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NUMERICAL SOLUTION OF AN INVERSE PROBLEM FOR THE MODIFIED ALIEV–PANFILOV MODEL

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We pose the inverse problem for the modified Aliev–Panfilov model, which involves determining the coefficient of a system of partial differential equations dependent on spatial variables from supplementary observations of the solution on the boundary. This inverse problem may be interpreted as a problem to find the shape and location of the cardiac region damaged by myocardial infarct. A numerical method is proposed for solving the problem and computer experiments illustrating its implementation are reported.

Keywords: Aliev–Panfilov mathematical model, inverse problem, numerical methods.

Introduction

Mathematical modeling methods are widely used in medicine to diagnose cardiac electrical activity. The transmission of electrical pulses in the heart is described using initial–boundary value problems for systems of evolutionary quasi-linear partial differential equations in two- or three-dimensional spatial geometry. The best known mathematical model qualitatively describing the transmission of electromagnetic excitation in the myocardium or the nerve system is the FitzHugh–Nagumo model [1, 2]. Several newer models provide more accurate description of the shape of the transmitted pulse, e.g., the Aliev–Panfilov model [3]. Some inverse problems for these models have been studied in [4–7] in the context of development of computer diagnostic methods in cardiology and some numerical methods have been proposed.

In the present article, we pose the inverse problem for the modified Aliev–Panfilov mathematical model: this problem involves determining the coefficient of a system of partial differential equations dependent on spatial variables from supplementary observations of the solution at the boundary. This inverse problem may be interpreted as a problem to find the shape and location of the region of the heart that has been damaged by myocardial infarct. A numerical method is proposed for the solution of the inverse problem and computer experiments illustrating its implementation are reported.

The Inverse Problem

Consider the Aliev–Panfilov model [3]

$$u_t = D\Delta u - ku(u-\alpha)(u-1) - uw, \qquad (x, y) \in G, \qquad t \in (0, T],$$

$$w_t = -\left(\varepsilon_0 + \frac{\mu_1 w}{u + \mu_2}\right)\left(w + ku(u - \alpha - 1)\right), \qquad (x, y) \in G, \qquad t \in (0, T],$$

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$$\frac{\partial u}{\partial n}(x, y, t) = 0, \qquad (x, y) \in \Gamma, \qquad t \in (0, T],$$

$$u(x, y, 0) = \varphi(x, y), \qquad (x, y) \in G,$$

$$w(x, y, 0) = 0, \qquad (x, y) \in G.$$

Here the function $u(x, y, t)$ is the transmembrane potential, the function $w(x, y, t)$ is the slow restoring variable associated with ion currents, $\varphi(x, y)$ is the initial potential perturbation, D, α, k, ε_0, μ_1, μ_2 are positive constants. G is a bounded region with the boundary Γ. This model is used to describe the transmission of an electromagnetic excitation in the myocardium, assuming homogeneity of the tissue characteristics responsible for current conduction and excitation of the medium.

Consider a modification of this model. Let the function $\chi(x, y) \in C^1(G)$ be such that it takes values close to zero over most of the region $H \subset G$ and values close to 1 over most of the region $G \backslash \bar{H}$. In other words, the main changes of the function $\chi(x, y)$ occur near the boundary of the region H. The modified Aliev–Panfilov model has the form

$$u_t = D\Delta u - \chi(x, y) ku(u - \alpha)(u - 1) - uw, \qquad (x, y) \in G, \qquad t \in (0, T], \tag{1}$$

$$w_t = -\left(\varepsilon_0 + \frac{\mu_1 w}{u + \mu_2}\right)(w + ku(u - \alpha - 1)), \qquad (x, y) \in G, \qquad t \in (0, T], \tag{2}$$

$$\frac{\partial u}{\partial n}(x, y, t) = 0, \qquad (x, y) \in \Gamma, \qquad t \in (0, T], \tag{3}$$

$$u(x, y, 0) = \varphi(x, y), \qquad (x, y) \in G, \tag{4}$$

$$w(x, y, 0) = 0, \qquad (x, y) \in G. \tag{5}$$

In the original Aliev–Panfilov model the nonlinear source $ku(u - \alpha)(u - 1)$ determines the excitability of the medium. In the modified Aliev–Panfilov model the nonlinear source $\chi(x, y) ku(u - \alpha)(u - 1)$ characterizes a medium capable of being excited in the region $G \backslash \bar{H}$ and incapable of being excited in the region H. Thus the mathematical model (1)–(5) may be applied to describe excitation processes in a heart part of which (the region H) has been damaged by myocardial infract. Such an approach for other excitation models has been considered in [5, 7].

