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# AN ALGORITHM FOR SOLVING THE FRACTIONAL VIBRATION EQUATION

**S. T. Mohyud-Din**[1] **and A. Yıldırım**[2]

In this paper, we present a framework to obtain the solutions to the fractional vibration equation by the homotopy perturbation method. The fractional derivative is described in the Caputo sense. Our method performs extremely well in terms of efficiency and simplicity. Numerical results are presented graphically showing the complete reliability of the proposed algorithm.

**Keywords:** homotopy perturbation method, Caputo derivative, fractional vibration equation, wave velocity, Mittag-Leffler function.

## 1. Introduction

In recent years, analysis of fractional differential equations, which are obtained from the classical differential equations in mathematical physics, engineering, vibration, and oscillation by replacing the second-order time derivative by a fractional derivative of order $\alpha$ satisfying $1 < \alpha \le 2$, has been a field of growing interest as evident from the literature survey. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Analytical methods used to solve these equations have very restricted applications, and the numerical techniques commonly used give rise to rounding off errors. Several mathematical methods including the Adomian decomposition method, modified decomposition method, variational iteration method, differential transform method, and homotopy perturbation method have been developed to obtain exact and approximate analytic solutions to differential equations of fractional order; see [1–8] and references therein. The basic motivation of this paper is the extension of the powerful algorithm of the homotopy perturbation method to solve the fractional vibration equation. This fractional vibration equation is obtained by replacing the second time derivative term in the corresponding vibration equation by a fractional derivative of order $\alpha$ with $1 < \alpha \le 2$. The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case $\alpha = 2$, the fractional vibration equation reduces to the standard vibration equation.

The homotopy perturbation method was introduced by He [9–13] by merging the standard homotopy and perturbation and has been applied to a wide class of diverse nonlinear problems of physical nature; see [9–33] and the references therein. The numerical results explicitly reveal the complete reliability and efficiency of the proposed iterative scheme.

## 2. Fractional Calculus

We give some basic definitions and properties of the fractional calculus theory, which are used further in this paper.

[1] Department of Mathematics, HITEC University, Taxila, Pakistan; e-mail: syedtauseefs@hotmail.com (corresponding author).
[2] Department of Mathematics, Ege University, İzmir, Turkey.

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***Definition 2.1.*** *A real function* $f(x)$, $x > 0$, *is said to be in the space* $C_\mu$, $\mu \in R$, *if there exists a real number* $p(>\mu)$ *such that* $f(x) = x^p f_1(x)$, *where* $f_1(x) \in C[0,\infty)$, *and it is said to be in the space* $C_\mu^m$ *if* $f^{(m)} \in C_\mu$, $m \in N$.

***Definition 2.2.*** *The Riemann–Liouville fractional integral operator of order* $\alpha \geq 0$, *of a function* $f \in C_\mu$, $\mu \geq -1$, *is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad \alpha > 0, \quad x > 0,$$

$$J^0 f(x) = f(x).$$

The properties of the operator $J^\alpha$ can be found in [33–36]; here we mention only the following: For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,

2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,

3. $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce the modified fractional differential operator $D^\alpha$ proposed by Caputo in his work on the theory of viscoelasticity [37].

***Definition 2.3.*** *The fractional derivative* $f(x)$ *in the Caputo sense is defined as*

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \tag{1}$$

*for* $m-1 < \alpha \leq m$, $m \in N$, $x > 0$, *and* $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

**Lemma 2.1.** *If* $m-1 < \alpha \leq m$, $m \in N$ *and* $f \in C_\mu^m$, $\mu \geq -1$, *then* $D^\alpha J^\alpha f(x) = f(x)$, *and,*

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivatives are considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider a fractional vibration equation, and the fractional derivatives are taken in the Caputo sense as follows:

***Definition 2.4.*** *For* $m$ *to be the smallest integer that exceeds* $\alpha$, *the Caputo time-fractional derivative operator of order* $\alpha > 0$ *is defined as*

$$D_t^{\alpha}u(t) = \frac{\partial^{\alpha}u(t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial t^m}d\tau & \text{for } m-1<\alpha<m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m} & \text{for } \alpha = m \in N. \end{cases} \tag{2}$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

## 3. Fractional Vibration Equation

We consider the fractional calculus version of the standard vibration equation in one dimension as

