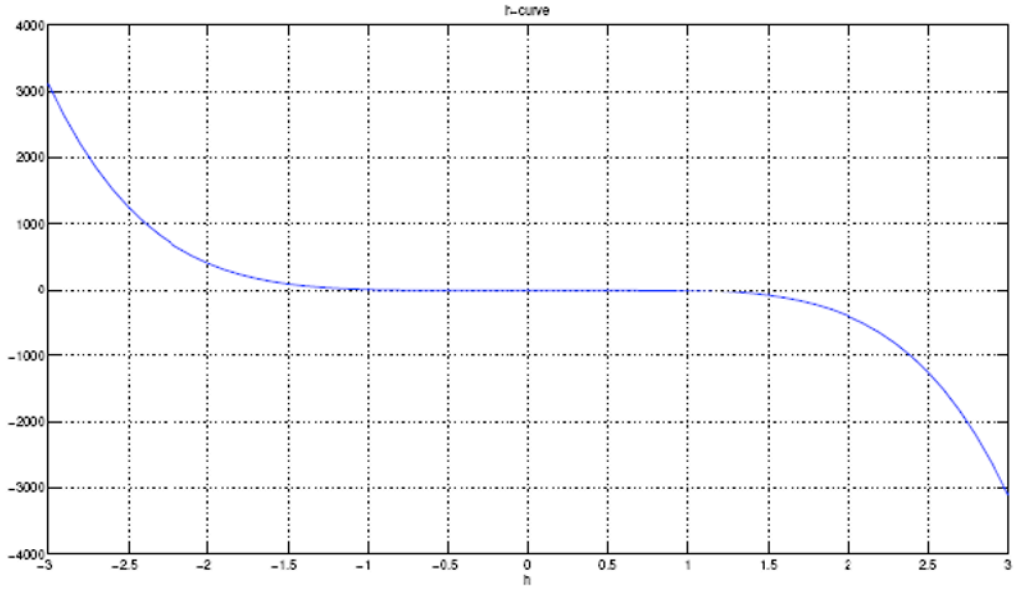

Fig. 1. The $\hbar_1$-curve for $y_1(t)$ at the 5th-order approximation with $t = 0.1$.

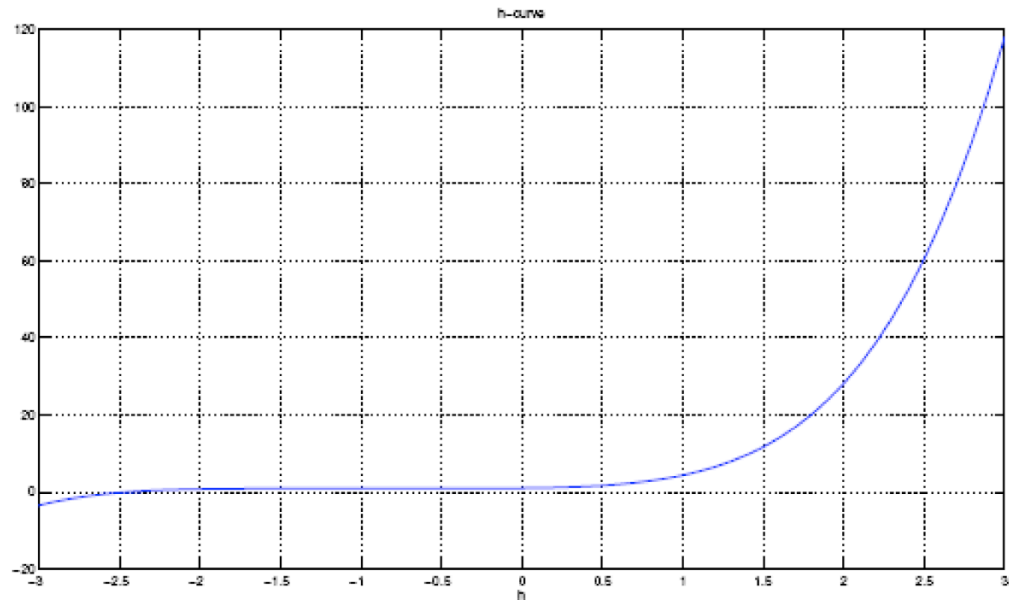

Fig. 2. The $\hbar_1$-curve for the $y_1(t)$ at the 5th-order approximation with $t = 0.1$.

As pointed out by Liao [4], in general, by means of the so-called $\hbar$-curve, it is straightforward to choose a proper value of $\hbar$ that ensures that the solution series is convergent. Figures 1 and 2 shows the $\hbar_1$-curve for $y_1(t)$ corresponding to $t = 0.1$, from which we can clearly identify an appropriate region for $\hbar_1$. In this way, we similarly draw the another $\hbar_i$-curves $(i = 2, 3)$ and choose several valid ones. The estimate errors for the solution of the HAM $(\hbar_i = -10^{-6})$ and the solution of the HPM $(\hbar_i = -1)$ for systems (11,13) for different values of n are presented in Tables 1–6. It is clear that the HAM is an efficient technique with greater accuracy for most cases in finding solutions, and our approximate solutions using the HAM are in good agreement with the exact values.