We assume that the boundary of the region H is specified by n parameters $\lambda_1, \ldots, \lambda_n$. Define the function $\chi(x, y; \lambda_1, \ldots, \lambda_n)$ as

$$\chi(x, y; \lambda_1, \ldots, \lambda_n) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\theta^2 g(x, y; \lambda_1, \ldots, \lambda_n)\right),$$

where $g(x, y; \lambda_1, \dots, \lambda_n)$ is a known function taking the values $g(x, y; \lambda_1, \dots, \lambda_n) < 0$, $(x, y) \in H$, and $g(x, y; \lambda_1, \dots, \lambda_n) > 0$, $(x, y) \in G \backslash \bar{H}$, and θ is a given constant.

We can now state the inverse problem for the modified model (1)–(5). Find the function $g(x, y; \lambda_1, \dots, \lambda_n)$ defining the boundary of the region H given the solutions of problem (1)–(5)

$$u_i(x, y, t) = \psi_i(x, y, t), \qquad (x, y) \in \Gamma, \qquad t \in [0, T], \qquad i = 1, \dots, m,$$

on the set $\Gamma \times [0, T]$, corresponding to different initial conditions $u_i(x, y, 0) = \varphi_i(x, y)$. The coefficients D, k, α, ε_0, μ_1, μ_2, θ and the functions $\varphi_i(x, y)$, $(x, y) \in G$, $i = 1, \dots, m$ are known.

Numerical Methods for Solving the Inverse Problem

Let us consider a numerical method for solving the inverse problem. Let $u_i(x, y, t; \bar{\lambda}_1, \dots, \bar{\lambda}_n)$, $i = 1, \dots, m$ be the solutions of problem (1)–(5) corresponding to the initial conditions $\varphi_i(x, y)$, $i = 1, \dots, m$, and $\bar{\chi} = \chi(x, y; \bar{\lambda}_1, \dots, \bar{\lambda}_n)$. Denote by $\bar{\psi}_i(x, y, t)$, $i = 1, \dots, m$, the values of $u_i(x, y, t; \bar{\lambda}_1, \dots, \bar{\lambda}_n)$ for $(x, y, t) \in \Gamma \times [0, T]$. We assume that the functions $\bar{\psi}_i(x, y, t)$, $i = 1, \dots, m$, are unknown, and instead we have the functions $\psi_{\delta i}(x, y, t)$, $i = 1, \dots, m$, such that

$$\sum_{i=1}^{m} \int_0^T \int_\Gamma (\psi_{\delta i}(x, y, t) - \bar{\psi}_i(x, y, t))^2 \, dl \, dt \le \delta^2.$$

As an approximate solution of the inverse problem we take the parameter values $\lambda_1, \dots, \lambda_n$, such that

$$\sum_{i=1}^{m} \int_0^T \int_\Gamma (u_i(x, y, t; \lambda_1, \dots, \lambda_n) - \psi_{\delta i}(x, y, t))^2 \, dl \, dt \le \delta^2.$$

Solution of the inverse problem thus reduces to minimizing the function

$$\Phi(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{m} \int_0^T \int_\Gamma (u_i(x, y, t; \lambda_1, \dots, \lambda_n) - \psi_{\delta i}(x, y, t))^2 \, dl \, dt.$$

We minimize $\Phi(\lambda_1, \dots, \lambda_n)$ by the gradient descent method.

Let us determine the gradient of the function $\Phi(\lambda_1, \dots, \lambda_n)$. To this end find its increment $\delta\Phi$. Define the functions $f_1(u) = ku(u - \alpha)(u - 1)$, $f_2(u) = ku(u - \alpha - 1)$, $f_3(u, w) = \varepsilon_0 + (\mu_1 w)/(u + \mu_2)$. Denote by λ the parameter vector $\lambda = (\lambda_1, \dots, \lambda_n)$, and by $\delta\lambda = (\delta\lambda_1, \dots, \delta\lambda_n)$ its increment. Assume that the function $\chi(x, y; \lambda)$ corresponds to the solution $\{u(x, y, t; \lambda), w(x, y, t; \lambda)\}$ of problem (1)–(5) and the function $\chi(x, y; \lambda + \delta\lambda)$ to the solution $\{u(x, y, t; \lambda + \delta\lambda), w(x, y, t; \lambda + \delta\lambda)\}$. Denote

$$p_i(x,y,t;\lambda,\delta\lambda) = u_i(x,y,t;\lambda+\delta\lambda) - u_i(x,y,t;\lambda),$$

$$q_i(x,y,t;\lambda,\delta\lambda) = w_i(x,y,t;\lambda+\delta\lambda) - w_i(x,y,t;\lambda).$$