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}, \qquad r \ge 0, \quad t \ge 0, \quad 1<\alpha\le 2, \tag{3}$$

which constitute the relation between the radial velocity of $u(r,t)$ to the fractional time derivative of order $\alpha$ $(1 < \alpha \le 2)$ of $u(r,t)$, and $c$ is the wave velocity of free vibration. It is easily seen that the whole hierarchy of moments $M_k = \left\langle r^k(t)\right\rangle$ have the same time dependence as for the fractional Brownian motion though their statistical features are quite different. Now taking the Laplace transform of Eq. (3), we get

$$s^{\alpha}\bar{u}(r,s) = c^2\left[\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}\right], \tag{4}$$

where $\bar{u}(r,s) = L\left[u(r,t)\right]$.

Equation (4) can be written as

$$r\frac{d^2}{dr^2}\bar{u}(r,s)+\frac{d}{dr}\bar{u}(r,s)-\frac{s^{\alpha}}{c^2}r\bar{u}(r,s) = 0. \tag{5}$$

Taking the series solution of $\bar{u}(r,s)$ as

$$\bar{u}(r,s) = \sum_{n=0}^{\infty}a_n r^{n+\rho}, \quad a_0 \ne 0, \quad \rho \text{ is real}, \tag{6}$$

we finally obtain

$$\bar{u}(r,s) = A\left(1+\ln r\right)+\frac{B}{c^2}s^{\alpha}r^2+o(s^{2\alpha}), \tag{7}$$

where $A$ and $B$ are constants.

Therefore,

$$u(r,s) \approx t^{-\alpha-1}, \tag{8}$$

which clearly exhibits the power law decay of $u(r,t)$ with $\alpha$ in contrast to the stretched exponential decay characteristic generally seen in fractional Brownian motion.

## 4. Solution Procedure

In this section the application of the homotopy perturbation method is discussed in solving the fractional vibration equation (3) with the initial conditions

$$u(r,0) = r^2, \tag{9}$$

$$\frac{\partial}{\partial t}u(r,0) = cr. \tag{10}$$

Equation (3) can be written as

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^{2-\alpha}}{\partial t^{2-\alpha}}\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right] \tag{11}$$

To solve Eqs. (9)–(11) by the homotopy perturbation method, we construct the following homotopy:

$$\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u_0}{\partial t^2}\right) = p\left(c^2\frac{\partial^{2-\alpha}}{\partial t^{2-\alpha}}\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]-\frac{\partial^2 u_0}{\partial t^2}\right). \tag{12}$$

Assume the solution of Eq. (12) to be in the form

$$u = u_0+pu_1+p^2u_2+p^3u_3+\dots. \tag{13}$$

Substituting Eq. (13) into Eq. (12) and collecting terms of the same power of $p$ give

$$p^0: \frac{\partial^2 u_0}{\partial t^2}-\frac{\partial^2 u_0}{\partial t^2} = 0, \tag{14}$$

$$p^1: \frac{\partial^2 u_1}{\partial t^2} = c^2\frac{\partial^{2-\alpha}}{\partial t^{2-\alpha}}\left[\frac{\partial^2 u_0}{\partial r^2}+\frac{1}{r}\frac{\partial u_0}{\partial r}\right]-\frac{\partial^2 u_0}{\partial t^2}, \tag{15}$$

$$p^2: \frac{\partial^2 u_2}{\partial t^2} = c^2 \frac{\partial^{2-\alpha}}{\partial t^{2-\alpha}}\left[\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r}\right], \tag{16}$$

$$p^3: \frac{\partial^2 u_3}{\partial t^2} = c^2 \frac{\partial^{2-\alpha}}{\partial t^{2-\alpha}}\left[\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r}\right], \quad \text{etc.} \tag{17}$$

The initial conditions admit the use of

$$u_0(r,t) = u(r,0) + t\frac{\partial}{\partial t}u(r,0) = r^2 + crt\,. \tag{18}$$

The solution reads

$$u_1(r,t) = \frac{4c^2 t^\alpha}{\Gamma(\alpha+1)}, \tag{19}$$

$$u_2(r,t) = \frac{c^3 t^{\alpha+1}}{r\Gamma(\alpha+2)}, \tag{20}$$

$$u_3(r,t) = \frac{c^5 t^{2\alpha+1}}{r^3\Gamma(2\alpha+2)}, \tag{21}$$

$$u_4(r,t) = \frac{9c^7 t^{3\alpha+1}}{r^5\Gamma(3\alpha+2)}, \tag{22}$$

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained.
The solution of Eqs. (9)–(11) can be obtained by setting $p = 1$ in Eq. (13):