Table 1
Error Estimates of $y_1(t)$ for Example 5.1

	VIM	HPM	ADM	HAM
$n = 3$	3.6571×10^{12}	1.0667×10^{-5}	1.0667×10^{-5}	4.0000×10^{-8}
$n = 4$	8.0248×10^{37}	1.0667×10^{-7}	1.0667×10^{-7}	4.0000×10^{-8}
$n = 5$	1.8696×10^{88}	$1.2800 \times 10^{+6}$	$1.2800 \times 10^{+6}$	4.0000×10^{-8}

Table 2
Error Estimates of $y_2(t)$ for Example 5.1

	VIM	HPM	ADM	HAM
$n = 3$	1.0971×10^{22}	$1.6000 \times 10^{+9}$	$1.6000 \times 10^{+9}$	4.0000×10^{-4}
$n = 4$	2.4074×10^{47}	$4.8000 \times 10^{+7}$	$4.8000 \times 10^{+7}$	3.9999×10^{-4}
$n = 4$	5.6048×10^{97}	$7.6800 \times 10^{+15}$	$7.6800 \times 10^{+15}$	3.9999×10^{-4}

Table 3
Error Estimates of $y_3(t)$ for Example 5.1

	VIM	HPM	ADM	HAM
$n = 3$	1.0971×10^{19}	$1.6000 \times 10^{+6}$	$1.6000 \times 10^{+6}$	1.6000×10^{-12}
$n = 4$	2.4074×10^{44}	$4.8000 \times 10^{+4}$	$4.8000 \times 10^{+4}$	4.8000×10^{-12}
$n = 5$	5.6087×10^{94}	$7.6800 \times 10^{+12}$	$7.6800 \times 10^{+12}$	9.6000×10^{-12}

Example 5.2. The second example is a system representing a nonlinear reaction, which was taken from Hull et al. [12, 14]

$$\begin{cases} \dfrac{dy_1}{dt} = -y_1, \\ \dfrac{dy_2}{dt} = y_1 - y_2^2, \\ \dfrac{dy_3}{dt} = y_2^2, \end{cases} \tag{13}$$

with the initial conditions

$$y_1(0) = 1, \qquad y_2(0) = 0, \qquad \text{and} \qquad y_3(0) = 0.$$

HAM Approach:

To solve the above system by the HAM, we obtain the system of m th-order deformation equations for $m \geq 1$ as follows:

$$\mathcal{L}_i \left[y_{i,m} - \chi_m y_{i,m-1} \right] = \mathcal{H}_i(t) h_i \mathcal{R}_{i,m} \left(\vec{y}_{1,m-1}, \vec{y}_{2,m-1}, \vec{y}_{3,m-1} \right), \quad i = 1,2,3, \tag{14}$$

where

$$\begin{cases} \mathcal{R}_{1,m}(\vec{y}_{1,m-1}, \vec{y}_{2,m-1}, \vec{y}_{3,m-1}) = \dfrac{dy_{1,m-1}}{dt} + y_{1,m-1}, \\ \mathcal{R}_{2,m}(\vec{y}_{1,m-1}, \vec{y}_{2,m-1}, \vec{y}_{3,m-1}) = \dfrac{dy_{2,m-1}}{dt} - y_{1,m-1} + \sum_{k=0}^{m-1} y_{2,k}\, y_{2,m-1-k}, \\ \mathcal{R}_{3,m}(\vec{y}_{1,m-1}, \vec{y}_{2,m-1}, \vec{y}_{3,m-1}) = \dfrac{dy_{3,m-1}}{dt} - \sum_{k=0}^{m-1} y_{2,k}\, y_{2,m-1-k}. \end{cases}$$

Now, the solution of the system of m th-order deformation equations (13) for $m \geq 1$ is as follows:

$$y_{i,m}(t) = \chi_m y_{i,m-1}(t) + h_i \int_0^t \left[\mathcal{H}_i(t_2) \mathcal{R}_{i,m} \left(\vec{y}_{1,m-1}, \vec{y}_{2,m-1}, \vec{y}_{3,m-1} \right) \right] dt_2. \tag{15}$$

We start with the initial approximations $y_{1,0}(t) = 1$ and $y_{2,0}(t) = y_{3,0}(t) = 0$; if we assume that $\mathcal{H}_i(t_2) = 1$, then by solving Eqs. (14) we easily obtain the approximate solution of Example (5.2),

$$y_1(t) = \sum_{j=0}^{+\infty} y_{1,j}(t) = [h_1]t + \left[\frac{1}{2} h_1^2 \right] t^2 + \left[(h_1 + 1) h_1 \right] t + \dots,$$

$$y_2(t) = \sum_{j=0}^{+\infty} y_{2,j}(t) = [-h_2]t - \left[\frac{1}{2} h_2 h_1 \right] t^2 - \left[(h_2 + 1) h_2 \right] t + \dots,$$

$$y_3(t) = \sum_{j=0}^{+\infty} y_{3,j}(t) = \left[-\frac{1}{3} \left(h_2^2 h_3 \right) \right] t^3 + \dots.$$