Note that

$$f_1(u_i+p_i)\,\chi(x,y;\lambda+\delta\lambda) - f_1(u_i)\chi(x,y;\lambda) = f_1(u_i)\sum_{j=1}^{n}\frac{\partial\chi}{\partial\lambda_j}\delta\lambda_j + f'_{1u}(u_i)\,p_i\chi(x,y;\lambda) + \tilde{R}_1,$$

$$(u_i+p_i)(w_i+q_i) - u_i w_i = w_i p_i + u_i q_i + \tilde{R}_2,$$

$$f_3(u_i+p_i, w_i+q_i)(w_i+q_i+f_2(u_i+p_i)) - f_3(u_i,w_i)(w_i+f_2(u_i))$$

$$= f_3(u_i,w_i)q_i + f_3(u_i,w_i)f'_{2u}(u_i)\,p_i + (f'_{3u}(u_i,w_i)\,p_i + f'_{3w}(u_i,w_i)\,q_i)(w_i+f_2(u_i)) + \tilde{R}_3,$$

where $\tilde{R}_i = O(\delta\lambda^2)$.

The functions p_i, q_i are solutions of the problem

$$\frac{\partial p_i}{\partial t} = D\Delta p_i - f_1(u_i)\sum_{j=1}^{n}\frac{\partial\chi}{\partial\lambda_j}\delta\lambda_j - f'_{1u}(u_i)\,p_i\chi(x,y;\lambda)$$

$$- w_i p_i - u_i q_i - \tilde{R}_1 - \tilde{R}_2, \quad (x,y)\in G, \quad t\in(0,T], \tag{6}$$

$$\frac{\partial q_i}{\partial t} = -f_3(u_i,w_i)(q_i + f'_{2u}(u_i)\,p_i) - (f'_{3u}(u_i,w_i)\,p_i$$

$$+ f'_{3w}(u_i,w_i)\,q_i)(w_i+f_2(u_i)) - \tilde{R}_3, \quad (x,y)\in G, \quad t\in(0,T], \tag{7}$$

$$\frac{\partial p_i}{\partial n}(x,y,t) = 0, \quad (x,y)\in\Gamma, \quad t\in(0,T], \tag{8}$$

$$p_i(x,y,0) = 0, \quad (x,y)\in G, \tag{9}$$

$$q_i(x,y,0) = 0, \quad (x,y)\in G. \tag{10}$$

Consider the increment of the function $\Phi(\lambda_1,\dots,\lambda_n)$:

$$\delta\Phi = \Phi(\lambda+\delta\lambda) - \Phi(\lambda) = \sum_{i=1}^{m}\int_0^T\int_\Gamma((u_i+p_i-\psi_{\delta i})^2 - (u_i-\psi_{\delta i})^2)\,dl\,dt$$

$$= \sum_{i=1}^{m}\int_0^T\int_\Gamma(2(u_i-\psi_{\delta i})p_i + p_i^2)\,dl\,dt. \tag{11}$$

Let us derive a different form for the increment of $\Phi(\lambda_1, \ldots, \lambda_n)$. Consider the functions $a_i(x, y, t)$, $b_i(x, y, t)$, which are solutions of the conjugate initial–boundary value problems

$$\frac{\partial a_i}{\partial t} = -D\Delta a_i + a_i\left(f'_{1u}(u_i)\chi(x, y; \lambda) + w_i\right) + b_i(f_3(u_i, w_i)f'_{2u}(u_i)$$

$$+f'_{3u}(u_i, w_i)(w_i + f_2(u_i))), \qquad (x, y) \in G, \qquad t \in [0, T), \tag{12}$$

$$\frac{\partial b_i}{\partial t} = a_i u_i + b_i\left(f_3(u_i, w_i) + f'_{3w}(u_i, w_i)(w_i + f_2(u_i))\right), \qquad (x, y) \in G, \qquad t \in [0, T), \tag{13}$$

$$D\frac{\partial a_i}{\partial n}(x, y, t) = 2(u_i - \psi_i), \qquad (x, y) \in \Gamma, \qquad t \in [0, T], \tag{14}$$

$$a_i(x, y, \mathrm{T}) = 0, \qquad (x, y) \in G, \tag{15}$$

$$b_i(x, y, \mathrm{T}) = 0 \qquad (x, y) \in G. \tag{16}$$

Introduce the integral

$$I = \sum_{i=1}^{m}\int_0^T\iint_G\Bigg[a_i\left(\frac{\partial p_i}{\partial t} - D\Delta p_i + f'_{1u}(u_i)p_i\chi(x, y; \lambda) + w_i p_i + u_i q_i\right)$$