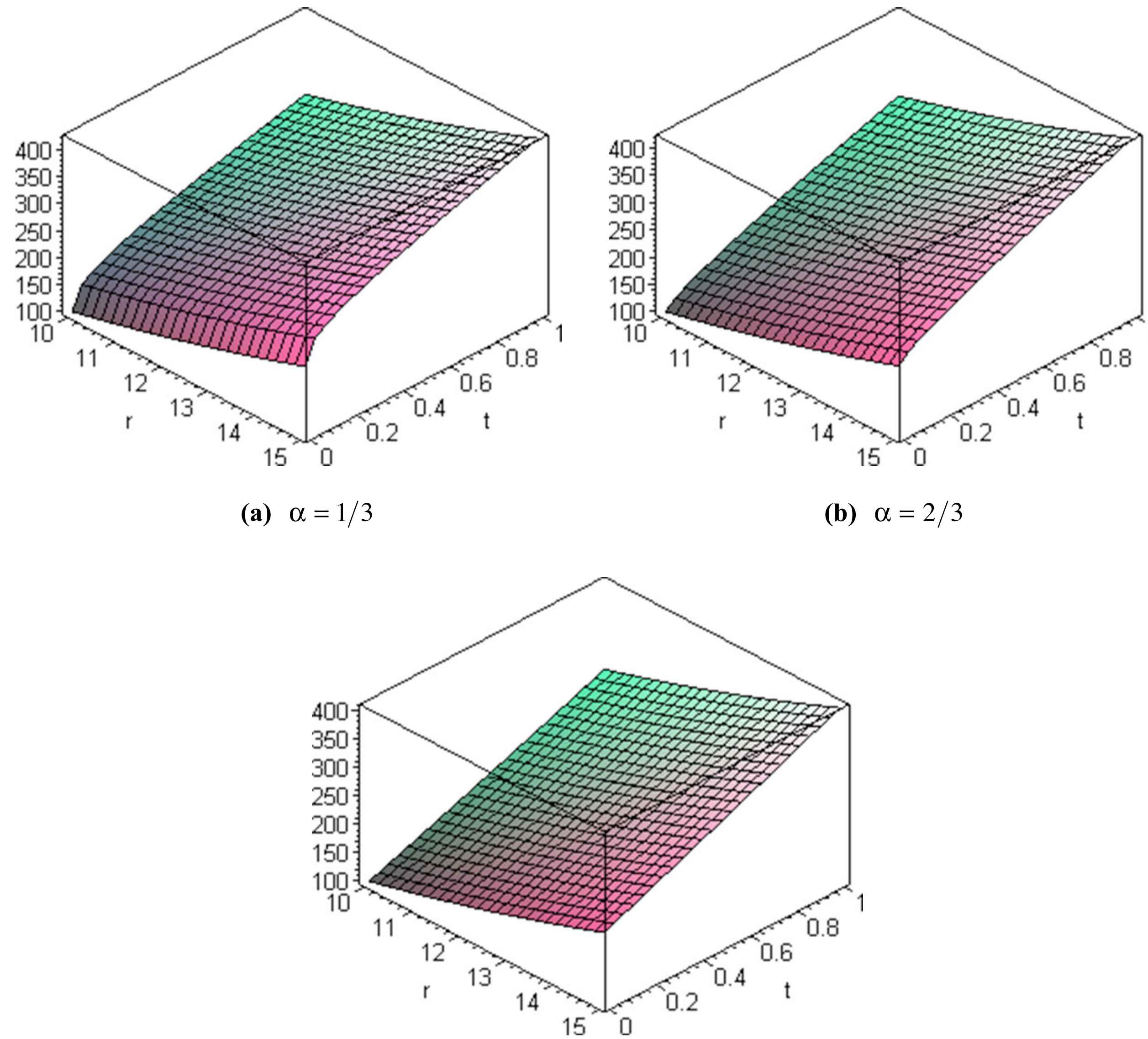
$$u = u_0 + u_1 + u_2 + u_3 + \dots. \tag{23}$$