In practice, all terms of the series $y_i(t) = \sum_{j=0}^{+\infty} y_{i,j}(t)$, $i = 1,2,3,$ cannot be determined, and so we use an approximation of the solution by the following truncated series:

$$\phi_m(t) = \sum_{j=0}^{m-1} y_{i,j}(t), \qquad \text{with} \qquad y_i(t) = \lim_{m \to +\infty} \phi_m(t), \qquad \text{for} \qquad i = 1,2,3.$$

Table 4
Error estimates of $y_1(t)$ for Example 5.2

	VIM	HPM	ADM	HAM
$n = 3$	1.6667×10^{-1}	1.6667×10^{-1}	1.6667×10^{-1}	1.0000×10^{-6}
$n = 4$	4.1667×10^{-2}	4.1667×10^{-2}	4.1667×10^{-2}	1.0000×10^{-6}
$n = 4$	8.3333×10^{-3}	8.3333×10^{-3}	8.3333×10^{-3}	9.9999×10^{-7}

Table 5
Error estimates of $y_2(t)$ for Example 5.2

	VIM	HPM	ADM	HAM
$n = 3$	4.2857×10^{-1}	1.6667×10^{-1}	1.6667×10^{-1}	1.0000×10^{-6}
$n = 4$	1.1680×10^{-1}	2.0833×10^{-1}	2.0833×10^{-1}	1.0000×10^{-6}
$n = 4$	2.9728×10^{-2}	2.5000×10^{-2}	2.5000×10^{-2}	9.9999×10^{-7}

Table 6
Error estimates of $y_3(t)$ for Example 5.2

	VIM	HPM	ADM	HAM
$n = 3$	2.6190×10^{-1}	3.3333×10^{-1}	3.3333×10^{-1}	3.3333×10^{-19}
$n = 4$	7.5131×10^{-2}	2.5000×10^{-1}	2.5000×10^{-1}	1.0000×10^{-18}
$n = 4$	2.1395×10^{-2}	1.6667×10^{-2}	1.6667×10^{-2}	2.0000×10^{-18}

VIM Approach:

Now, we apply the VIM for this example as done in [16], but there are some mistakes in their solutions. For this goal, the correction functionals for y_1, y_2, and y_3 in the VIM approach can be expressed, respectively, as follows:

$$\begin{cases} y_{1,n+1}(t) = y_{1,n}(t) + \int_0^t \lambda_1 \left[(y_{1,n})_\tau(\tau) + y_{1,n}(\tau) \right] d\tau, \\ y_{2,n+1}(t) = y_{2,n}(t) + \int_0^t \lambda_2 \left[(y_{2,n})_\tau(\tau) - y_{1,n}(\tau) + \left(y_{2,n}(\tau) \right)^2 \right] d\tau, \\ y_{3,n+1}(t) = y_{3,n}(t) + \int_0^t \lambda_3 \left[(y_{3,n})_\tau(\tau) - \left(y_{2,n}(\tau) \right)^2 \right] d\tau, \end{cases}$$

where the subscript n indicates the nth-order approximation, and λ_i, $i = 1, 2, 3$, is a general Lagrangian multiplier. In the same way as Example (5.1), the Lagrange multipliers can be identified as $\lambda_1 = \lambda_2 = \lambda_3 = -1$, and then we have the following solutions (we perform the calculations until the third-order approximation):

$$\begin{cases} y_{1,1}(t) = 1 - t, \\ y_{1,2}(t) = 1 - t + \dfrac{1}{2}t^2, \\ \vdots \end{cases} \qquad \begin{cases} y_{2,1}(t) = t, \\ y_{2,2}(t) = t - \dfrac{1}{2}t^2 - \dfrac{1}{3}t^3, \\ \vdots \end{cases} \qquad \begin{cases} y_{3,1}(t) = 0, \\ y_{3,2}(t) = \dfrac{1}{3}t^3. \\ \vdots \end{cases}$$

The error estimates for approximate solutions of this example obtained by the VIM, HPM, and HAM approach are shown in Tables 4–6, which tell us that the HAM approach is a efficient technique with respect to VIM and HPM.

6. Conclusion

In this paper, the HAM has been successfully applied for solving some systems of equations, and the results of this method are compared with the approaches obtained by the VIM and the HPM. The HAM provides us with a convenient way to control the convergence of approximation series; this is a fundamental qualitative difference in analysis between the HAM and other methods. The examples in this paper are further confirmation of the flexibility and potential of the HAM for complicated nonlinear problems in science and engineering. Comparison results with VIM and HPM in both examples reveal that the HAM is very powerful, efficient, and easy to use in obtaining approximate solutions of nonlinear problems.

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