$$+b_i\left(\frac{\partial q_i}{\partial t} + f_3(u_i, w_i)(q_i + f'_{2u}(u_i)p_i) + (f'_{3u}(u_i, w_i)p_i + f'_{3w}(u_i, w_i)q_i)\right.$$

$$\left.\times(w_i + f_2(u_i))\right) + p_i\left(\frac{\partial a_i}{\partial t} + D\Delta a_i - f'_{1u}(u_i)a_i\chi(x, y; \lambda) - w_i a_i\right.$$

$$\left. - b_i f_3(u_i, w_i)f'_{2u}(u_i) - f'_{3u}(u_i, w_i)(w_i + f_2(u_i))b_i\right)$$

$$+ q_i\left(\frac{\partial b_i}{\partial t} - a_i u_i - b_i\left(f_3(u_i, w_i) + f'_{3w}(u_i, w_i)(w_i + f_2(u_i))\right)\right)dx\,dy\,dt. \tag{17}$$

Clearly

$$I = \sum_{i=1}^{m}\int_0^T\iint_G[(a_i p_i + b_i q_i)_t - (Da_i\Delta p_i - Dp_i\Delta a_i)]\,dx\,dy\,dt$$

Applying Green's formula and the initial and boundary conditions (8)–(10), (14)–(16), we obtain

$$I = \sum_{i=1}^{m} \iint_G (a_i p_i + b_i q_i)\Big|_{t=0}^{t=T} dx\,dy - \sum_{i=1}^{m} \int_0^T \int_\Gamma \left(Da_i \frac{\partial p_i}{\partial n} - Dp_i \frac{\partial a_i}{\partial n} \right) dl\,dt$$

$$= \sum_{i=1}^{m} \int_0^T \int_\Gamma (p_i 2(u_i - \psi_i))\,dl\,dt. \tag{18}$$

On the other hand, from (17), (6), (7), (12), and (13) we obtain that

$$I = -\sum_{i=1}^{m} \int_0^T \iint_G a_i \left(f_1(u_i) \sum_{j=1}^{n} \frac{\partial \chi}{\partial \lambda_j} \delta\lambda_j + \tilde{R}_1 \right) dx\,dy\,dt. \tag{19}$$

Using (18) and (19), we can write the expression (11) for the increment of $\Phi(\lambda)$ in the form

$$\delta\Phi = \sum_{i=1}^{m} \left[\int_0^T \iint_G -a_i \left(f_1(u_i) \sum_{j=1}^{n} \frac{\partial \chi}{\partial \lambda_j} \delta\lambda_j + \tilde{R}_1 \right) dx\,dy\,dt + \int_0^T \int_\Gamma p_i^2\,dl\,dt \right].$$

Ignoring terms of second order of smallness, we obtain the following expression for the gradient:

$$\frac{\partial \Phi}{\partial \lambda_j} = -\sum_{i=1}^{m} \int_0^T \iint_G a_i f_1(u_i) \chi_{\lambda_j}(x, y; \lambda)\,dx\,dy\,dt, \qquad 1 \le j \le n$$

This gradient is now used to construct the gradient descent method for the minimization of the function $\Phi(\lambda_1, \dots, \lambda_n)$. The iterative process stops as soon as $\Phi(\lambda_1, \dots, \lambda_n) \le \delta^2$.

As the functions $\varphi_i(x, y)$ we take the localized perturbations

$$\varphi_i(x, y) = \exp\{-((x - x_i)^2 + (y - y_i)^2)/\sigma^2\}.$$

The first approximation for the gradient descent method is obtained by enumerating a small number of parameter tuples λ and choosing the tuple λ^0 that minimizes $\Phi(\lambda^0)$. The set for the initial enumeration is constructed so that the union of the regions described by the parameter tuples generates a maximal cover of the region G.

Computer Experiments

The numerical method described for the solution of the inverse problem has been applied to determine elliptical regions H. We present a scheme of the computer experiments and some numerical results produced by the method.