Thus the exact solution may be obtained by using

$$u(r,t) = \sum_{n=0}^{\infty} u_n(r,t)\,, \tag{24}$$

$$= r^2 + crt + \frac{4c^2 t^\alpha}{\Gamma(\alpha+1)} + \frac{c^3 t^{\alpha+1}}{r\Gamma(\alpha+2)} + \frac{c^5 t^{2\alpha+1}}{r^3\Gamma(2\alpha+2)} + \frac{9c^7 t^{3\alpha+1}}{r^5\Gamma(3\alpha+2)} + \dots, \tag{25}$$

$$= r^2 + \frac{4c^2 t^\alpha}{\Gamma(\alpha+1)} + crtE_{\alpha,2}\left(\frac{c^2}{r^2}kt^\alpha\right), \tag{26}$$




**(a)** $\alpha = 1/3$ **(b)** $\alpha = 2/3$

**(c)** $\alpha = 1$

**Fig. 1.** Plot of $u(r,t)$ with respect to $r$ and $t$ at $c = 5$.

where

$$k^n = [1.3.5\ldots(2n-3)]^2$$

and

$$E_{\alpha,b}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha + b)}$$

is the generalized Mittag-Leffler function [38].

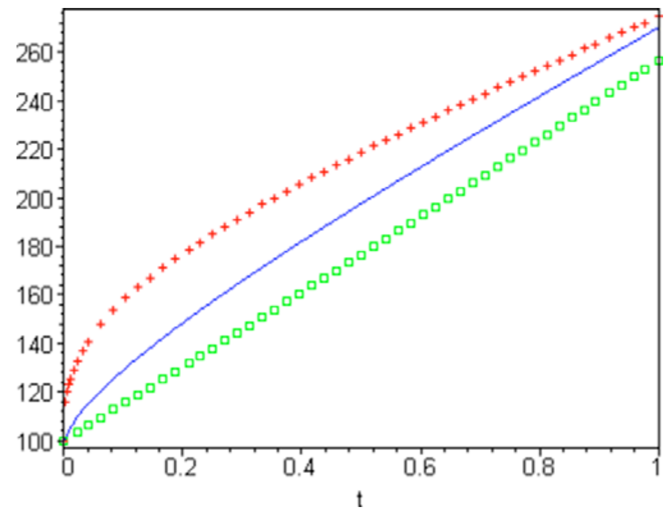
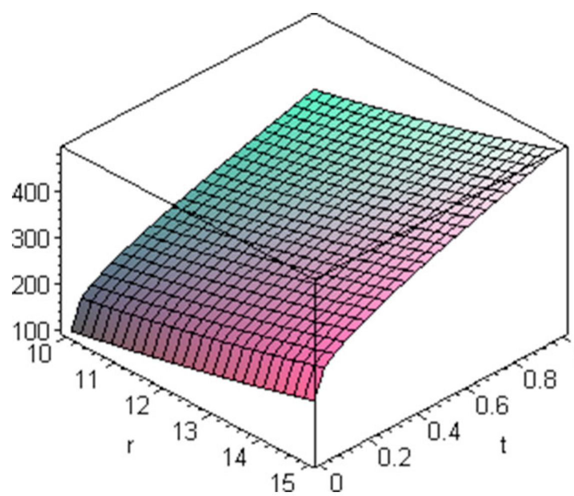
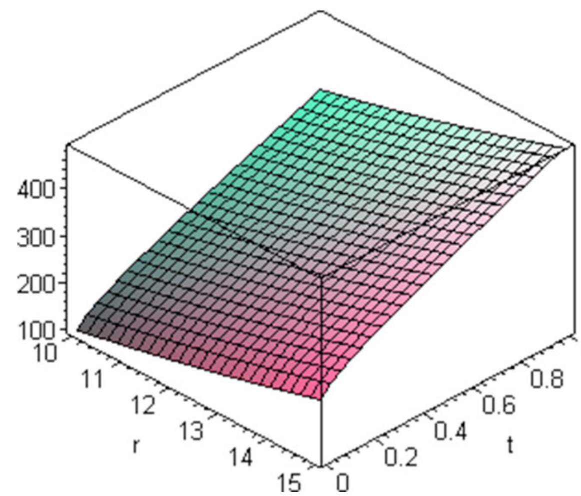