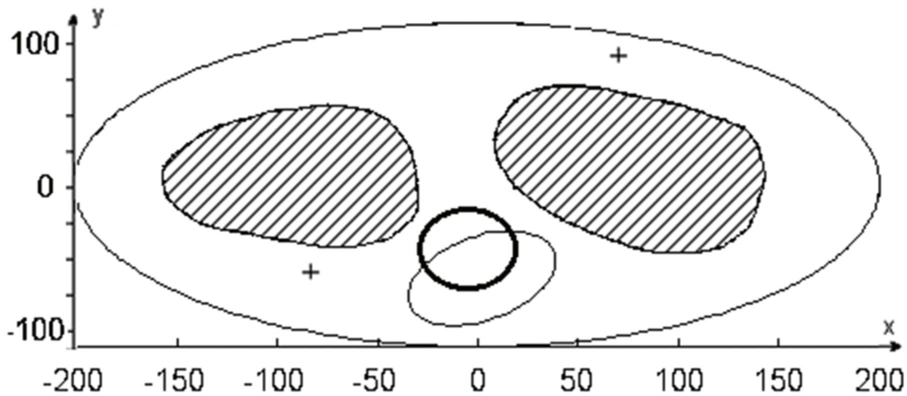

Fig. 1

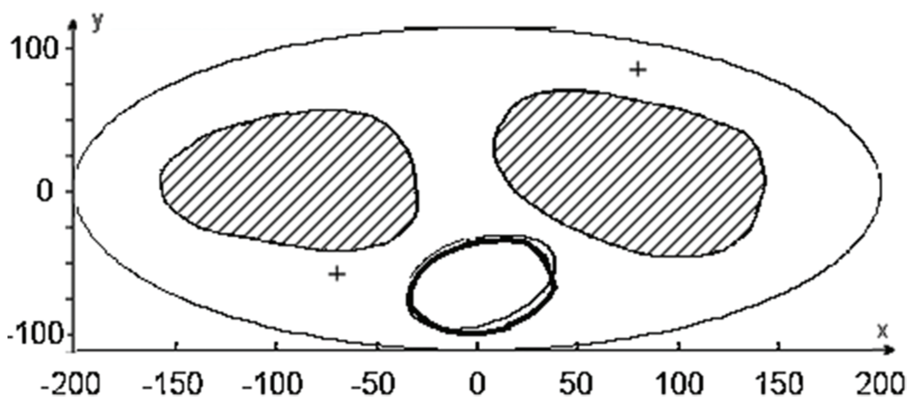

Fig. 2

Direct problems for the modified Aliev–Panfilov model (1)–(5) have been solved in the region G approximating the heart section (see Figs. 1, 2) by the finite element method; the deal.II library[3] has been used to develop the software implementation. An order of 10^5 finite elements were used in our computations. The following parameter values were used in all computer experiments: $D = 1$, $k = 8$, $\alpha = 0.15$, $\varepsilon_0 = 0.002$, $\mu_1 = 0.2$, $\mu_2 = 0.3$. By solving the direct problem for one or several initial conditions, we evaluated $\bar{\psi}_i(x, y, t)$ on the boundary $(x, y) \in \Gamma$, $t \in [0, T]$; an error was then injected into these functions to obtain $\psi_{\delta i}(x, y, t)$ such that

$$\sum_{i=1}^{m} \int_0^T \int_\Gamma (u_i(x, y, t; \lambda_1, \dots, \lambda_n) - \psi_{\delta i}(x, y, t))^2 \, dl \, dt = \delta^2 .$$

The computation error was set at

$$\delta = 0.02 \sqrt{\sum_{i=1}^{m} \int_0^T \int_\Gamma \bar{\psi}_i^{\,2}(x, y, t) \, dl \, dt} .$$

[3] A Finite Element Differential Equations Analysis Library (http://www.dealii.org/).

Then these functions were applied to solve the inverse problem by the proposed numerical method. In our computer experiments we solved inverse problems reconstructing elliptical regions H parametrized by five parameters. The function g was taken in the form

$$g(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = \left(\frac{((x-\lambda_1)\cos(\lambda_5) - (y-\lambda_2)\sin(\lambda_5))}{\lambda_3} \right)^2 + \left(\frac{(x-\lambda_1)*\sin(\lambda_5) + (y-\lambda_2)*\cos(\lambda_5)}{\lambda_4} \right)^2 - 1 .$$

A similar problem for the FitzHugh–Nagumo model has been investigated in [7]. The results show that solutions of at least two direct problems are needed in order to determine an elliptical region. For the Aliev–Panfilov model we accordingly also reconstruct the elliptical region from the solutions of two direct problems. Figs. 1, 2 show the results of one computer experiment. The shaded parts are cutouts in the region G corresponding to heart ventricles; crosses mark the centers of localization of the initial distribution functions φ_i; the thin curve is the exact sought region, the bold curve marks the first approximation in Fig. 1 and the final result produced by the proposed method in Fig. 2.

The result shows that the method is effective for a region G with internal cutouts and for cases when the sought region and the excitation source are located on different sides of the internal cutouts.

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