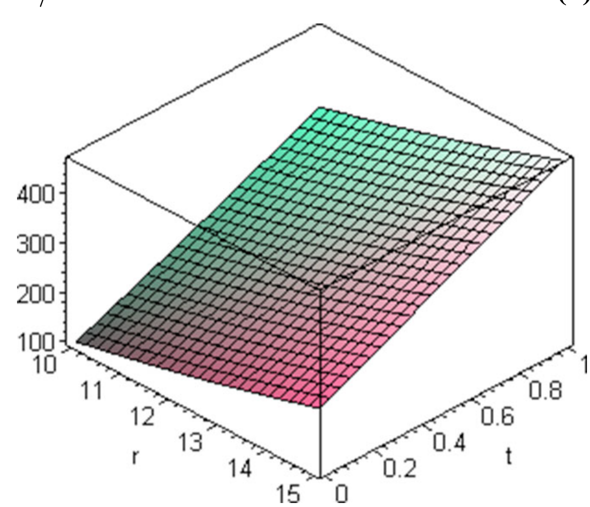

**Fig. 2.** Plot of $u(r,t)$ vs. $t$ for different values of $\alpha$ at $r = 10$ and $c = 5$; $\alpha$: (+) $\alpha = 1/3$, (–) $\alpha = 2/3$, (□) $\alpha = 1$.




**(a)** $\alpha = 1/3$




**(b)** $\alpha = 2/3$




**(c)** $\alpha = 1$

**Fig. 3.** Plot of $u(r,t)$ with respect to $r$ and $t$ at $c = 6$.

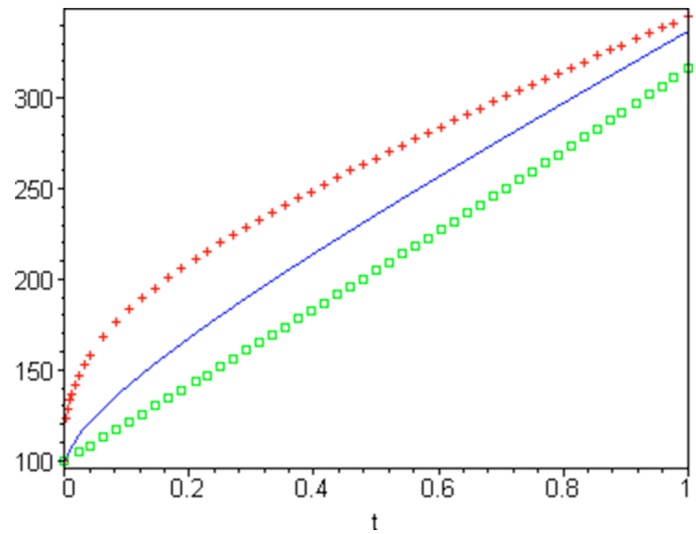



**Fig. 4.** Plot of $u(r, t)$ vs. $t$ for different values of $\alpha$ at $r = 10$ and $c = 6$; $\alpha$: (+) $\alpha = 1/3$, (–) $\alpha = 2/3$, (□) $\alpha = 1$.

## 5. Numerical Results and Discussion

In this section, numerical results of the displacement for various values of radii of the membrane and time are presented through Figs. 1–4. It is observed from Figs. 1 and 3 that the displacement increases with increase of both $r$ and $t$ for both wave velocities $c = 5$ and $c = 6$. It is also seen from Figs. 2 and 4 that the displacement rapidly increases with increase of $t$ and $c$ both at a fixed value of the radius of the membrane $(r = 10)$ but decreases with increase of the fractional time derivative $\alpha$, which is in complete agreement with the fact described in Section 3. The numerical calculations and figures are made using Maple software (Version 10).

## 6. Conclusions

The homotopy perturbation method is very powerful in finding solutions for various physical, vibration, and oscillation problems. The main interest is in finding numerical solutions of vibration equation. It is seen that our method is efficient for finding the solutions of a higher degree of accuracy. Our method is direct and straightforward and avoids voluminous calculations. Also the homotopy perturbation method facilitates computational work, for which it gives the required solution faster in comparison with other methods [1–6]. Another important part of the study is to explain the decay of $u(r, t)$ with increase in the fractional time derivative $\alpha$, which has been accomplished by the authors. The author strongly believes that the present study in solving fractional vibration equations for very large membranes constitutes a significant change from the usual approach and thus will considerably benefit engineers working in this field.

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